Computability

Paolo Baldan
Department of Mathematics
University of Padua

September 22, 2024

This is a collection of exam exercises, roughly organised by thematic areas. The exercises often come along with a solution, which is sometimes fully detailed and in some other cases only sketched.

The exercises that can be used for the preparation of the intermediate test are marked by a "(p)".

Please report any mistake you might find.

1 URM machine

Exercise 1.1(p). Consider a variant, denoted URM$^-$, of the URM machine obtained replacing the successor instruction $S(n)$ with a predecessor instruction $P(n)$. Executing $P(n)$ replaces the content r_n of register n with $r_n \dot{-} 1$. Determine the relation between the set $\mathcal{C}^-$ of the functions computable by a URM$^-$ machine and the set $\mathcal{C}$ of functions computable by a standard URM machine. Is one contained in the other? Is the inclusion strict? Justify your answer.

Exercise 1.2(p). Consider a variant of the URM machine where the jump and successor instructions are replaced by the instruction $JI(m, n, t)$ which compare the content r_m and r_n of of registers R_m and R_n and then:

- if $r_m = r_n$, increment register R_m and jump to the address t (it is intended that if t is outside the program, the execution of the program halts).
- otherwise, continue with the next instruction.

Describe the relation between the set $\mathcal{C}'$ of the functions computable by the new machine and the set $\mathcal{C}$ of the functions that can be computed by a standard URM machine. Is one included in the other? Is the inclusion strict? Justify your answers.

Exercise 1.3(p). Consider a variant URM^s of URM machine obtained by removing the successor $S(n)$ and jump $J(m, n, t)$ instructions, and inserting the instruction $JS(m, n, t)$, which compares the contents of register m and n, and if they coincide, it jumps to instruction t, otherwise it increments the m-th register and executes the next instruction. Determine the relation between the set $\mathcal{C}^s$ of functions computable by a URM^s machine and the set $\mathcal{C}$ of functions computable by a standard URM machine. Is one included in the other? Is the inclusion strict? Justify your answers.

Exercise 1.4(p). Consider the subclass of URM programs where, if the i-th instruction is a jump instruction $J(m, n, t)$, then $t > i$. Prove that the functions computable by programs in such subclass are all total.

Exercise 1.5. Consider a variant of the URM machine, which includes the jump and transfer instructions and two new instructions

- *A(m, n)* which adds to register m the content of register n, i.e., $r_m \leftarrow r_m + r_n$;
- *C(n)* which replaces the value in register n by its sign, i.e., $r_n \leftarrow sg(r_n)$.

Determine the relation between the set $\mathcal{C}'$ of the functions computable with the new machine and the set $\mathcal{C}$ of the functions that can be computed with the URM machine. Is one included in the other? Is the inclusion strict? Justify your answers.

Exercise 1.6(p). Consider a variant URM^m of the URM machine obtained by removing the successor instruction $S(n)$ and adding the instruction $M(n)$, which stores in the nth register the value $1 + \min\{r_i \mid i \leqslant n\}$, i.e., the successor of the least value contained in registers with index less than or equal to n. Determine the relation between the set $\mathcal{C}^m$ of functions computable by the URM^m machine and the set $\mathcal{C}$ of the functions computable by the ordinary URM machine. Is one included in the other? Is the inclusion strict? Justify your answers.

Exercise 1.7(p). Define the operation of primitive recursion and prove that the set $\mathcal{C}$ of URM-computable functions is closed with respect to this operation.

2 Primitive Recursive Functions

Exercise 2.1(p). Give the definition of the set $\mathcal{PR}$ of recursive primitive functions and, using only the definition, prove that the function $pow2 : \mathbb{N} \to \mathbb{N}$, defined by $pow2(y) = 2^y$, is primitive recursive.

Exercise 2.2(p). Give the definition of the set $\mathcal{PR}$ of primitive recursive functions and, using only the definition, prove that the the characteristic function χ_A of the set $A = \{2^n - 1 : n \in \mathbb{N}\}$ is primitive recursive. You can assume, without proving it, that sum, product, sg and $\overline{sg}$ are in $\mathcal{PR}$.

Exercise 2.3(p). Give the definition of the set $\mathcal{PR}$ of primitive recursive functions and, using only the definition, prove that the $\chi_{\mathbb{P}}$, the characteristic function of the set of even numbers $\mathbb{P}$ is primitive recursive.

Exercise 2.4(p). Give the definition of the set $\mathcal{PR}$ of primitive recursive functions and, using only the definition, prove the function $half : \mathbb{N} \to \mathbb{N}$, defined by $half(x) = x/2$, is primitive recursive.

Exercise 2.5(p). Give the definition of the set $\mathcal{PR}$ of primitive recursive functions and, using only the definition, prove that $p_2 : \mathbb{N} \to \mathbb{N}$ defined by $p_2(y) = |y - 2|$ is primitive recursive.

3 SMN Theorem

Exercise 3.1(p). State the smn theorem and prove it (it is sufficient to provide the informal argument using encode/decode functions).

