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Concurrent semantics for fusions: Weak prime domains and connected event structures ☆

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ABSTRACT

Stable event structures, and their duality with prime algebraic domains, represent a landmark of concurrency theory, since they provide a neat characterisation of causality in computations. As such, they have been used for defining the concurrent semantics of many formalisms, from Petri nets to (linear) graph rewriting systems.
Stability however is restrictive for formalisms with "fusion", where a computational step may merge parts of the state. This happens e.g. for graph rewriting systems with non-linear rules, which are used to cover some relevant applications (such as the graphical encoding of calculi with name passing).
Guided by the need of giving semantics to such formalisms, we leave aside stability and characterise a class of domains, referred to as weak prime domains, naturally generalising prime algebraic domains. We then identify a corresponding class of event structures, that we call connected event structures, via a duality result formalised as an equivalence of categories.

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1. Introduction

For a long time stable/prime event structures and their duality with prime algebraic domains have been considered one of the landmarks of concurrency theory, providing a clear characterisation of causality in software systems. They have been used to provide a concurrent semantics to a wide range of foundational formalisms, from Petri nets [1] to linear graph rewriting systems [2–4] and process calculi [5–7]. They are one of the standard tools for the formal treatment of (true, i.e., non-interleaving) concurrency. See, e.g., [8] for a reasoned survey on the use of such causal models. Recently, they have been used in the study of concurrency in weak memory models [9–11] and for process mining and differencing [12].

In order to endow a chosen formalism with an event structure semantics, a standard construction consists in viewing the class of computations as a partial order. An element of the order is thus some sort of configuration, i.e., an execution trace up to an equivalence that identifies traces differing only for the order of independent steps (e.g., interchange law [13] in term rewriting, shift equivalence [14] in graph rewriting or permutation equivalence [15] in λ-calculus), and the order relates two computations when the latter is an extension of the former. Events are then identified with configurations consisting of a maximal computation step (e.g., a transition of a CCS process, a firing for a Petri net, or a reduction for a λ-term) with all its causes. As a simple example, consider the CCS process $a.c \mid b$. The corresponding transition system is

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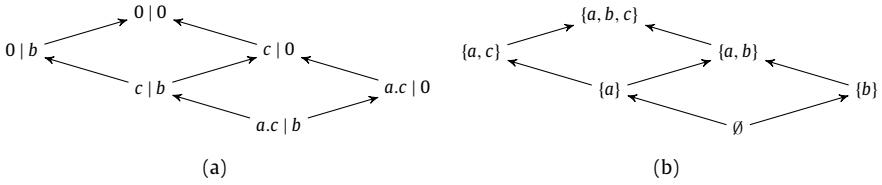
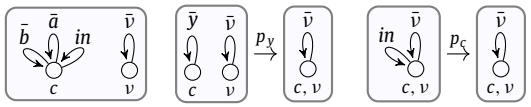
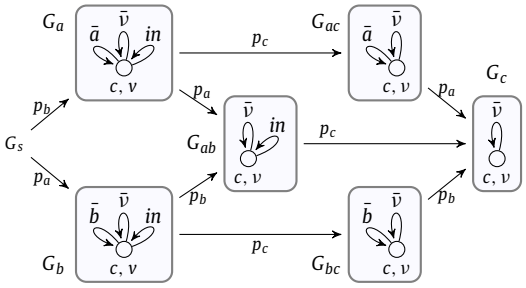


(a) (b)

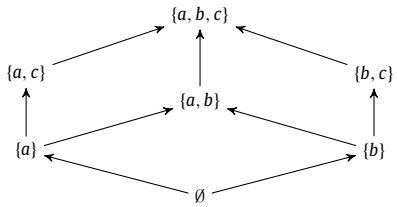
Fig. 1. The (a) transition system and (b) domain of configurations of the process $a.c \mid b$.



(a) The start graph G_s and the rules p_y ($y \in \{a, b\}$) and p_c.



(b) The possible rewrites.



(c) The domain of configurations.

Fig. 2. A graph rewriting system with fusions.

depicted in Fig. 1a. We can identify the states of the computation with the sets of actions executed and obtain the partial order depicted in Fig. 1b. The fact that each computation step in a configuration has a uniquely determined set of causes, a property that for event structures is called *stability*, allows one to characterise such elements, order-theoretically, as the prime elements: if they are included in a join they must be included in one of the elements that are joined. For example, in Fig. 1b, the events correspond to the configurations $\{a\}$ (transition a with empty set of causes), $\{a, c\}$ (transition c caused by a) and $\{b\}$ (transition b with empty set of causes). Each element of the partial order of configurations can be reconstructed uniquely as the join of the primes under it, so that the partial order is prime algebraic. This duality between event structures and domains of configurations can be nicely formalised in terms of an equivalence between the categories of prime event structures and prime algebraic domains [1,16].

The set up described so far fails when moving to formalisms where a computational step can merge parts of the state. This happens, e.g., in nominal calculi where, as a result of name passing, the received name is identified with a local one at the receiver [17,18] or in the modelling of bonding in biological/chemical processes [19]. Whenever we think of the state of the system as some kind of graph with the dynamics described by graph rewriting, this means that rewriting rules are non-linear (more precisely, in the jargon of the double pushout approach [20], left-linear but possibly not right-linear). In general terms, the point is that, in the presence of fusions, the same event can be enabled by different minimal sets of events, thus preventing the identification of a proper notion of causality.

As an example, consider the graph rewriting system in Fig. 2. The start graph G_s and the rewriting rules p_a, p_b, and p_c are reported in Fig. 2a. Observe that rules p_y, where y can be either a or b, delete edge $\bar{y}$ and merge nodes c and ν. The possible rewrites are depicted in Fig. 2b. For instance, applying p_a to G_s we get the graph G_b. Now, p_b can still be

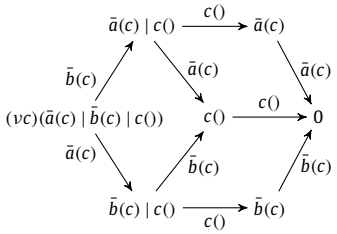


Fig. 3. The possible transitions of the π-calculus process $(\nu c)(\bar{a}(c) \mid \bar{b}(c) \mid c())$.

applied to G_b matching its left-hand side non-injectively, thus getting graph G_{ab}. Similarly, we can apply first p_b and then p_a, obtaining again G_{ab}. Observe that at least one between p_a and p_b must be applied to enable p_c, since the latter rule requires nodes c and ν to be merged. Note also that in a situation where all the three rules p_a, p_b, and p_c are applied, since p_a and p_b are independent, it is not possible to define a proper notion of causality. We only know that at least one between p_a and p_b must be applied before p_c. The corresponding domain of configurations, reported in Fig. 2c, is naturally derived from the possible rewrites in Fig. 2b.

The graph rewriting system of Fig. 2a is a (simplified) representation of the π-calculus process $(\nu c)(\bar{a}(c) \mid \bar{b}(c) \mid c())$. Rules p_y, for $y \in \{a, b\}$, represent the execution of $\bar{y}(c)$ that outputs on channel y the restricted name c. The first rule that is executed extrudes name c, while the second is just a standard output. The name c is available outside the scope only after the extrusion, and after that the input prefix $c()$ can be consumed. Fig. 3 shows the possible transitions of the process, which correspond one-to-one to the possible rewrites of Fig. 2b.

The impossibility of modelling these situations with stable event structures is well-known (see, e.g., [16] for a general discussion, [2] for graph rewriting systems or [17] for the π-calculus). One has to drop the stability requirement and replace causality by an enabling relation $\vdash$. More precisely, in the specific case we would have $\emptyset \vdash a$, $\emptyset \vdash b$, $\{a\} \vdash c$, $\{b\} \vdash c$.

