Introduction to Partial Differential Equations (PDEs) - Part I

Master Degree in Mathematical Engineering

Course Description - A.Y. 2016-2017

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1. **Basic concepts and definitions**

   Notations.

   Order of an equation. Classification of equations: linear, semilinear, quasilinear and fully nonlinear PDEs.

   Discussion of some linear models: transport equation, diffusion equation, wave equation, Laplace's equation, Black-Scholes equation, Schrödinger's equation, Maxwell's equations.

   Discussion of some nonlinear models: Burgers' equation, Eikonal equation, minimal surface equation, Navier-Stokes equations.

   Well-posed problems.

2. **First-order PDEs**

   Transport equation with constant coefficients. Initial-value problem for homogeneous and nonhomogeneous equation. Classical and weak solutions.

   Conservation laws. Discussion of some models: Burgers equation, vehicular traffic. Method of characteristics to construct classical solutions.

   Weak distributional solutions. Equivalence between classical and weak solutions for C^1 solutions. Rankine-Hugoniot conditions. Characterization of weak solutions that are piecewise continuous with finitely many curves of discontinuities. Non uniqueness of weak solutions. Admissibility conditions for weak solutions: vanishing viscosity, stability conditions on discontinuities. Riemann problem for conservation law with convex flux. Construcion of solutions: rarefaction waves, shock waves.

3. **Wave equation**

   Classification of second order linear PDEs: elliptic, parabolic, hyperbolic PDEs.

   Physical interpretation of the wave equation.

   D'Alembert's formula for solutions of the Cauchy problem of the one-dimensional wave equation on the line. Progressive and regressive waves. Characteristic lines, domain of dependence (principle of causality) and of influence of the solution. No regularization effect on the solutions. D'Alembert's formula for solutions of the mixed problem with homogeneous Dirichlet boundary conditions relative to the one-dimensional wave equation on the half-line and on a bounded interval (reflection method).

   Method of separation of variables to solve the mixed problem for the one-dimensional wave equation on a bounded interval with homogeneous Dirichlet boundary conditions.

Method of spherical means for solutions of the Cauchy problem of the wave equation in $\mathbb{R}^n$, $n \geq 2$. The Euler-Poisson-Darboux equation. Kirkhoff's formula for solutions of the Cauchy problem of the wave equation in $\mathbb{R}^3$. Method of descent. Poisson's formula for solutions of the Cauchy problem of the wave equation in $\mathbb{R}^2$. Huygen's principle.

Duhamel's principle to obtain solutions of the Cauchy problem for the nonhomogeneous wave equation. The case of the one-dimensional nonhomogeneous wave equation on the line.

Energy methods. Uniqueness of solutions of the mixed problem for the nonhomogeneous wave equation with Dirichlet boundary conditions. Domain of dependence and finite speed of propagation.

4. Laplace's equation and harmonic functions

Physical interpretation of Laplace's equation. Harmonic functions (C^2-solutions of Laplace's equation). General properties. Invariance of Laplace's equation under translations and rotations of the domain variables. Derivation of the fundamental solution. Newtonian potential. Solution of Poisson's equation on the whole space $\mathbb{R}^n$ for a smooth density with compact support or for a bounded and Hölder continuous density (without proof) . Interpretation of the Laplacian of the fundamental solution as minus the Dirac's delta.

Green's representation formula for solutions of the Poisson equation with Dirichlet boundary conditions. Definition of Green's function and Poisson's Kernel for an open domain of $\mathbb{R}^n$. Properties of the Green function: positivity, symmetry. Interpretation of the Laplacian of the Green function as minus the Dirac's delta.

Construction of Green's function for an half-space. Poisson's formula for harmonic functions on an half-space. Solution of the Laplace's equation with Dirichlet boundary conditions on an half-space.

Construction of Green's function for a ball. Poisson's formula for harmonic functions on a ball. Solution of the Laplace's equation with Dirichlet boundary conditions on a ball of arbitrary radius. Method of separation of variables to solve the Laplace's equation with Dirichlet boundary conditions on a two-dimensional ball of arbitrary radius.

Definition of Neumann function for an open domain of $\mathbb{R}^n$. Green's representation formula for solutions of the Poisson's equation with Neumann boundary conditions.

Distributions. Definition and some examples: L^p_{loc}-functions, fundamental solution of Laplace's equation, Dirac's delta. Convolutions and mollifiers: definition and properties. Injectivity of the map that associates to any function in L^p_{loc} the corresponding distribution.

Definition of weak derivative of a distribution. Examples: absolute value, fundamental solution of Laplace's equation. Sobolev spaces $W^{1,p}(\Omega), W^{1,p}_0(\Omega)$, $H^1(\Omega), H^1_0(\Omega)$. Basic properties. Variational formulation of homogeneous Dirichlet problem for the Laplacian in a bounded domain of $\mathbb{R}^n$. Existence of weak solutions based on Lax-Milgram Theorem. Regularity of weak solutions (without proof). Recovery of classical solutions.

Relation between harmonic functions on $\mathbb{R}^2$ and holomorphic functions.

Mean-value formulas for harmonic functions over the sphere and the entire ball. Koebe Theorem: continuous functions enjoying the mean-value property are $\mathcal{C}^\infty$ and harmonic.

Local a-priori bounds on the derivatives of harmonic functions (Cauchy estimates). Analiticity of harmonic functions (without proof). Regularity of $H^1(\mathbb{R}^n)$ weak solutions of Laplace equations. Liouville's Theorem for lower (or upper) bounded harmonic functions on $\mathbb{R}^n$.

Harnack's inequality. Convergence theorems for sequence of harmonic functions: uniformly convergent on compact sets; monotone and pointwise bounded (Harnack's Convergence Theorem); equicontinuous and pointwise bounded (Ascoli- Arzelà Theorem)

Sub-harmonic and super-harmonic functions. Strong maximum (minimum) principles for sub-harmonic (super-harmonic) functions on connected domains. Weak maximum (minimum) principles for sub-harmonic (super-harmonic) functions on bounded domains.

Reference Textbooks

1. S. Salsa, *Partial Differential Equations in Action: From Modelling to Theory*, Springer, Universitext, Milano, 2008.
2. L. C. Evans *Partial Differential Equations*, AMS, 2010 (2^{nd} edition).