Exercise 3.2(p). State the theorem s-m-n and use it to prove that it exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $|W_{s(x)}| = 2x$ and $|E_{s(x)}| = x$.

Exercise 3.3. State the smn theorem and use it to prove that there exists a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ such that $W_{s(x,y)} = \{z : x * z = y\}$

Exercise 3.4(p). Prove that there is a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$ it holds that $W_{k(n)} = \mathbb{P} = \{x \in \mathbb{N} \mid x \text{ even}\}$ and $E_{k(n)} = \{x \in \mathbb{N} \mid x \geqslant n\}$.

Exercise 3.5. State the smn theorem. Use it to prove it exists a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that $W_{k(n)} = \{x \in \mathbb{N} \mid x \geqslant n\}$ e $E_{k(n)} = \{y \in \mathbb{N} \mid y \text{ even}\}$ for all $n \in \mathbb{N}$.

4 Decidability and Semidecidability

Exercise 4.1. Prove the "structure theorem" of semidecidable predicates, i.e., show that a predicate $P(\vec{x})$ is semidecidable if and only if there exists a decidable predicate $Q(\vec{x}, y)$ such that $P(\vec{x}) \equiv \exists y.\ Q(\vec{x}, y)$.

Exercise 4.2. Prove the "projection theorem", i.e., show that if the predicate $P(x, \vec{y})$ is semidecidable then also $\exists x.\, P(x, \vec{y})$ is semi-decidable. Does the converse implication hold? Is it the case that if $P(x, \vec{y})$ is decidable then also $\exists x.\, P(x, \vec{y})$ is decidable? Give a proof or a counterexample.

5 Numerability and diagonalization

Exercise 5.1(p). Consider the set F_0 of functions $f : \mathbb{N} \to \mathbb{N}$, possibly partial, such that $cod(f) \subseteq \{0\}$. Is the set F_0 countable? Justify your answer.

Exercise 5.2(p). A function $f : \mathbb{N} \to \mathbb{N}$ is called *total increasing* when it is total and for each $x, y \in \mathbb{N}$, if $x < y$ then $f(x) < f(y)$. Prove that the set of total increasing functions is not countable.

Exercise 5.3(p). A function $f : \mathbb{N} \to \mathbb{N}$ is called *total increasing* when it is total and for each $x, y \in \mathbb{N}$, if $x \leqslant y$ then $f(x) \leqslant f(y)$. It is called binary if $cod(f) \subseteq \{0, 1\}$. Is the set of binary total increasing functions countable? Justify your answer.

6 Functions and Computability

Exercise 6.1(p). Define a function $f : \mathbb{N} \to \mathbb{N}$ total and not computable such that $f(x) = x$ for infinite arguments $x \in \mathbb{N}$ or prove that such a function cannot exist.

Exercise 6.2(p). Say that a $f function : \mathbb{N} \to \mathbb{N}$ is *increasing* if it is total and for each $x, y \in \mathbb{N}$, if $x \leqslant y$ then $f(x) \leqslant f(y)$. Is there an increasing function which is not computable? Justify your answer.

Exercise 6.3(p). Are there two functions $f, g : \mathbb{N} \to \mathbb{N}$ with g not computable such that the composition $f \circ g$ (defined by $(f \circ g)(x) = f(g(x))$) is computable? And requiring that f is also not computable, can the composition $f \circ g$ be computable? Justify your answer, giving examples or proving non-existence.

Exercise 6.4(p). Is there a function $f : \mathbb{N} \to \mathbb{N}$ with finite range, total and increasing (i.e. $f(x) \leqslant f(y)$ for $x \leqslant y$) and not computable? Justify your answer with an example or a proof of non-existence. What if we relax the requirement of totality?

Exercise 6.5(p). Say that a function $f : \mathbb{N} \to \mathbb{N}$ is *decreasing* if it is total and for each $x, y \in \mathbb{N}$, if $x \leqslant y$ then $f(x) \geqslant f(y)$. Is there a decreasing function which is not computable? Justify your answer.

Exercise 6.6(p). Say if there can be a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that for any other non-computable function $g : \mathbb{N} \to \mathbb{N}$ the function $f + g$ defined by $(f + g)(x) = f(x) + g(x)$ is computable. Justify your answer (providing an example of such f, if it exists, or proving that cannot exist).

Exercise 6.7. Say if there can be a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that there exists a non-computable function $g : \mathbb{N} \to \mathbb{N}$ for which the function $f+g$ (defined by $(f+g)(x) = f(x)+g(x)$) is computable. Justify your answer (providing an example of such f, if it exists, or proving that cannot exist).

Exercise 6.8(p). Say if there can be a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that $dom(f) \cap img(f)$ is finite. Justify your answer (providing an example of such f, if it exists, or proving that cannot exist).

Exercise 6.9. Is there non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that $dom(f) \cap img(f)$ is empty? Justify your answer (providing an example of such f, if it exists, or proving that cannot exist).