The questions that we try to answer is: what can be retained of the duality between events structures and domains, when dealing with formalisms with fusions? Which are the properties of the domain of computations that arise in this setting? What are the event structure counterparts?

The domain of configurations of the example suggests that an event is still a computation that cannot be decomposed as the join of other computations. Hence, in order-theoretical terms, it is an irreducible. However, due to instability, irreducibles are not necessarily primes: two different irreducibles can represent the same computation step with different minimal enablings, in a way that an irreducible can be included in a computation that is the join of two computations without being included in any of the two. For instance, in the example above, $\{a, c\}$ is an irreducible, corresponding to the execution of c enabled by a, and it is included in $\{a\} \sqcup \{b, c\} = \{a, b, c\}$, although neither $\{a, c\} \subseteq \{a\}$ nor $\{a, c\} \subseteq \{b, c\}$. Uniqueness of decomposition of an element in terms of (downward closed sets of) irreducibles also fails, e.g., $\{a, b, c\} = \{a\} \sqcup \{b\} \sqcup \{a, c\} = \{a\} \sqcup \{b\} \sqcup \{b, c\}$: the irreducibles $\{a, c\}$ and $\{b, c\}$ can be used interchangeably in the decomposition of $\{a, b, c\}$.

Building on the previous observation, we introduce an equivalence on irreducibles identifying those that can be used interchangeably in the decompositions of an element (intuitively, different minimal enablings of the same computation step). This is used to define a weaker notion of primality (up to interchangeability) that allows us to characterise the class of domains suited for modelling the semantics of formalisms with fusions as the class of weak prime algebraic domains.

Given a weak prime algebraic domain, a corresponding event structure can be obtained by taking as events the set of irreducibles, quotiented under the (transitive closure of the) interchangeability relation. The resulting class of event structures is a (mild) restriction of the general event structures in [16] that we call connected event structures. Categorically, we get an equivalence between the category of weak prime algebraic domains and the one of connected event structures, generalising the equivalence between prime algebraic domains and prime event structures.

Interestingly, we can also show that the category of general event structures [16] coreflects into our category of weak prime algebraic domains. Therefore our notion of weak prime algebraic domain can be seen as a novel characterisation of the partial order of configurations of such event structures that is alternative to those based on intervals in [21,22]. It represents a natural generalisation of the one for prime event structures, with irreducibles (instead of primes) having a tight connection with events. The correspondence is established, at a categorical level, as a coreflection of categories: to the best of our knowledge, this has not been done before in the literature.

As mentioned above, weak prime domains, corresponding to possibly unstable event structures, satisfy the same conditions as prime domains, corresponding to stable event structures, up to an equivalence on irreducibles. This suggests the possibility of viewing unstable event structures as stable ones up to an equivalence on events. We show how this can be formalised leading to a set up closely related to the framework of prime event structures with equivalence [23,24].

Various notions of transition systems and automata embodying a notion of independence or concurrency between computations have been proposed and their relation with event structures thoroughly investigated. Among the early models we recall automata with concurrency [25] and transition systems with independence [26]. Since the finite configurations of an event structure can be seen as a transitions system, the question naturally arises as to whether these can be characterised axiomatically as a class of suitably enriched transition systems. A positive answer for prime event structures is given in [27] with respect to a class of (occurrence) transition systems with independence. However, the adoption there of a notion of

independence that relates not necessarily co-initial transitions naturally led to axioms with a "global" flavour. The quest of a characterisation in terms of local axioms, posed in [27], has brought to the notion of prime graphs in [28]. Recently, in connection with abstract rewriting and concurrent games, a slightly different yet equivalent characterisation has been rediscovered in [29], where prime event structures are shown to correspond exactly to a class of asynchronous graphs. Roughly, an asynchronous graph is a transition system where some squares are declared to commute, meaning that the coinitial edges of the square are concurrent and each one can follow the other. Here we show that a larger class of asynchronous graphs, obtained by removing one axiom, characterises the transition systems associated with general event structures (the correspondence being one-to-one if restricted to connected event structures).

The rest of the paper is structured as follows. In Section 2 we recall the basics of (prime) event structures and their correspondence with prime algebraic domains. In Section 3 we introduce weak prime algebraic domains. In Section 4 we define connected event structures, and we establish a categorical relation between weak prime domains and (connected) event structures. In Section 5 we present a characterisation of our proposal in terms of a formalism reminiscent of prime event structures with equivalence [23,24]. We also discuss and formalise the relation of our work with alternative characterisations of the domains of event structures based on intervals and on asynchronous graphs. Finally, in Section 6 we wrap up the main contributions of the paper and we sketch further advances and some connections with related works.

In order to streamline the presentation, the proofs of some technical results are confined to an appendix.

2. Background: domains and event structures

In this section we review the basics of event structures, as introduced in [16], and their duality with partial orders. A thorough discussion of event structures and their relation with other models of concurrency can be found in [26], while for a comprehensive treatment of domains in computer science the reader is referred to [30,31].

2.1. Event structures

For the sake of presentation, we focus on event structures with binary conflict. Most results can be easily rephrased for event structures with non-binary conflict expressed by means of a consistency predicate (This is explicitly discussed in [32]). Given a set X we denote by $\mathbf{2}^X$ and $\mathbf{2}^X_{fin}$ the powerset and the set of finite subsets of X, respectively. For $m, n \in \mathbb{N}$, we denote by $[m, n]$ the set $\{m, m+1, \ldots, n\}$.

Definition 2.1 *(Event structure).* An *event structure* (ES for short) is a tuple $\langle E, \vdash, \# \rangle$ such that

- E is a set of events;
- $\vdash \subseteq \mathbf{2}^E_{fin} \times E$ is the *enabling* relation, satisfying $X \vdash e$ and $X \subseteq Y$ implies $Y \vdash e$;
- $\# \subseteq E \times E$ is the conflict relation.

A subset $X \subseteq E$ is *consistent* if $\neg(e \# e')$ for all $e, e' \in X$.

An ES $\langle E, \vdash, \# \rangle$ is often denoted simply by E. Computations are captured by the notion of configuration.

Definition 2.2 *(Configuration, live event structure).* Let $\langle E, \vdash, \# \rangle$ be an ES. A *configuration* of E is a consistent subset $C \subseteq E$ that is *secured*, i.e., such that for all $e \in C$ there are $e_1, \ldots, e_n \in C$ with $e_n = e$ such that $\{e_1, \ldots, e_{k-1}\} \vdash e_k$ for all $k \in [1, n]$ (in particular, $\emptyset \vdash e_1$). The set of configurations of an ES E is denoted by $Conf(E)$ and the subset of *finite* configurations by $Conf_{fin}(E)$. An ES is *live* if conflict is *saturated*, i.e., for all $e, e' \in E$, if there is no $C \in Conf(E)$ such that $\{e, e'\} \subseteq C$ then $e \# e'$, and moreover for all $e \in E$ it holds $\neg(e \# e)$.

Remark 2.3. In live ES, the fact that conflict is saturated corresponds to inheritance of conflict in prime event structures. Moreover, the absence of self-conflicts implies that each event appears in some configuration (intuitively, it is executable). In the rest of the paper, we restrict to live ES, hence the qualification "live" is omitted.

In this setting, two events are *concurrent* when they are consistent and enabled by the same configuration. Since the enabling predicate is over finite sets of events, we can consider minimal sets of events enabling a given one.

Definition 2.4 *(Minimal enabling).* Let $\langle E, \vdash, \# \rangle$ be an ES. Given a configuration $C \in Conf(E)$ and an event $e \in E$ we write $C \vdash_0 e$ and call it a *minimal enabling* for e, when $C \cup \{e\} \in Conf(E)$ (hence $C \cup \{e\}$ consistent and $C \vdash e$), and for any other configuration $C' \subseteq C$, if $C' \vdash e$ then $C' = C$.

