Introduction to Partial Differential Equations (PDEs) - Part I
Master Degree in Mathematical Engineering

Course Description - A.Y. 2015-2016
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1. **Basic concepts and definitions**

   Notations.

   Order of an equation. Classification of equations: linear, semilinear, quasilinear and fully nonlinear PDEs.

   Discussion of some linear models: transport equation, diffusion equation, wave equation, Laplace's equation, Black-Scholes equation, Schrödinger's equation, Maxwell's equations.

   Discussion of some nonlinear models: Burgers' equation, Eikonal equation, minimal surface equation, Navier-Stokes equations.

   Well-posed problems.

   Transport equation with constant coefficients. Initial-value problem for homogeneous and nonhomogeneous equation. Classical and weak solutions.

   Classification of second order linear equations: elliptic, parabolic, hyperbolic PDEs.

2. **Laplace's equation and harmonic functions**

   Physical interpretation of Laplace's equation. Harmonic functions (C^2-solutions of Laplace's equation). General properties. Invariance of Laplace's equation under translations and rotations of the domain variables. Derivation of the fundamental solution. Newtonian potential. Solution of Poisson's equation on the whole space $\mathbb{R}^n$ for a smooth density with compact support or for a bounded and Hölder continuous density (without proof) .

   Definition of Green's function and Poisson's Kernel for an open domain of $\mathbb{R}^n$. Green's representation formula for solutions of the Poisson equation with Dirichlet boundary conditions. Symmetry of Green's function.

   Construction of Green's function for an half-space. Poisson's formula for harmonic functions on an half-space. Solution of the Laplace's equation with Dirichlet boundary conditions on an half-space.

   Construction of Green's function for a ball. Poisson's formula for harmonic functions on a ball. Solution of the Laplace's equation with Dirichlet boundary conditions on a ball (without proof).

   Definition of Neumann function for an open domain of $\mathbb{R}^n$. Green's representation formula for solutions of the Poisson's equation with Neumann boundary conditions.

   Distributions. Definition and some examples: L^p_{loc}-functions, fundamental solution of Laplace's equation, Dirac's delta. Injectivity of the map that associates to any function in L^p_{loc} the corresponding distribuiton.

Definition of weak derivative of a distribution. Examples: absolute value, fundamental solution of Laplace's equation. Sobolev spaces $W^{1,p}(\Omega), W_0^{1,p}(\Omega)$, $H^1(\Omega), H_0^1(\Omega)$. Basic properties. Variational formulation of homogeneous Dirichlet problem for the Laplacian in a bounded domain of $\mathbb{R}^n$. Existence of weak solutions based on Lax-Milgram Theorem. Regularity of weak solutions (without proof). Recovery of classical solutions.

Relation between harmonic functions on $\mathbb{R}^2$ and holomorphic functions.

Mean-value formulas for harmonic functions over the sphere and the entire ball. Koebe Theorem: continuous functions enjoying the mean-value property are $\mathcal{C}^\infty$ and harmonic. Local a-priori bounds on the derivatives of harmonic functions (Cauchy estimates). Analiticity of harmonic functions (without proof). Regularity of $H^1(\mathbb{R}^n)$ weak solutions of Laplace equations. Liouville's Theorem for lower (or upper) bounded harmonic functions on $\mathbb{R}^n$.

Harnack's inequality. Convergence theorems for sequence of harmonic functions: uniformly convergent on compact sets; monotone and pointwise bounded (Harnack's Convergence Theorem); equicontinuous and pointwise bounded (Ascoli- Arzelà Theorem)

Sub-harmonic and super-harmonic functions. Strong maximum (minimum) principles for sub-harmonic (super-harmonic) functions on connected domains. Weak maximum (minimum) principles for sub-harmonic (super-harmonic) functions on bounded domains. Positivity and comparison principle. Uniqueness and stability of solutions of the Dirichlet problem for the Poisson's equation on bounded domains. Counterexamples to uniqueness of solutions in unbounded domains.

Representation formula of bounded solutions to Poisson's equation in the whole space $\mathbb{R}^n$ for a smooth density with compact support. Uniqueness results for solutions of the Poisson's equation on complementary sets of bounded domains among solutions growing less than logarithmically at infinity ($n = 2$) or vanishing at infinity ($n \geq 2$). Uniqueness of bounded solutions of the Dirichlet problem for the the Poisson's equation on an half-space (symmetrization procedure).

Characterization of sub-harmonic and super-harmonic functions. Properties of sub-harmonic and super-harmonic functions. Harmonic lifting of sub-harmonic functions. Perron's Theorem: existence of generalized solutions of the Dirichlet problem for the Laplace's equation with bounded boundary data on open bounded domains. Regular boundary points with respect to the Laplacian: local and global barrier functions at a boundary point. Wiener's Theorem: existence of solutions of the Dirichlet problem for the Laplace's equation with continuous boundary data on open bounded domains with regular boundary. Geometric criterions for the existence of barriers at a boundary point: external sphere and cone conditions.

3. Heat (or diffusion) equation

Physical interpretation of the diffusion equation. General properties. Superposition principle. Temporal irreversibility and symmetries: invariance of Heat equation under translations and parabolic dilations of the domain variables, and under reflections of the space variables. Mixed initial boundary value problems for the heat equation. Boundary conditions of Dirichlet, Neumann and Robin (or radiation) type. Parabolic interior and boundary.

Analysis of the steady state solution and of the transient regime for the heat equation on a bounded interval, with constant initial data and Dirichlet boundary conditions. Method of separation of variables to solve the mixed problem for the homogeneous heat equation with homogeneous Dirichlet boundary conditions. Convergence of the series (and of its derivatives) constructed by the method of separation of variables. Attainability of the initial condition in the least square (or L^2) sense and in the uniform convergence topology on compact subsets of the domain not containing the right-hand side.

Energy method to establish uniqueness of solutions of the mixed problem for the nonhomogeneous heat equation on bounded C^1-domains with Dirichlet boundary conditions and initial condition satisfied in the L^2 sense (without proof).

Reference Textbooks

1. S. Salsa, *Partial Differential Equations in Action: From Modelling to Theory*, Springer, Universitext, Milano, 2008.
2. L. C. Evans *Partial Differential Equations*, AMS, 2010 (2^{nd} edition).