Exercise 6.10. Is there a total non-computable function $f : \mathbb{N} \to \mathbb{N}$, such that its image $cod(f) = \{y \mid \exists x \in \mathbb{N}.\ f(x) = y\}$ is finite? Provide an example or show that such a function does not exists.

Exercise 6.11(p). Prove that the function $f : \mathbb{N} \to \mathbb{N}$, defined as

$$f(x) = \begin{cases} \varphi_x(x) & \text{if } x \in W_x \\ x & \text{otherwise} \end{cases}$$

is not computable.

Exercise 6.12(p). Say if there is a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that, for infinite $x \in \mathbb{N}$ it holds

$$f(x) = \varphi_x(x)$$

If the answer is negative, provide a proof, if the answer is positive, provide an example of such a function.

Exercise 6.13. Say if there is a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(x) \neq \varphi_x(x)$$

only on a single argument $x \in \mathbb{N}$. If the answer is negative provide a proof, if the answer is positive give an example of such a function.

Exercise 6.14. Is there non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(x) \neq \varphi_x(x)$$

only on a single $x \in \mathbb{N}$? If the answer is negative provide a proof of non-existence, otherwise give an example of such a function.

Exercise 6.15. Is there a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that $cod(f)$ is the set $\mathbb{P}$ of even numbers? Justify your answer response (providing an example of such f, if it exists, or proving that it does not exist).

Exercise 6.16. Say if there is a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that the set $D = \{x \in \mathbb{N} \mid f(x) \neq \phi_x(x)\}$ is finite. Justify your answer.

Exercise 6.17. Say if there are total computable functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(x) \neq \varphi_x(x)$ for each $x \in K$ and $g(x) \neq \varphi_x(x)$ for each $x \notin K$. Justify your answer by providing a example or by proving non-existence.

Exercise 6.18. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} 2x + 1 & \text{if } \varphi_x(x) \downarrow \\ 2x \dot{-} 1 & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Exercise 6.19(p). Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} x & \text{sg } \forall y \leqslant x.\ \varphi_y \text{total} \\ 0 & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Exercise 6.20. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} x + 2 & \text{if } \varphi_x(x) \downarrow \\ x \dot{-} 1 & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Exercise 6.21. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} \varphi_x(x+1) + 1 & \text{if } \varphi_x(x+1) \downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Exercise 6.22. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_y(y) \downarrow \text{ for each } y \leqslant x \\ 0 & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Exercise 6.23. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } \varphi_x(x) \downarrow \\ x + 1 & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Exercise 6.24. A function $f : \mathbb{N} \to \mathbb{N}$ is called *almost total* if it is undefined on a finite set of points. Is there an almost total and computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f \subseteq \chi_K$? Justify your answer by giving an example of such a function in case it exists or a proof of non-existence, otherwise.

Exercise 6.25. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is *almost constant* if there is a value $k \in \mathbb{N}$ such that the set $\{x \mid f(x) \neq k\}$ is finite. Is there an almost constant function which is not computable? Adequately motivate your answer.

Exercise 6.26. Is there a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ with the property that $f(x) = x^2$ for all $x \in \mathbb{N}$ such that $\varphi_x(x) \downarrow$? Justify your answer by providing an example of such function, if it exists, or by proving that it does not exist, otherwise.

Exercise 6.27(p). Is there a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that for any non-computable function $g : \mathbb{N} \to \mathbb{N}$ the function $f * g$ (defined as $(f*g)(x) = f(x) \cdot g(x)$) is computable?

Justify your answer (providing an example of such f, if it exists, or proving that it does not exist).

Exercise 6.28(p). Define a function $f : \mathbb{N} \to \mathbb{N}$ total and not computable such that $f(x) = x/2$ for each even $x \in \mathbb{N}$ or prove that such a function does not exist.

Exercise 6.29. Is there a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that the function $g : \mathbb{N} \to \mathbb{N}$ defined, for each $x \in \mathbb{N}$, by $g(x) = f(x) \dot{-} x$ is computable? Provide an example or prove that such a function does not exist.

Exercise 6.30(p). Is there may be a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that for each non-computable function $g : \mathbb{N} \to \mathbb{N}$ the function $f + g$ (defined by $(f + g)(x) = f(x) + g(x)$) is computable? Justify your answer (providing an example of such f, if it exists, or proving that cannot exist).

Exercise 6.31. Is there a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $dom(f) = K$ and $cod(f) = \mathbb{N}$? Justify your answer.

Exercise 6.32. Let A be a recursive set and let $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ be computable functions. Prove that the function $f : \mathbb{N} \to \mathbb{N}$ defined below is computable:

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \notin A \end{cases}$$

Does the result hold if we weaken the hypotheses and assume A only r.e.? Explain how the proof can be adapted, if the answer is positive, or provide a counterexample, otherwise.

Exercise 6.33(p). Is there a total, non-computable function such that $img(f) = \{f(x) \mid x \in \mathbb{N}\}$ is the set Pr of Prime numbers? Justify your answer.