The classes of stable and prime ES represent our starting point and play an important role in the paper.

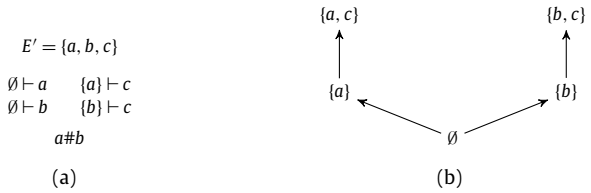


Fig. 4. A stable non-prime event structure and its domain of configurations.

Definition 2.5 *(Stable and prime event structures).* An ES $\langle E, \vdash, \# \rangle$ is *stable* if $X \vdash e$, $Y \vdash e$, and $X \cup Y \cup \{e\}$ consistent imply $X \cap Y \vdash e$. It is *prime* if $X \vdash e$ and $Y \vdash e$ imply $X \cap Y \vdash e$.

For stable ES, given a configuration C and an event $e \in C$, there is a unique minimal configuration $C' \subseteq C$ such that $C' \vdash_0 e$. The set C' can be seen as the set of causes of the event e in the configuration C. This gives a well-defined notion of causality that is local to each configuration. In a prime ES, for any event e there is a unique minimal enabling $C \vdash_0 e$, thus providing a global notion of causality. In general, in possibly unstable ES, due to the presence of consistent *or-enablings*, there might be distinct minimal enablings in the same configuration.

Example 2.6. A simple example of unstable ES is the one associated with the running example discussed in the introduction (see Fig. 2). The set of events is $\{a, b, c\}$, the conflict relation $\#$ is the empty one and the minimal enablings are $\emptyset \vdash_0 a$, $\emptyset \vdash_0 b$, $\{a\} \vdash_0 c$, and $\{b\} \vdash_0 c$. Thus, event c has two minimal enablings and these are consistent, hence $\{a, b\} \vdash c$. The corresponding configurations are reported in Fig. 2c.

Observe that, instead, if we add a conflict between a and b then $\{a, b\}$ and $\{a, b, c\}$ are no longer configurations (see Fig. 4). The event structure is trivially stable since no event has distinct minimal consistent enablings.

The class of ES can be turned into a category.

Definition 2.7 *(Category of event structures).* A morphism of ES $f : E_1 \to E_2$ is a partial function $f : E_1 \to E_2$ such that for all $C_1 \in Conf(E_1)$ and $e_1, e_1' \in E_1$ with $f(e_1)$, $f(e_1')$ defined

- if $f(e_1) \# f(e_1')$ then $e_1 \# e_1'$;
- if $f(e_1) = f(e_1')$ then $e_1 = e_1'$ or $e_1 \# e_1'$;
- if $C_1 \vdash_1 e_1$ then $f(C_1) \vdash_2 f(e_1)$.

We denote by ES the category of ES and their morphisms and by sES and pES, respectively, the full subcategories of stable and prime ES.

2.2. Domains

A preordered or partially ordered set $\langle D, \sqsubseteq \rangle$ is often denoted simply as D, omitting the (pre)order relation. We denote by $\preceq$ the immediate predecessor relation, i.e., for $x, y \in D$, we write $x \preceq y$ whenever $x \sqsubseteq y$ and for all $z \in D$ if $x \sqsubseteq z \sqsubseteq y$ then $z \in \{x, y\}$. A subset $X \subseteq D$ is *consistent* if it has an upper bound $d \in D$ (i.e., $x \sqsubseteq d$ for all $x \in X$). Given $x, y \in D$, we write $x \frown y$ when $\{x, y\}$ is consistent. A subset $X \subseteq D$ is *pairwise consistent* if $x \frown y$ for all $x, y \in X$. A subset $X \subseteq D$ is *directed* if $X \neq \emptyset$ and every pair of elements in X has an upper bound in X. We say that D is *complete* if every directed subset has a least upper bound in D.

A subset $X \subseteq D$ is an *ideal* if it is directed and downward closed. Given an element $x \in D$, we write $\downarrow x$ to denote the *principal ideal* $\{y \in D \mid y \sqsubseteq x\}$ generated by x. Given a partial order D, its *ideal completion*, denoted by $\mathsf{Idl}(D)$, is the set of ideals of D, ordered by subset inclusion. The *least upper bound* and the *greatest lower bound* of a subset $X \subseteq D$ (if they exist) are denoted by $\bigsqcup X$ and $\bigsqcap X$, respectively.

Definition 2.8 *(Domains).* A partial order D is *coherent* if for all pairwise consistent $X \subseteq D$ the least upper bound $\bigsqcup X$ exists. An element $d \in D$ is *compact* if for all directed $X \subseteq D$, $d \sqsubseteq \bigsqcup X$ implies $d \sqsubseteq x$ for some $x \in X$. The set of compact elements of D is denoted by $\mathsf{K}(D)$. A coherent partial order D is *algebraic* if for all $x \in D$ we have $x = \bigsqcup(\downarrow x \cap \mathsf{K}(D))$. We say that D is *finitary* if for all $a \in \mathsf{K}(D)$ the set $\downarrow a$ is finite. We refer to algebraic finitary coherent partially ordered sets as *domains*.

Note that every domain has a bottom element, which is given by $\bot = \bigsqcup \emptyset$, and that in every domain all the non-empty subsets have a meet. In fact, if $\emptyset \neq X \subseteq D$, then $\bigsqcap X = \bigsqcup L(X)$ where $L(X) = \{y \mid \forall x \in X.\, y \sqsubseteq x\}$ is the set of lowerbounds of X, which is pairwise consistent since dominated by all $x \in X$. And it is also easy to see that the finite joins of compact elements are compact.

For a domain D we can think of its elements as "pieces of information" expressing the states of evolution of a process. Compact elements represent states that are reached after a finite number of steps. Thus algebraicity essentially says that infinite computations can be approximated with arbitrary precision by finite ones. More formally, when D is algebraic, it is determined by $\mathsf{K}(D)$, i.e., $D \simeq \mathsf{Idl}(\mathsf{K}(D))$.

For an ES, the configurations ordered by subset inclusion form a domain. When the ES is stable, if a minimal enabling is included in the join of different configurations, then it is necessarily included in one of the configurations. In order-theoretical terms, minimal enablings are prime elements, and thus they represent the building blocks of computations.

Definition 2.9 *(Primes and prime algebraicity).* Let D be a domain. A *complete prime* is an element $p \in D$ such that for all pairwise consistent $X \subseteq D$, if $p \sqsubseteq \bigsqcup X$ then $p \sqsubseteq x$ for some $x \in X$. The set of complete prime elements of D is denoted by $pr(D)$. The domain D is *prime algebraic* (or simply *prime*) if for all $x \in D$ we have $x = \bigsqcup(\downarrow x \cap pr(D))$.

Prime domains can be also characterised as coherent finitary distributive complete partial orders [16]. Note that complete primes are compact (since each directed set is pairwise consistent). Since we will only use complete primes, the qualification "complete" will be omitted.

Prime domains are the domain theoretical counterpart of stable and prime ES. For a stable ES $\langle E, \#, \vdash \rangle$, the partial order $\langle Conf(E), \subseteq \rangle$ is a prime domain, denoted $\mathcal{D}_S(E)$. Conversely, given a prime domain D, the triple $\langle pr(D), \#, \vdash \rangle$, where $p \# p'$ if $\neg(p \frown p')$ and $X \vdash p$ when $(\downarrow p \cap pr(D)) \setminus \{p\} \subseteq X$, is a prime ES, denoted $\mathcal{E}_S(D)$.

This correspondence can be elegantly formulated at the categorical level [16]. We recall the notion of domain morphism.

