$\boxed{\text{Es 1}}$ $\displaystyle\int_1^{+\infty} \frac{e^{\alpha x}}{\ln(1+x^2)}\,dx = I_\alpha, \quad \alpha \in \mathbb{R}$

$\alpha < 0$, $\displaystyle\lim_{x\to+\infty} \frac{e^{\alpha x}}{\ln(1+x^2)} \cdot x^2 = 0 \Rightarrow I_\alpha$ è convergente

per confronto asintotico con $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ convergente.

$\alpha = 0$, $\displaystyle\int_2^{+\infty} \frac{1}{\ln(1+x^2)}\,dx \sim \int_2^{+\infty} \frac{dx}{\ln x} > \int_2^{+\infty} \frac{dx}{\ln x}$ divergente a $+\infty$

$\Rightarrow I_0 = +\infty$

$\alpha > 0$, $\displaystyle\lim_{x\to+\infty} \frac{e^{\alpha x}}{\ln(1+x^2)} = +\infty \Rightarrow I_\alpha$ divergente a $+\infty$

perché l'integranda non è infinitesima

$\Rightarrow I_\alpha$ è $\begin{cases} \text{convergente} & \text{se } \alpha < 0 \\ \text{divergente a } +\infty & \text{se } \alpha \geq 0 \end{cases}$

$\boxed{\text{Es. 2}}$ $\displaystyle \ell = \lim_{n\to\infty} \frac{\sqrt{n^2+1} - n - \ln\left(1+\frac{1}{2n}\right)}{\cos\left(\frac{1}{n}\right) - 1} = \lim_{n\to\infty} \frac{N}{D}$

$\sqrt{n^2+1} - n = n\left(\left(1+\frac{1}{n^2}\right)^{1/2} - 1\right) = n\left(\frac{1}{2n^2} + o\left(\frac{1}{n^3}\right)\right) = \frac{1}{2n} + o\left(\frac{1}{n^2}\right)$

$\ln\left(1+\frac{1}{n}\right) = \frac{1}{2n} - \frac{1}{8n^2} + o\left(\frac{1}{n^2}\right) \Rightarrow \boxed{N = \frac{1}{8n^2} + o\left(\frac{1}{n^2}\right)}$

$D = -\frac{1}{2n^2} + o\left(\frac{1}{n^3}\right) \Rightarrow \boxed{\ell} = \displaystyle\lim_{n\to\infty} \frac{\frac{1}{8n^2}}{-\frac{1}{2n^2}} = \boxed{-4.}$

$\boxed{\text{Es. 3}}$ $f(x) = \arctan\left(\dfrac{\sqrt{3}\,x}{1+x+x^2}\right)$

i) $\mathrm{Dom}(f) = \mathbb{R}$, $1+x+x^2$ ha discriminante $\Delta = 1-4 = -3 < 0$

ii) $\displaystyle\lim_{x\to\pm\infty} f(x) = 0 \Rightarrow$ $\boxed{y = 0 \text{ asintoto orizzontale bilatero per } x \to \pm\infty}$

iii)

$$f'(x) = \frac{1}{\left(1+\frac{3x^2}{(1+x+x^2)^2}\right)} \cdot \frac{\sqrt{3}x+\sqrt{3}+\sqrt{3}x^2-\sqrt{3}x-2\sqrt{3}x}{(1+x+x^2)^2} = \frac{\sqrt{3}(1-x^2)}{3x^2+(1+x+x^2)^2} =$$

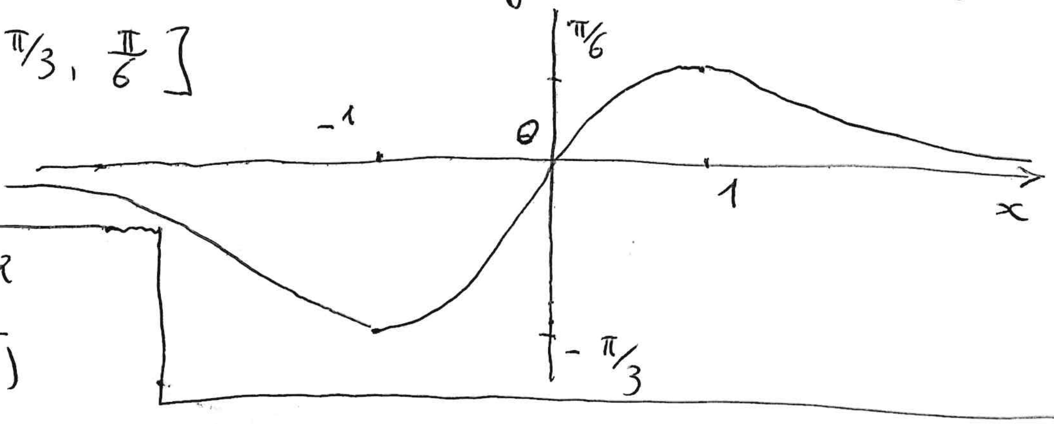
$$= \frac{\sqrt{3}(1-x^2)}{(1+2x+6x^2+2x^3+x^4)} \Rightarrow \{f' \geq 0\} = [-1,1]$$