7 Reduction, Recursiveness and Recursive Enumerability

Exercise 7.1. Prove that a set A is recursive if and only if there is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $x \in A$ if and only if $f(x) > x$.

Exercise 7.2. Prove that a set A is recursive if and only if there are two total computable functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for each $x \in \mathbb{N}$

$$x \in A \text{ if and only if } f(x) > g(x).$$

Exercise 7.3. Prove that a set A is recursive if and only if $A \leqslant_m \{0\}$.

Exercise 7.4. Let $A \subseteq \mathbb{N}$ be a set and let $f : \mathbb{N} \to \mathbb{N}$ be a computable function. Prove that if A is r.e. then $f(A) = \{y \in \mathbb{N} \mid \exists x \in A.\ y = f(x)\}$ is r.e. Is the converse also true? That is, from $f(A)$ r.e. can we deduce that A is r.e.?

Exercise 7.5. Let A be a recursive set and $f : \mathbb{N} \to \mathbb{N}$ be a total computable function. Is it true, in general, that $f(A)$ is r.e.? Is it true that $f(A)$ is recursive? Justify your answers with a proof or counterexample.

Exercise 7.6. Let $A \subseteq \mathbb{N}$ be a set and let $f : \mathbb{N} \to \mathbb{N}$ be a computable function. Prove that if A is recursive then $f^{-1}(A) = \{x \in \mathbb{N} \mid f(x) \in A\}$ is r.e. Is the set $f^{-1}(A)$ also recursive? For the latter give a proof or provide a counterexample.

Exercise 7.7. Prove that a set A is r.e. if and only if $A \leqslant_m K$.

Exercise 7.8. Prove that a set A is r.e. if and only if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $A = img(f)$ (remember that $img(f) = \{y : \exists z.\ y = f(z)\}$).

Exercise 7.9. Given a function $f : \mathbb{N} \to \mathbb{N}$, define the predicate $P_f(x, y) \equiv$ "$f(x) = y''$, i.e., $P_f(x, y)$ is true if $x \in dom(f)$ and $f(x) = y$. Prove that f is computable if and only if the predicate $P_f(x, y)$ is semi-decidable.

Exercise 7.10. Let $A \subseteq \mathbb{N}$. Prove that A is recursive and infinite if and only if it is the image of a function $f : \mathbb{N} \to \mathbb{N}$ computable, total and strictly increasing (i.e., such that for each $x, y \in \mathbb{N}$, if $x < y$ then $f(x) < f(y)$).

Exercise 7.11. Let $\pi : \mathbb{N}^2 \to \mathbb{N}$ be the function encoding pairs of natural numbers into the natural numbers. Prove that a function $f : \mathbb{N} \to \mathbb{N}$ is computable if and only if the set $A_f = \{\pi(x, f(x))\ x \in \mathbb{N}\}$ is recursively enumerable.

Exercise 7.12. Prove that a set $A \subseteq \mathbb{N}$ is recursive if and only if $A \leqslant_m \{0\}$.

Exercise 7.13. Let $A \subseteq \mathbb{N}$ be a non-empty set. Prove that A is recursively enumerable if and only if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $dom(f)$ is the set of prime numbers and $img(f) = A$.

Exercise 7.14. Let $\mathcal{A} \subseteq \mathcal{C}$ be a set of computable functions such that, denoted by $\mathbf{0}$ and $\mathbf{1}$ the constant functions 0 and1, respectively, we have $\mathbf{0} \notin \mathcal{A}$ and $\mathbf{1} \in \mathcal{A}$. Define $A = \{x : \varphi_x \in \mathcal{A}\}$ and show that either A is not or $\bar{A}$ is not r.e.

Exercise 7.15. Establish whether an index $x \in \mathbb{N}$ can exist such that $\bar{K} = \{2^y - 1 : y \in E_x\}$. Justify your answer.

Exercise 7.16. Given two sets $A, B \subseteq \mathbb{N}$ what $A \leqslant_m B$ means. Prove that given $A, B, C \subseteq \mathbb{N}$ the following hold:

a. if $A \leqslant_m B$ and $B \leqslant_m C$ then $A \leqslant_m C$;

b. if $A \neq \mathbb{N}$ then $\varnothing \leqslant_m A$.

Exercise 7.17. Given two sets $A, B \subseteq \mathbb{N}$ define what $A \leqslant_m B$ means. Is it the case that $A \leqslant_m A \cup \{0\}$ for all sets A? If the answer is positive, provide a proof, otherwise, a counterexample. In the second case, identify a condition (specifying whether it is only sufficient or also necessary) that make $A \leqslant_m A \cup \{0\}$ true.

Exercise 7.18. Given two sets $A, B \subseteq \mathbb{N}$ define what $A \leqslant_m B$ means. Prove that, given any $A \subseteq \mathbb{N}$, we have A r.e. iff $A \leqslant_m K$.