Definition 2.10 *(Category of prime domains).* Let D_1, D_2 be prime domains. A morphism $f : D_1 \to D_2$ is a total function such that for all consistent $X_1 \subseteq D_1$ and $d_1, d_1' \in D_1$

1. if $d_1 \preceq d_1'$ then $f(d_1) \preceq f(d_1')$;
2. $f(\bigsqcup X_1) = \bigsqcup f(X_1)$;
3. if $X_1 \neq \emptyset$ then $f(\bigsqcap X_1) = \bigsqcap f(X_1)$.

We denote by pDom the category of prime domains and their morphisms.

The correspondence is then captured by the result below.

Theorem 2.11 *(Duality). There are functors $\mathcal{D}_S$: sES $\to$ pDom and $\mathcal{E}_S$: pDom $\to$ sES establishing a coreflection. It restricts to an equivalence of categories between* pDom *and* pES.

3. Weak prime algebraic domains

We argued before that in order to properly capture the semantics of computational formalisms with fusions the requirement of stability needs to be omitted. In this section we show that domains arising in absence of stability can be characterised by resorting to a weakened notion of prime element.

We start recalling the notion of irreducible element.

Definition 3.1 *(Irreducibles).* Let D be a domain. A *complete irreducible* of D is an element $x \in D$ such that, for all pairwise consistent $X \subseteq D$, if $x = \bigsqcup X$ then $x \in X$. The set of complete irreducibles of D is denoted by $ir(D)$ and, for $d \in D$, we define $ir(d) = \downarrow d \cap ir(D)$.

Observe that complete irreducibles in a domain are compact. In fact, if i is a complete irreducible, by algebraicity, $i = \bigsqcup \downarrow i \cap \mathsf{K}(D)$ whence $i \in \downarrow i \cap \mathsf{K}(D)$. Conversely, we have the following.

Lemma 3.2 *(Irreducibility and compactness). Let D be a domain and $d \in \mathsf{K}(D)$. Then d is a complete irreducible iff for all $x, y \in D$, $x \frown y$ and $d = x \sqcup y$ implies $d = x$ or $d = y$.*

Since in this paper we will refer only to complete irreducibles, the qualification "complete" will be omitted. Irreducibles in domains have a simple characterisation.

Lemma 3.3 *(Unique predecessor for irreducibles). Let D be a domain and $i \in D$. Then $i \in ir(D)$ iff it has a unique immediate predecessor.*

The unique predecessor of an irreducible will play an important role, hence we introduce a notation.

Definition 3.4 *(Immediate predecessor).* Let D be a domain and $i \in ir(D)$. We denote by $p(i)$ the (unique) immediate predecessor of i.

We next observe that any domain is actually irreducible algebraic, namely it can be generated by the irreducibles.

Proposition 3.5 *(Domains are irreducible algebraic). Let D be a domain and $d \in D$. Then it holds $d = \bigsqcup ir(d)$.*

We next observe that every prime is an irreducible and, if D is a prime domain, then also the converse holds, i.e., irreducibles coincide with primes.

Proposition 3.6 *(Irreducibles vs. primes). Let D be a domain. Then $pr(D) \subseteq ir(D)$. Moreover, D is a prime domain iff $pr(D) = ir(D)$.*

Quite intuitively, in the domain of configurations of an ES the irreducibles are minimal enablings of events. For instance, in the domain depicted in Fig. 2c the irreducibles are $\{a\}$, $\{b\}$, $\{a, c\}$, and $\{b, c\}$. For stable ES, the domain is prime and thus, as observed above, irreducibles coincide with primes. This fails in unstable ES, as we can see in our running example: while $\{a\}$ and $\{b\}$ are primes, the two minimal enablings of c, namely $\{a, c\}$ and $\{b, c\}$, are not. In fact, $\{a, c\} \subseteq \{a\} \sqcup \{b, c\}$, but neither $\{a, c\} \subseteq \{a\}$ nor $\{a, c\} \subseteq \{b, c\}$.

The key observation is that in general an event corresponds to a class of irreducibles, like $\{a, c\}$ and $\{b, c\}$ in our example. Additionally, two irreducibles corresponding to the same event can be used, to a certain extent, interchangeably for building the same configuration. For instance, $\{a, b, c\} = \{a, b\} \sqcup \{a, c\} = \{a, b\} \sqcup \{b, c\}$. We next formalise this intuition, i.e., we interpret irreducibles in a domain as minimal enablings of some event and we identify classes of irreducibles corresponding to the same event.

We start by observing that, in a prime domain, any element admits a unique decomposition in terms of downward closed sets of irreducibles (or, equivalently, of primes).

Lemma 3.7 *(Unique decomposition in prime domains). Let D be a prime domain and $X, X' \subseteq ir(D)$ downward closed sets of irreducibles. If $\bigsqcup X = \bigsqcup X'$ then $X = X'$.*

The result above no longer holds in domains arising in the presence of fusions. For instance, in the domain in Fig. 2c, $X = \{\{a\}, \{b\}, \{a, c\}\}$, $X' = \{\{a\}, \{b\}, \{b, c\}\}$ and $X'' = \{\{a\}, \{b\}, \{b, c\}, \{a, c\}\}$ are all decompositions for $\{a, b, c\}$. The idea is to identify irreducibles that can be used interchangeably in a decomposition.

Definition 3.8 *(Interchangeability).* Let D be a domain and $i, i' \in ir(D)$. We write $i \leftrightarrow i'$ if $i \frown i'$ and for all $X \subseteq ir(D)$ such that $X \cup \{i\}$ and $X \cup \{i'\}$ are downward closed and consistent we have $\bigsqcup(X \cup \{i\}) = \bigsqcup(X \cup \{i'\})$.

In words, $i \leftrightarrow i'$ means that i and i' produce the same effect when added to a decomposition that already includes their predecessors. Hence, intuitively, i and i' correspond to the execution of the same event with different and consistent enablings.

We next give some characterisations of interchangeability.

Lemma 3.9 *(Characterising $\leftrightarrow$). Let D be a domain and $i, i' \in ir(D)$. Then the following are equivalent*

1. $i \leftrightarrow i'$;
2. $i \frown i'$ *and for all* $d \in \mathsf{K}(D)$ *if* $p(i), p(i') \sqsubseteq d$ *then* $d \sqcup i = d \sqcup i'$;
3. $i \frown i'$ *and* $i \sqcup p(i') = p(i) \sqcup i'$.

The interchangeability relation is clearly reflexive and symmetric, but not transitive in general: in the domain of Fig. 5, using the characterisation in Lemma 3.9(3) one can easily see that $i \leftrightarrow i_1$ and $i_1 \leftrightarrow i'$ but not $i \leftrightarrow i'$, simply because it is not the case that $i \frown i'$. More interestingly, in the domain of Fig. 6, we have $i \leftrightarrow i_1$, $i_1 \leftrightarrow i_2$, $i_2 \leftrightarrow i'$, hence $i \leftrightarrow^* i'$. However, despite the fact that $i \frown i'$, it does not hold $i \leftrightarrow i'$, since $p(i) \sqcup i' \neq i \sqcup p(i')$. This shows that the intuition that interchangeable irreducibles correspond to the execution of the same event with different and consistent enablings is still not properly captured. Since i and i' represent the execution of the same event and they are consistent, one would expect that they are interchangeable.

The next definition formalises two additional properties that a domain must enjoy to provide $\leftrightarrow$ the intended meaning.

Definition 3.10 *(Interchangeable domain).* Let D be a domain. We say that D is *interchangeable* when

1. for all $i, i' \in ir(D)$, if $i \leftrightarrow^* i'$ and $p(i) \frown p(i')$ then $i \leftrightarrow i'$;
2. for all $i, i', j, j' \in ir(D)$, if $i \leftrightarrow^* i' \frown j' \leftrightarrow^* j$, $p(i) \frown j$, and $p(j) \frown i$ then $i \frown j$.