$\Rightarrow$ $\boxed{\begin{array}{l} f \text{ è crescente su } [-1,1] \\ f \text{ è decrescente su }]-\infty,-1] \text{ e su } [1,+\infty[\end{array}}$

iv) $x=1$ punto di massimo assoluto, $\max f = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$x=-1$ punto di minimo assoluto, $\min f = \arctan(-\sqrt{3}) = -\frac{\pi}{3}$

v) $\mathrm{Im}(f) = [-\pi/3, \frac{\pi}{6}]$



Es. 4

$$F(x) = \int_{x^2-4x+3}^{-x^2+3x-2} \frac{dt}{\ln(1+|t|)}$$

i) $\mathrm{Dom}(F) =]2,3[$

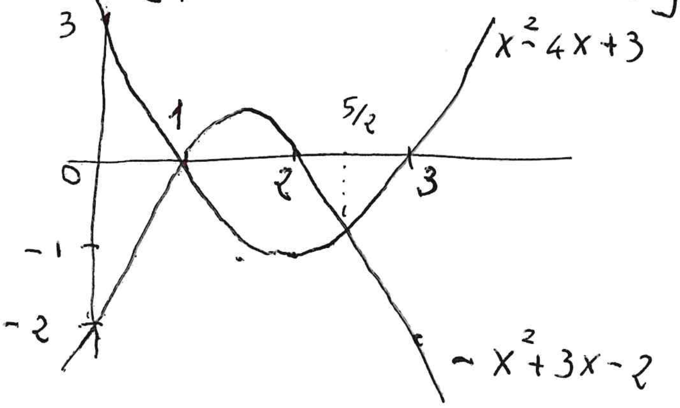
$\int \frac{1\, dt}{\ln(1+|t|)} \underset{t\to 0}{\sim} \int \frac{1}{|t|}\, dt$ che non è integrabile in $t=0$ $\Rightarrow$

$\mathrm{Dom}(F) = \{x \,;\, x \notin [\min\{x^2-4x+3, -x^2+3x-2\}, \max\{x^2-4x+3, -x^2+3x-2\}]\}$

x^2-4x+3, $x_{1,2} = 2 \pm 1 = \begin{cases} 3 \\ 1 \end{cases}$

$-x^2+3x-2$, $x_{1,2} = \frac{-3\pm 1}{-2} = \begin{cases} 2 \\ 1 \end{cases}$

$\Rightarrow$



$\Rightarrow \boxed{\mathrm{Dom}(F) =]2,3[}$

$F(x) \geq 0 \iff -x^2+3x-2 \geq x^2-4x+3 \iff 2x^2-7x+5 \leq 0$

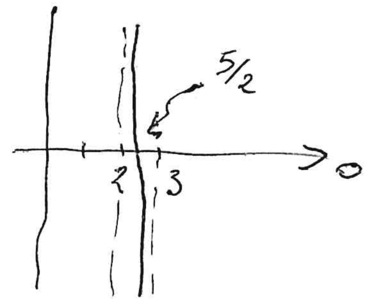
$2x^2-7x+5$, $x_{1,2} = \frac{7\pm\sqrt{49-40}}{4} = \begin{cases} \frac{5}{2} \\ 1 \end{cases}$ $\Rightarrow \boxed{\{F \geq 0\} =]2, \frac{5}{2}]}$

ii) $\lim_{x \to 2^+} F(x) = \int_{-1} \frac{dt}{\ln(1+|t|)} = +\infty \Rightarrow$ $\boxed{x = 2 \text{ asintoto verticale}}$

$\lim_{x \to 3^-} F(x) = \int_0^{-2} \frac{dt}{\ln(1+|t|)} = -\infty \Rightarrow$ $\boxed{x = 3 \text{ asintoto verticale}}$

$$F'(x) = \frac{(-2x+3)}{\ln(1+|x^2-3x+2|)} - \frac{(2x-4)}{\ln(1+|x^2-4x+3|)} < 0 \quad \forall x \in \,]2,3[$$

$\Rightarrow$ F è decrescente



Es. 5 $y' = (e^x - e^{-x})y + \operatorname{senh}(x)$

i) sol. eq. omogenea associata $y' = (e^x - e^{-x})y$;

$$\boxed{\Psi_c(x) = c\, e^{(2\cosh(x))}, \quad c \in \mathbb{R}}$$

ii) sol. eq. completa: calcoliamo $\int e^{-2\cosh(x)} \operatorname{senh}(x)\,dx = \frac{e^{-2\cosh(x)}}{-2}$

$$\Rightarrow \boxed{\Psi_c(x) = e^{2\cosh(x)}\left[c - \frac{e^{-2\cosh(x)}}{2}\right] = c\,e^{2\cosh(x)} - \frac{1}{2}, \quad c \in \mathbb{R}}$$

iii) Pb di Cauchy $\begin{cases} y' = (e^x - e^{-x})y + \operatorname{senh}(x) \\ y(0) = 1 \end{cases}$

$1 = \Psi_c(0) = c\,e^2 - \frac{1}{2} \Rightarrow c = \frac{3}{2} e^{-2} \Rightarrow$

$$\boxed{\Psi(x) = \frac{e^{2\cosh(x)}}{2}\left[3e^{-2} - e^{-2\cosh(x)}\right] = \frac{3}{2} e^{2(\cosh(x)-1)} - \frac{1}{2}}$$