Exercise 7.19. Prove that a set $A \subseteq \mathbb{N}$ is recursive if and only if A and $\bar{A}$ are r.e.

Exercise 7.20. State and prove Rice's theorem(without using the second recursion theorem).

Exercise 7.21. Define what it means for a set $A \subseteq \mathbb{N}$ to be saturated and prove that K is not is saturated.

Exercise 7.22. Let$\mathcal{A} \subseteq \mathcal{C}$be a set of functions computable and let$f \in \mathcal{A}$ such that for any function over$\theta \subseteq f$ is worth $\theta \notin \mathcal{A}$. Prove that $A = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{A}\}$ is not r.e.

8 Characterization of sets

Exercise 8.1. Study the recursiveness of the set $A = \{x \in \mathbb{N} : |W_x| \geqslant 2\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.2. Study the recursiveness of the set $A = \{x \in \mathbb{N} : x \in W_x \cap E_x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.3. Study the recursiveness of the set

$$B = \{x \mid x \in W_x \cup E_x\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Exercise 8.4. Study the recursiveness of the set $A = \{x \in \mathbb{N} : W_x \subseteq \mathbb{P}\}$, where $\mathbb{P}$ is the set of even numbers, i.e. establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.5. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \exists y, z \in \mathbb{N}.\ z > 1 \ \wedge \ x = y^z\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.6. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \phi_x(y) = y \text{ for infinitely many } y\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.7. Study the recursiveness of the set $A = \{x \in \mathbb{N} : W_x \subseteq E_x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.8. Study the recursiveness of the set $A = \{x \in \mathbb{N} : |W_x| > |E_x|\}$, i.e. establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.9. Study the recursiveness of the set $A = \{x \in \mathbb{N} \mid \varphi_x(y) = x * y \text{ per some } y\}$, that is to say if A e $\bar{A}$are recursive/recursively enumerable.

Exercise 8.10. Study the recursiveness of the set $A = \{x \in \mathbb{N} \mid |W_x \cap E_x| = 1\}$, i.e., establish if A e $\bar{A}$are recursive/recursively enumerable.

Exercise 8.11. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is *strictly increasing* when for each $y, z \in \text{dom}(f)$, if $y < z$ then $f(y) < f(z)$. Study the recursiveness of the set $A = \{x \mid \varphi_x \text{ sharply increasing}\}$, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.12. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is *almost total* if it is undefined on a finite set of points. Study the recursiveness of the set $A = \{x \mid \varphi_x \text{ almost total}\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.13. Study the recursiveness of the set $A = \{x \in \mathbb{N} : W_x \cap E_x = \emptyset\}$, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.14. Given a set $X \subseteq \mathbb{N}$, we define $X + 1 = \{x + 1 : x \in X\}$. Study the recursiveness of the set $A = \{x \in \mathbb{N} : E_x = W_x + 1\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.15. Let $\mathbb{P}$ be the set of even numbers. Prove that indicated with $A = \{x \in \mathbb{N} : E_x = \mathbb{P}\}$, we have $\bar{K} \leqslant_m A$.

Exercise 8.16. Study the recursiveness of the set $\mathbb{A} = \{x \in \mathbb{N} : \varphi_x(x) \downarrow \ \wedge \ \varphi_x(x) < x + 1\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.17. Study the recursion of the set $A = \{x \in \mathbb{N} : x \in W_x \ \wedge \ \varphi_x(x) = x^2\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.18. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \exists k \in \mathbb{N}.\ \varphi_x(x+3k) \uparrow\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.19. Study the recursiveness of the set $A = \{x \in \mathbb{N} : W_x = \overline{E_x}\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.20. Study the recursiveness of the set

$$B = \{\pi(x, y) \ \ P_x(x) \downarrow \ \text{in less than } y \text{steps}\},$$

i.e., establish whether B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.21. Given $A = \{x \mid \varphi_x \text{ is total}\}$, show that $\bar{K} \leqslant_m A$.

Exercise 8.22. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \varphi_x(y) = y \text{for infinities} y\}$, that is, say if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.23. Given a subset $X \subseteq \mathbb{N}$ define $F(X) = \{0\} \cup \{y, y+1 \mid y \in X\}$. Studying recursiveness of the set $A = \{x \in \mathbb{N} : W_x = F(E_x)\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.24. Study the recursiveness of the set

$$B = \{x \mid k \cdot (x+1) \in W_x \cap E_x \text{for each } k \in \mathbb{N}\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Exercise 8.25. Let $\varnothing$ be the always undefined function. Study the recursiveness of the set $A = \{x \mid \varphi_x = \varnothing\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.26. Study the recursiveness of the set $A = \{x \ \ \forall y.\ \text{if } y + x \in W_x \text{ then } y \leqslant \varphi_x(y+x)\}$, i.e., establish whether A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.27. Study the recursiveness of the set $A = \{x \mid \varphi_x(y+x) \downarrow \quad \text{for some } y \geqslant 0\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.28. Let $X \subseteq \mathbb{N}$ be finite, $X \neq \varnothing$ and define $A_X = \{x \in \mathbb{N} : W_x = E_x \cup X\}$. Study the recursiveness of A, i.e., say if A_X and $\bar{A_X}$ are recursive/recursively enumerable.