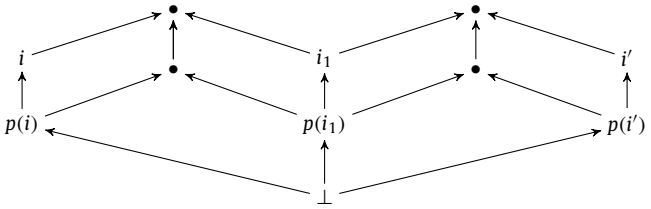


Fig. 5. Interchangeability need not be transitive.

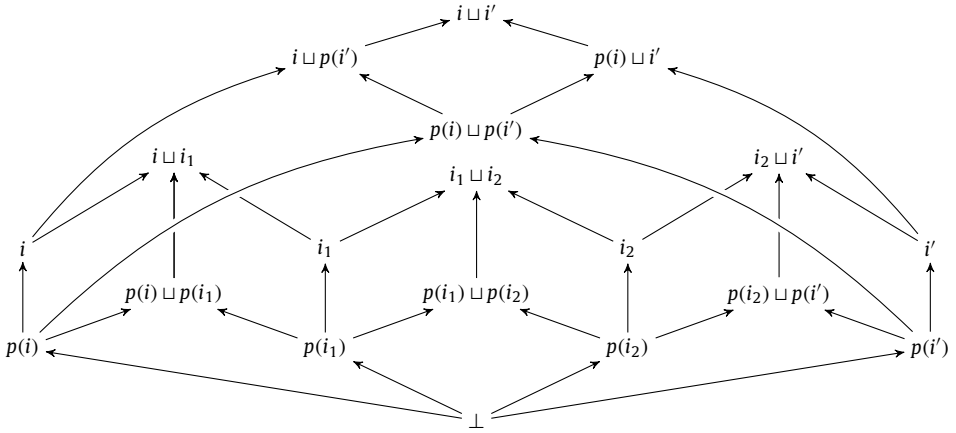


Fig. 6. A domain that is not interchangeable, since Definition 3.10(1) is violated.

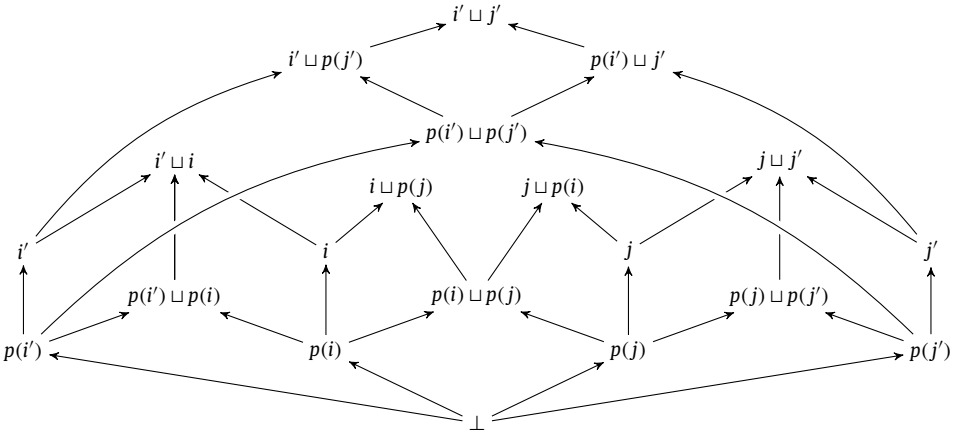
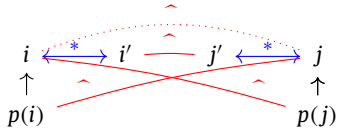


Fig. 7. A domain that is not interchangeable, since Definition 3.10(2) is violated.

The latter axiom is graphically illustrated below.



Property (1) is motivated by the discussion above. It intuitively asks that whenever i and i' represent the execution of the same event and they are consistent, then they are interchangeable. Property (2) can be read as follows: if i, i' and j, j' represent the same events and i', j' are consistent, the only source of inconsistency between i and j is in their enablings. In other words, either i and j are consistent or it must be that $p(i)$ is inconsistent with j, or i is inconsistent with $p(j)$. A situation in which this property fails is illustrated in Fig. 7.

We define weak primes: they weaken the property of prime elements, requiring that it holds up to interchangeability.

Definition 3.11 *(Weak prime).* Let D be a domain. A *weak prime* of D is an element $i \in ir(D)$ such that for all pairwise consistent $X \subseteq D$, if $i \sqsubseteq \bigsqcup X$ then there exist $i' \in ir(D)$ and $d \in X$ such that $i \leftrightarrow i'$ and $i' \sqsubseteq d$. We denote by $wpr(D)$ the set of weak primes of D.

Clearly, since interchangeability is reflexive, any prime is a weak prime. Moreover, in prime domains interchangeability turns out to be the identity and thus also the converse holds.

Lemma 3.12 *(Weak primes in prime domains). Let D be a prime domain. Then $\leftrightarrow$ is the identity and $wpr(D) = pr(D)$.*

We argue that the domain of configurations arising in the presence of fusions are characterised domain-theoretically as interchangeable domains where irreducibles are weak primes, i.e., that the domain is algebraic with respect to weak primes.

Definition 3.13 *(Weak prime algebraic domains).* Let D be an interchangeable domain. It is *weak prime algebraic* (or simply *weak prime*) if for all $d \in D$ it holds $d = \bigsqcup(\downarrow d \cap wpr(D))$.

Observe that weak prime domains are assumed to be interchangeable. This hypothesis will actually play a role only when relating weak prime domains and event structures in Section 4.2. However, in order to simplify the presentation we preferred to assume it since the beginning.

In the same way as prime domains are domains where all irreducibles are primes (see Proposition 3.6), we can provide a characterisation of weak prime domains in terms of coincidence between irreducibles and weak primes.

Proposition 3.14 *(Weak prime domains, again). Let D be an interchangeable domain. Then D is weak prime iff all its irreducibles are weak primes.*

Proof. Let D be an interchangeable domain. We know, by Lemma 3.5, that D is irreducible algebraic. If all irreducibles are weak primes, then clearly D is also weak prime algebraic. Conversely, if it is weak prime algebraic, then for any irreducible $i \in ir(D)$, we have that $i = \bigsqcup(\downarrow i \cap wpr(D))$. Since i is irreducible, this implies $i \in \downarrow i \cap wpr(D) \subseteq wpr(D)$, as desired. □

We finally introduce a category of weak prime domains by defining a notion of morphism.

Definition 3.15 *(Category of weak prime domains).* Let D_1, D_2 be weak prime domains. A weak prime domain morphism $f : D_1 \to D_2$ is a total function such that for all consistent $X_1 \subseteq D_1$ and $d_1, d_1' \in D_1$

1. if $d_1 \preceq d_1'$ then $f(d_1) \preceq f(d_1')$;
2. $f(\bigsqcup X_1) = \bigsqcup f(X_1)$;
3. if $d_1 \frown d_1'$ and $d_1 \sqcap d_1' \preceq d_1$ then $f(d_1 \sqcap d_1') = f(d_1) \sqcap f(d_1')$.

We denote by wDom the category of weak prime domains and their morphisms.