Exercise 8.29. Let $A = \{x \in \mathbb{N} : W_x \cap E_x \ \neq \emptyset\}$. Study the recursiveness of A, i.e., say if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.30. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \forall k \in \mathbb{N}.\ x + k \in W_x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.31. A partial function $f : \mathbb{N} \to \mathbb{N}$ is called injective when for each $x, y \in dom(f)$, if $f(x) = f(y)$ then $x = y$. Study the recursiveness of the set $A = \{x \ \ \varphi_x \text{ injective}\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.32. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \exists y \in E_x. \exists z \in W_x.\ x = y * z\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.33. Study the recursiveness of the set $A = \{x \in \mathbb{N} : x \in W_x \ \wedge \ \varphi_x(x) > x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.34. Let f be a total computable function such that $img(f) = \{f(x) : x \in \mathbb{N}\}$ is infinite. Study the recursiveness of the set

$$A = \{x \ \ \exists y \in W_x.\ x < f(y)\},$$

i.e., establish if A e $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.35. Study the recursiveness of the set $B = \{x \in \mathbb{N} : x \in E_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.36. Study the recursiveness of the set $V = \{x \in \mathbb{N} : W_x \text{ infinity}\}$, i.e., establish if V and $\overline{V}$ are recursive/recursively enumerable.

Exercise 8.37. Study the recursiveness of the set $V = \{x \in \mathbb{N} : \exists y \in W_x.\ \exists k \in \mathbb{N}.\ y = k \cdot x\}$, i.e., establish if V and $\overline{V}$ are recursive/recursive enumerable.

Exercise 8.38. Study the recursiveness of the set $V = \{x \in \mathbb{N} : |W_x| > 1\}$, i.e., establish if V and $\overline{V}$ are recursive/recursive enumerable.

Exercise 8.39. Let P be the set of even numbers and Pr the set of prime numbers. Show that $P \leqslant_m Pr$ and $Pr \leqslant_m P$.

Exercise 8.40. Let $f : \mathbb{N} \to \mathbb{N}$ be a fixed total computable function. Study the recursiveness of the set $B = \{x \in \mathbb{N} \ \ f(x) \in E_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.41. Let $f : \mathbb{N} \to \mathbb{N}$ be a fixed total computable function. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ img(f) \cap E_x \neq \emptyset\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable. Please note that $img(f) = \{f(x) \mid x \in \mathbb{N}\}$.

Exercise 8.42. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ E_x \not\supseteq W_x\}$, i.e., establish if B e $\bar{B}$are recursive/recursively enumerable.

Exercise 8.43. Let $B = \{x \mid \forall m \in \mathbb{N}.\ m \cdot x \in W_x\}$. Study the recursiveness of the B set, that is to say if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.44. Given $A = \{x \mid \varphi_x \text{ is total}\}$, show that $\bar{K} \leqslant_m A$.

Exercise 8.45. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ \exists y > x.\ y \in E_x\}$, i.e., establish if$B$ and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.46. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ \forall y > x.\ 2y \in W_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.47. Study the recursiveness of the set $B = \{x \in N : 1 \leqslant |E_x| \leqslant 2\}$, i.e., establish if B e $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.48. Study the recursiveness of the set $A = \{x \in \mathbb{N} \mid \mathbb{P} \subseteq W_x\}$, i.e., establish if$A$ and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.49. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ \varphi_x(y) = y^2 \text{ for infinitive } y\}$, i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Exercise 8.50. Given $X \subseteq \mathbb{N}$, indicate by $2X$ the set $2X = \{2x : x \in X\}$. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ 2W_x \subseteq E_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Exercise 8.51. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ W_x \supseteq Pr\}$, where $Pr \subseteq \mathbb{N}$is the set of the prime numbers, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.52. Classify the following set from the point of view of recursiveness

$$B = \{\pi(x, y)\ \ P_x \text{ stops on input } x \text{ in more than } y \text{ steps}\},$$

where $\pi : \mathbb{N}^2 \to \mathbb{N}$ is the pair encoding function, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.53. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is symmetric in the interval $[0, 2k]$ if $dom(f) \supseteq$

$[0, 2k]$ and for each $y \in [0, k]$ we have $f(y) = f(2k - y)$. Study the recursiveness of the set

$$A = \{x \in \mathbb{N} : \exists k > 0.\ \varphi_x \text{ symmetric in } [0, 2k]\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.54. Given $X \subseteq \mathbb{N}$ define $inc(X) = X \cup \{x + 1 : x \in X\}$. Classify the following set from the point of view of recursiveness $B = \{x \in \mathbb{N} : inc(W_x) = E_x\}$, i.e. say if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.55. Classify the following set from the point of view of recursiveness

$$B = \{x \ \ \varphi_x(0) \uparrow \ \vee \varphi_x(0) = 0\},$$

i.e., establish if B and $\bar{B}$are recursive/recursively enumerable.

Exercise 8.56. A function $f : \mathbb{N} \to \mathbb{N}$ is said *increasing* when for each $x, y \in dom(f)$, if $x < y$ then $f(x) < f(y)$. Define $B = \{x \in \mathbb{N} : \varphi_x \text{ increasing}\}$ and show that $\overline{K} \leqslant_m B$.

Exercise 8.57. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is k-bounded if $\forall x \in dom(f)$ we have $f(x) < k$. For each $k \in \mathbb{N}$, study the recursiveness of the set

$$A_k = \{x \in \mathbb{N} : \varphi_x\ k\text{- bounded}\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.58. Classify the following set from the point of view of recursiveness $B = \{x + y : x, y \in \mathbb{N} \ \wedge \ \varphi_x(y) \uparrow\}$, i.e., establish whether B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.59. Let f be a total computable function. Classify the following set from the point of view of recursiveness $B_f = \{x \in \mathbb{N} \ \ \varphi_x(y) = f(y) \text{ for infinitives} y\}$, i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Exercise 8.60. Let f be a total computable function, different from the identity. Classify the following set from the point of view of recursiveness $B_f = \{x \in \mathbb{N} \ \ \varphi_x = f \circ \varphi_x\}$, i.e., establish if B_f and $\bar{B}_f$ are recursive/recursively enumerable.

Exercise 8.61. Study the recursiveness of the set $B = \{x \in \mathbb{N} : \exists k \in \mathbb{N}.\ k \cdot x \in W_x\}$, i.e. establish whether B and $\overline{B}$ are recursive/recursively enumerable.

Exercise 8.62. Classify from the point of view of recursiveness the set $B = \{x \in \mathbb{N} : \forall k \in \mathbb{N}.\ k + x \in W_x\}$, i.e., establish if B and $\overline{B}$ are recursive/recursively enumerable.

Exercise 8.63. Classify from the point of view of recursiveness the set $V = \{x \in \mathbb{N} : E_x \text{ infinite}\}$, i.e., establish if V and $\overline{V}$ are recursive/recursively enumerable.

Exercise 8.64. Classify the following set from the point of view of recursiveness $B = \{x \in \mathbb{N} \;\; x \in W_x \backslash \{0\}\}$, i.e. establish if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.65. Classify the following set from the point of view of recursiveness

$$A = \{x \mid W_x \backslash E_x \text{ infinite}\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.66. Classify the following set from the point of view of recursiveness $B = \{x \in N : |W_x \backslash E_x| \geqslant 2\}$, i.e., establish if B e $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.67. Classify the following set from the point of view of recursiveness $B = \{x \in \mathbb{N} : \exists k \in \mathbb{N}.\ \forall y \geqslant k.\ \varphi_x(y) \downarrow\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.68. Classify the following set from the point of view of recursiveness $B = \{x \in \mathbb{N} \;\; x > 0 \;\wedge\; x/2 \notin E_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.69. Classify the following set from the point of view of recursiveness

$$B = \{x \in \mathbb{N} : \forall y \in W_x. \exists z \in W_x. (y < z) \;\wedge\; (\varphi_x(y) > \varphi_x(z))\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Exercise 8.70. Classify the following set from the point of view of recursiveness

$$B = \{x \in \mathbb{N} : \forall y \in W_x. \exists z \in W_x. (y < z) \;\wedge\; (\varphi_x(y) < \varphi_x(z))\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Exercise 8.71. Classify the following set from the point of view of recursiveness

$$A = \{x \mid W_x \cup E_x = \mathbb{N}\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.72. Classify the following set from the point of view of recursiveness

$$B = \{x \mid \exists k \in \mathbb{N}.\ kx \in W_x\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Exercise 8.73. Given $X, Y \subseteq \mathbb{N}$ define $X + Y = \{x + y \mid x \in X \;\wedge\; y \in Y\}$. Study the recursiveness of the set

$$B = \{x \mid x \in W_x + E_x\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Exercise 8.74. Classify from the point of view of recursiveness the set $A = \{x \in \mathbb{N} : W_x \cap E_x = \mathbb{N}\}$, i.e., say if A and $\bar{A}$ are recursive/recursively enumerable.

9 Second recursion theorem

Exercise 9.1. State and prove the second recursion theorem.

Exercise 9.2. State the second recursion theorem and use it to prove that K is not is recursive.

Exercise 9.3. State the Second Recursion Theorem and use it for proving that there exists $x \in \mathbb{N}$ such that $\varphi_x(y) = y^x$, for each $y \in \mathbb{N}$.

Exercise 9.4. State the Second Recursion Theorem and use it for proving that there exists $n \in \mathbb{N}$ such that $W_n = E_n = \{x \cdot n : x \in \mathbb{N}\}$.