Compared with the notion of morphism for prime domains in Definition 2.10 (from [16]), we still require the preservation of $\preceq$ and $\sqcup$ of consistent sets (conditions (1) and (2)). However, the third condition, i.e., preservation of $\sqcap$, is weakened to preservation in some cases. General preservation of meets is indeed not expected in the presence of fusions. Consider, e.g., the ES in Example 2.6. Take another ES $E' = \{c\}$ with $\emptyset \vdash c$ and the morphism $f : E \to E'$ that forgets a and b, i.e., $f(c) = c$ and $f(a)$, $f(b)$ undefined. Then the natural extension of f to configurations would not preserve meets: $f(\{a, c\}) \sqcap f(\{b, c\}) = \{c\} \sqcap \{c\} = \{c\} \neq f(\{a, c\} \sqcap \{b, c\}) = f(\emptyset) = \emptyset$. Intuitively, the condition $d_1 \sqcap d_1' \preceq d_1$ means that d_1' includes the computation modelled by d_1 possibly apart from a final step, hence $d_1 \sqcap d_1'$ coincides with d_1 when such step is removed. Since domain morphisms preserve immediate precedence (i.e., single steps), the same happens for $f(d_1)$, which models a computation included in $f(d_1')$, possibly apart for the execution of a final step. The meet $f(d_1) \sqcap f(d_1')$ is $f(d_1)$ without such step, and thus it is expected to coincide with $f(d_1 \sqcap d_1')$.

Note that condition (3) above can be equivalently stated by asking that if $d_1 \frown d_1'$ and $d_1 \sqcap d_1' \prec d_1$ then $f(d_1 \sqcap d_1') = f(d_1) \sqcap f(d_1')$. In fact, if $d_1 \sqcap d_1' = d_1$ then $d_1 \sqsubseteq d_1'$. Hence, by monotonicity $f(d_1) \sqsubseteq f(d_1')$, and thus $f(d_1 \sqcap d_1') = f(d_1) = f(d_1) \sqcap f(d_1')$, as desired. In the following we will often use this form of the condition.

In general we only have

$$f(\bigsqcap X_1) \sqsubseteq \bigsqcap f(X_1)$$

In fact, for all $x_1 \in X_1$, we have $\bigsqcap X_1 \sqsubseteq x_1$, hence $f(\bigsqcap X_1) \sqsubseteq f(x_1)$ and thus $f(\bigsqcap X_1) \sqsubseteq \bigsqcap f(X_1)$. Still, when restricted to prime domains, also the converse inequality holds and our notion of morphism boils down to the original one, i.e., the full subcategory of wDom having prime domains as objects is pDom.

Theorem 3.16 (pDom *as a subcategory of* wDom*). The category of prime domains* pDom *is the full subcategory of* wDom *having prime domains as objects.*

4. Weak prime domains and connected event structures

In this section we show that, relaxing the stability assumption, we can generalise the duality result described in the previous section, linking weak prime domains and suitably defined classes of ES.

4.1. From event structures to weak prime domains

We show that the set of configurations of an ES, ordered by subset inclusion, is a weak prime domain where the compact elements are the finite configurations. Moreover, the correspondence can be lifted to a functor. We also identify a subclass of ES that we call *connected* ES and that are the exact counterpart of weak prime domains (in the same way as prime ES correspond to prime algebraic domains).

Definition 4.1 *(Configurations of an event structure, ordered).* Let E be an ES. We define $\mathcal{D}(E) = \langle Conf(E), \subseteq \rangle$. Given an ES morphism $f : E_1 \to E_2$, its image $\mathcal{D}(f) : \mathcal{D}(E_1) \to \mathcal{D}(E_2)$ is defined as $\mathcal{D}(f)(C_1) = \{f(e_1) \mid e_1 \in C_1\}$.

We need some technical facts, collected in the following lemma. Recall that in the setting of unstable ES we can have distinct consistent minimal enablings for an event. The following notation will be useful.

Definition 4.2 *(Connected enablings).* Let E be an ES, $C, C' \in Conf(E)$ and $e \in E$. When $C \vdash_0 e$, $C' \vdash_0 e$, and $C \cup C' \cup \{e\}$ is consistent, we write $C \overset{e}{\frown} C'$. We denote by $\overset{e}{\frown}^*$ the transitive closure of the relation $\overset{e}{\frown}$.

Note that, whenever $C \vdash_0 e$ and $C' \vdash_0 e$, requiring $C \cup C' \cup \{e\}$ consistent amounts to require $C \cup C'$ consistent, since conflict is binary.

Lemma 4.3 *(Properties of the domain of configurations). Let $\langle E, \vdash, Con \rangle$ be an* ES*. Then*

1. *$\mathcal{D}(E)$ is a domain, $\mathsf{K}(\mathcal{D}(E)) = Conf_{fin}(E)$, join is union, and $C \prec C'$ iff $C' = C \cup \{e\}$ for some $e \in E \setminus C$;*
2. *$C \in Conf(E)$ is irreducible iff $C = C' \cup \{e\}$ and $C' \vdash_0 e$; in this case we denote C as $\langle C', e \rangle$;*
3. *for $C \in Conf(E)$, we have $ir(C) = \{\langle C', e' \rangle \mid e' \in C \ \wedge \ C' \subseteq C \ \wedge \ C' \vdash_0 e'\}$; moreover $p(\langle C', e' \rangle) = C'$;*
4. *for $\langle C_1, e_1 \rangle, \langle C_2, e_2 \rangle \in ir(\mathcal{D}(E))$, we have $\langle C_1, e_1 \rangle \leftrightarrow \langle C_2, e_2 \rangle$ iff $e = e_1 = e_2$ and $C_1 \overset{e}{\frown} C_2$;*
5. *$\mathcal{D}(E)$ is interchangeable.*

Concerning point 1, observe that the meet in the domain of configurations is $C \sqcap C' = \bigcup\{C'' \in Conf(E) \mid C'' \subseteq C \ \wedge \ C'' \subseteq C'\}$, which is usually smaller than the intersection. For instance, in Fig. 2, $\{a, c\} \sqcap \{b, c\} = \emptyset \neq \{c\}$. Point 2 says that irreducibles are configurations of the form $C \cup \{e\}$ that admits a secured execution in which the event e appears as the last one and cannot be switched with any other. In other words, irreducibles are minimal enablings of events. Point 3 characterises the irreducibles in a configuration. According to point 4, two irreducibles are interchangeable when they are different minimal enablings for the same event.

Proposition 4.4 *(The domain of configurations is weak prime). Let E be an* ES*. Then $\mathcal{D}(E)$ is a weak prime domain. Moreover, given two* ES *E_1 and E_2 and a morphism $f : E_1 \to E_2$, its image $\mathcal{D}(f) : \mathcal{D}(E_1) \to \mathcal{D}(E_2)$ is a weak prime domain morphism.*

Proof. We know that $\mathcal{D}(E)$ is a domain (Lemma 4.3(1)) and that it is interchangeable (Lemma 4.3(5)).

In order to show that $\mathcal{D}(E)$ is a weak prime domain, we exploit the characterisation in Proposition 3.14, i.e., we prove that all irreducibles are weak primes. Consider an irreducible I, which by Lemma 4.3(2) is of the shape $I = \langle C, e \rangle$ with $C \vdash_0 e$, and suppose that $I \subseteq \bigsqcup X$ for some $X \subseteq \mathcal{D}(E)$. In particular, this means that $e \in \bigsqcup X$ and thus there is $C' \in X$ such that $e \in C'$. In turn, we can consider a minimal enabling of e in C', i.e., a minimal $C'' \subseteq C'$ such that $C'' \vdash_0 e$, and we have that $I'' = \langle C'', e \rangle$ is an irreducible $I'' \subseteq C'$. Since I and I'' are consistent, as they are both included in $\bigsqcup X$, then $C \overset{e}{\frown} C''$ and by Lemma 4.3(4) $I \leftrightarrow I''$.

We prove that for an ES morphism $f : E_1 \to E_2$, its image $\mathcal{D}(f) : \mathcal{D}(E_1) \to \mathcal{D}(E_2)$ is a weak prime domain morphism.