Exercise 9.5. State the Second Recursion Theorem and use it for proving that $x \in \mathbb{N}$exists such that $\varphi_x(y) = x + y$.

Exercise 9.6. State the Second Recursion Theorem and use it for proving that there exists $x \in \mathbb{N}$ such that $\varphi_x(y) = x - y$.

Exercise 9.7. State the second recursion theorem and use it for proving that there exists a $n \in \mathbb{N}$ such that φ_n is total and $|E_n| = n$.

Exercise 9.8. State the second recursion theorem and use it for proving that the function $\Delta : \mathbb{N} \to \mathbb{N}$, defined by $\Delta(x) = \min\{y : \varphi_y \neq \varphi_x\}$, is not computable.

Exercise 9.9. State the second recursion theorem and use it for proving that, if we indicate by e_0 an index of the function always undefined $\emptyset$ and by e_1 an index of the identity function, the function $h : \mathbb{N} \to \mathbb{N}$, defined by

$$h(x) = \begin{cases} e_0 & \text{if } \varphi_x \text{ is total} \\ e_1 & \text{otherwise} \end{cases}$$

is not computable.

Exercise 9.10. State the Second Recursion Theorem and use it for proving that there exists an index $x \in \mathbb{N}$ such that

$$\varphi_x(y) = \begin{cases} y^2 & \text{if } x \leqslant y \leqslant x+2 \\ \uparrow & \text{otherwise} \end{cases}$$

Exercise 9.11. State the second recursion theorem and use it for proving that the set $C = \{x : 2x \in W_x \cap E_x\}$ is not saturated.

Exercise 9.12. State the second recursion theorem. Use it for proving that the set $C = \{x \in \mathbb{N} \mid x \in E_x\}$ not saturated.

Exercise 9.13. Let e_0 and e_1 be indices for the function always undefined $\varnothing$ and the constant 1, respectively. State the Second Recursion Theorem and use it to prove that the function $g : \mathbb{N} \to \mathbb{N}$ defined as below, is not computable:

$$g(x) = \begin{cases} e_0 & \varphi_x \text{ total} \\ e_1 & \text{otherwise} \end{cases}$$

Exercise 9.14. State the second recursion theorem. Prove that, given a function $f : \mathbb{N} \to \mathbb{N}$ total computable injective, the set $C_f = \{x : f(x) \in W_x\}$ is not saturated.

Exercise 9.15. State the second recursion theorem. Use it for proving that if C is a set such that $C \leqslant_m \overline{C}$, then C is not saturated.

Exercise 9.16. State the Second Recursion Theorem and use it for proving that there is an index$and \in \mathbb{N}$ such that

$$\varphi_e(y) = \begin{cases} y+e & \text{if } y \text{ multiple of} e \\ \uparrow & \text{otherwise} \end{cases}$$

Exercise 9.17. State the second recursion theorem. Use it for proving that every function f which is not total, but undefined only on a single point, i.e. $dom(f) = \mathbb{N} \backslash \{k\}$ for some $k \in \mathbb{N}$, admits a fixed point, i.e., there is $x \neq k$ such that $\varphi_x = \varphi_{f(x)}$.

Exercise 9.18. State the Second Recursion Theorem and use it for proving that there is $n \in \mathbb{N}$ such that $W_n = E_n = \{x \cdot n : x \in \mathbb{N}\}$.

Exercise 9.19. Prove that there exists $n \in \mathbb{N}$ such that $\varphi_n = \varphi_{n+1}$ and also $m \in \mathbb{N}$ such that $\varphi_m \neq \varphi_{m+1}$.

Exercise 9.20. State the second recursion theorem. Use it for proving that the set $B = \{x \in \mathbb{N} : \exists k \in \mathbb{N}.\ k \cdot x \in W_x\}$ is not saturated.

Exercise 9.21. State the second recursion theorem. Use it for proving that the set $C = \{x \in \mathbb{N} : \varphi_x(x) = x^2\}$ is not saturated.

Exercise 9.22. State the second recursion theorem and use it for proving that there is an index k such that $W_k = \{k * i \mid i \in \mathbb{N}\}$.

Exercise 9.23. State the second recursion theorem. Use it for proving that the set $C = \{x \in \mathbb{N} : [0, x] \subseteq W_x\}$ is not saturated.

Exercise 9.24. State the second recursion theorem and use it for proving that there is an index $n \in \mathbb{N}$ such that $\varphi_{p_n} = \varphi_n$, where p_n is the n-th prime number.

Exercise 9.25. State the second recursion theorem. Use it for proving that there is an index x such that $W_x = \{kx \mid k \in \mathbb{N}\}$.

Exercise 9.26. State the second recursion theorem. Use it for prove that there is an index $e \in \mathbb{N}$ such that $W_e = \{e^n : n \in \mathbb{N}\}$.

Exercise 9.27. Use the second recursion theorem to prove that the following set is not saturated

$$C = \{x \mid W_x = \mathbb{N} \ \wedge \ \varphi_x(0) = x\}.$$