- $C_1 \preceq C_1'$ implies $\mathcal{D}(f)(C_1) \preceq \mathcal{D}(f)(C_1')$
  Since $\mathcal{D}(f)(C_i) = \{f(d_i) \mid d_i \in C_i\}$ and by Lemma 4.3(1) $C_1 \preceq C_1'$ iff $C_1' = C_1 \cup \{e_1\}$ for some event e_1, the result follows immediately.

Fig. 8. Non-connected ES do not uniquely determine a domain.

- for $X_1 \subseteq \mathcal{D}(E_1)$ consistent, $\mathcal{D}(f)(\bigsqcup X_1) = \bigsqcup \mathcal{D}(f)(X_1)$
  Since $\mathcal{D}(f)$ takes the image as set and $\bigsqcup$ on consistent sets is union, the result follows.
- for $C_1, C_1' \in \mathcal{D}(E_1)$ consistent $C_1 \sqcap C_1' \prec C_1$ it holds $f(C_1 \sqcap C_1') = f(C_1) \sqcap f(C_1')$
  Since $C_1 \sqcap C_1' \prec C_1$, by Lemma 4.3(1) we have that $C_1 = (C_1 \sqcap C_1') \cup \{e_1\}$ for some $e_1 \notin C_1 \sqcap C_1'$. Clearly $e_1 \notin C_1'$, otherwise we would have $C_1 \subseteq C_1'$ and thus $C_1 \sqcap C_1' = C_1$. Therefore in this case, the meet coincides with intersection, $C_1 \sqcap C_1' = C_1 \cap C_1' = C_1 \setminus \{e_1\}$. Since for the events in $C_1 \cup C_1'$, by definition of event structure morphism, f is injective, we have that $f(C_1) \cap f(C_1') = f(C_1 \cap C_1')$. As a general fact, $f(C_1) \sqcap f(C_1') \subseteq f(C_1) \cap f(C_1')$. Therefore, putting things together, we conclude

$$f(C_1) \sqcap f(C_1') \subseteq f(C_1) \cap f(C_1') = f(C_1 \cap C_1') = f(C_1 \sqcap C_1')$$

  The converse inequality holds in any domain (as observed after Definition 3.15) and thus the result follows. □

A special role is played by the subclass of *connected* ES that will be shown to be the counterpart of weak prime domains.

Definition 4.5 *(Connected event structure).* An ES is *connected* if whenever $C \vdash_0 e$ and $C' \vdash_0 e$ then $C \overset{e}{\frown}{}^{*} C'$. We denote by cES the full subcategory of ES having connected ES as objects.

In words, different minimal enablings for the same event must be pairwise connected by a chain of consistency. Equivalently, for each event e the set of minimal enablings, say $M_e = \{C \mid C \vdash_0 e\}$, endowed with the relation $\overset{e}{\frown}$ is a connected graph. Intuitively, as discussed in more detail below, if M_e were not connected, then we could split event e into different instances, one for each connected component, without changing the associated domain.

For instance, the ES in Example 2.6 is a connected ES. Only event c has two minimal enablings $\{a\} \vdash_0 c$ and $\{b\} \vdash_0 c$ and obviously $\{a\} \overset{c}{\frown} \{b\}$. Instead, the event structure in Fig. 4 is not connected since there are two minimal enablings of event c, i.e., $\{a\} \vdash_0 c$ and $\{b\} \vdash_0 c$ and it does not hold $\{a\} \overset{c}{\frown}{}^{*} \{b\}$ since they are not consistent.

Clearly, prime ES are also connected ES. More precisely, we have the following.

Proposition 4.6 *(Primality = stability + connectedness). Let E be an ES. Then E is prime iff it is stable and connected.*

Proof. The fact that a prime ES is stable and connected follows immediately from the definitions. Conversely, let E be a stable and connected ES. We show that E is prime, i.e., each $e \in E$ has a unique minimal enabling. Let $C, C' \in Conf(E)$ be minimal enablings for e, i.e., $C \vdash_0 e$ and $C' \vdash_0 e$. Since E is connected $C \overset{e}{\frown}{}^{*} C'$. Let $C \overset{e}{\frown} C_1 \overset{e}{\frown} \ldots \overset{e}{\frown} C_n \overset{e}{\frown} C'$. Then by stability we get that $C = C_1 = \ldots = C_n = C'$. □

The defining property of connected ES allows one to recognise that two minimal enablings are relative to the same event by only looking at the partially ordered structure and thus, as we will see, from the domain of configurations of a connected ES we can recover an ES isomorphic to the original one and vice versa (see Theorem 4.14). In general, this is not possible. For instance, consider again the ES E' in Fig. 4 and the corresponding domain of configurations, which for the reader convenience is replicated on the left of Fig. 8. Intuitively, it is not possible to recognise that $\{a, c\}$ and $\{b, c\}$ are different minimal enablings of the same event. In fact, by considering the ES E'' with events $E'' = \{a, b, c_1, c_2\}$ such that $a \# b$, the minimal enablings are again $\emptyset \vdash_0 a$, $\emptyset \vdash_0 b$, $\{a\} \vdash_0 c_1$, and $\{b\} \vdash_0 c_2$. Hence we would get an isomorphic domain of configurations (see Fig. 8, right).

4.2. From weak prime domains to connected event structures

We show how to get an ES from a weak prime domain. As anticipated, events are equivalence classes of irreducibles, where the equivalence is (the transitive closure of) interchangeability.

In order to properly relate domains to the corresponding ES we need to prove some properties of irreducibles and of the interchangeability relation in weak prime domains.

Domains are irreducible algebraic (see Proposition 3.5), hence any element is determined by the irreducibles under it. The difference between two elements is thus somehow captured by the irreducibles that are under one element and not under the other. This motivates the following definition.

Definition 4.7 *(Irreducible difference).* Let D be a domain and $d, d' \in \mathsf{K}(D)$ such that $d \sqsubseteq d'$. Then we define $\delta(d', d) = ir(d') \setminus ir(d)$.

The immediate precedence relation intuitively relates domain elements corresponding to configurations that differ for the execution of a single event. We can indeed show that whenever $d \prec d'$ the irreducible difference of d' and d consists of a set of irreducibles which are pairwise interchangeable, hence, intuitively corresponding to the same event.

Lemma 4.8 *(Immediate precedence and irreducibles/2). Let D be a weak prime domain and $d, d' \in D$ such that $d \preceq d'$. Then for all $i, i' \in \delta(d', d)$*

1. *$d' = d \sqcup i$;*
2. *if $i \sqsubseteq i'$ then $i = i'$;*
3. *$i \leftrightarrow i'$.*

In a prime domain, an element admits a unique decomposition in terms of primes (see Lemma 3.7). Here the same holds for irreducibles but only up to interchangeability. Given a domain D and an irreducible $i \in ir(D)$, we denote by $[i]_{\leftrightarrow^*}$ the corresponding equivalence class. For $X \subseteq ir(D)$ we define $[X]_{\leftrightarrow^*} = \{[i]_{\leftrightarrow^*} \mid i \in X\}$.

Proposition 4.9 *(Unique decomposition up to $\leftrightarrow$). Let D be a weak prime domain, $d \in \mathsf{K}(D)$, and $X \subseteq D$ a downward closed and consistent set such that $[X]_{\leftrightarrow^*} \subseteq [ir(d)]_{\leftrightarrow^*}$. Then $d = \bigsqcup X$ iff $[X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$.*

We explicitly observe that, by the above result, if $X = [ir(d)]_{\leftrightarrow^*}$ for some $d \in \mathsf{K}(D)$ then d is uniquely determined by X.
We now have all the tools needed for mapping our domains to an ES.

Definition 4.10 *(Event structure for a weak prime domain).* Let D be a weak prime domain. The ES $\mathcal{E}(D) = \langle E, \#, \vdash \rangle$ is defined as follows

- $E = [ir(D)]_{\leftrightarrow^*}$;
- $e \# e'$ if there is no $d \in \mathsf{K}(D)$ such that $e, e' \in [ir(d)]_{\leftrightarrow^*}$;
- $X \vdash e$ if there is $i \in e$ such that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} \subseteq X$.

Given a morphism $f : D_1 \to D_2$, its image $\mathcal{E}(f) : \mathcal{E}(D_1) \to \mathcal{E}(D_2)$ is defined for $[i_1]_{\leftrightarrow^*} \in E_1$ as $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*}$, where $i_2 \in \delta(f(i_1), f(p(i_1)))$, and $\mathcal{E}(f)([i_1]_{\leftrightarrow^*})$ is undefined if $f(p(i_1)) = f(i_1)$.

The events in $\mathcal{E}(D)$ are equivalence classes of irreducibles. Two events e, e' are consistent (not in conflict) when there is some compact element d such that $e, e' \in [ir(d)]_{\leftrightarrow^*}$. Spelled out, this means that there are irreducibles $i \in e$ and $i' \in e'$ such that $i, i' \sqsubseteq d$, i.e., there are minimal enablings of the events e and e' in the same configuration. Finally, an event e is enabled by a set X when X includes, up to interchangeability, all the predecessors of e.
Note that the definition above is well-given: in particular, there is no ambiguity in the definition of the image of a morphism, since by Lemma 4.8(3) we easily conclude that for all $i_2, i_2' \in \delta(f(i_1), f(p(i_1)))$, it holds $i_2 \leftrightarrow i_2'$ (this is argued in detail in the proof of Lemma 4.11).

Lemma 4.11 *(From weak prime domains to event structures). Let D be a weak prime domain. Then $\mathcal{E}(D)$ is an ES. Moreover, given two weak prime domains D_1, D_2 and a morphism $f : D_1 \to D_2$, its image $\mathcal{E}(f) : \mathcal{E}(D_1) \to \mathcal{E}(D_2)$ is an ES morphism.*

Since in a prime domain irreducibles coincide with primes (Proposition 3.6), $\leftrightarrow$ is the identity (Lemma 3.12) and $\delta(d', d)$ is a singleton when $d \prec d'$, the construction above produces the prime ES $\mathsf{pES}(D)$ as defined in Section 2.
Given a weak prime domain D, the finite configurations of the ES $\mathcal{E}(D)$ exactly correspond to the elements in $\mathsf{K}(D)$. Moreover, in such ES we have a minimal enabling $C \vdash_0 e$ when there is an irreducible in e (recall that events are equivalence classes of irreducibles) such that C contains all and only (the equivalence classes of) its predecessors.

Lemma 4.12 *(Compacts vs. configurations). Let D be a weak prime domain and $C \subseteq \mathcal{E}(D)$ a finite set of events. Then C is a configuration in the ES $\mathcal{E}(D)$ iff there exists a unique $d \in \mathsf{K}(D)$ such that $C = [ir(d)]_{\leftrightarrow^*}$. Moreover, for all $e \in \mathcal{E}(D)$ we have that $C \vdash_0 e$ iff $C = [ir(i) \setminus \{i\}]_{\leftrightarrow^*}$ for some $i \in e$.*

Given the lemma above, it is now possible to state how weak prime domains relate to connected ES.

Proposition 4.13 *(From weak prime domains to connected ES). Let D be a weak prime domain. Then $\mathcal{E}(D)$ is a connected ES.*

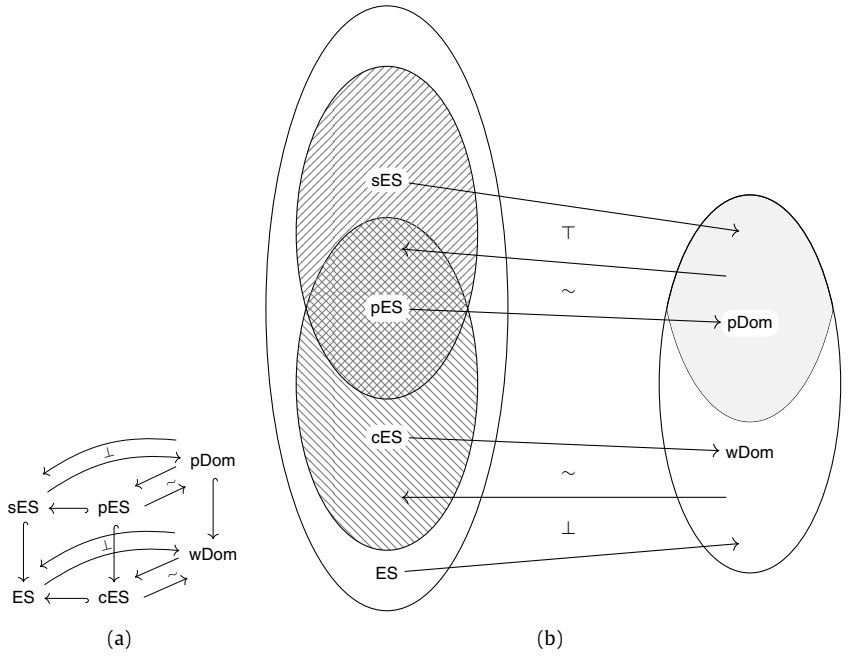


Fig. 9. A summary of the relations among classes of ES and domains.

Proof. We have to show that if $X \vdash_0 e$ and $X' \vdash_0 e$, then $X \overset{e}{\frown}{}^{*} X'$. Note that, by Lemma 4.12, from $X \vdash_0 e$ and $X' \vdash_0 e$, we deduce that there exists $i, i' \in e$ such that $[ir(i)\setminus\{i\}]_{\leftrightarrow^*} = X$ and $[ir(i')\setminus\{i'\}]_{\leftrightarrow^*} = X'$. Since $i, i' \in e$ we deduce that $i \leftrightarrow^* i'$, namely $i = i_0 \leftrightarrow i_1 \leftrightarrow \ldots \leftrightarrow i_n = i'$. We proceed by induction on n. The base case $n = 0$ is trivial. If $n > 0$ then from $i \leftrightarrow i_1 \leftrightarrow^* i'$ we have that $i_1 \in e$ and, if we let $X_1 = [ir(i_1)\setminus\{i_1\}]_{\leftrightarrow^*}$, then $X_1 \vdash_0 e$. By inductive hypothesis, we know that $X_1 \overset{e}{\frown}{}^{*} X'$. Moreover, since $i \leftrightarrow i_1$, the irreducibles i and i_1 are consistent. Hence, by definition of conflict in $\mathcal{E}(D)$, also $X \cup X_1 \cup \{e\}$ is consistent and hence $X \overset{e}{\frown} X_1$. Therefore $X \overset{e}{\frown}{}^{*} X'$, as desired. □

4.3. Relating categories of models

We show that, at a categorical level, the constructions taking a weak prime domain to an ES and an ES to a domain (the domain of its configurations) establish a coreflection between the corresponding categories. This becomes an equivalence when it is restricted to the full subcategory of connected ES.

Theorem 4.14 *(Coreflection of* ES *and* wDom*). The functors* $\mathcal{D} : \mathsf{ES} \to \mathsf{wDom}$ *and* $\mathcal{E} : \mathsf{wDom} \to \mathsf{ES}$ *form a coreflection* $\mathcal{E} \dashv \mathcal{D}$*. It restricts to an equivalence between* wDom *and* cES*.*

The above result indirectly provides a way of turning a general ES into a connected ES.

Corollary 4.15 *(From general to connected* ES*). The functors* $\mathcal{C} : \mathsf{ES} \to \mathsf{cES}$ *defined by* $\mathcal{C} = \mathcal{E} \circ \mathcal{D}$ *and the inclusion* $\mathcal{I} : \mathsf{cES} \to \mathsf{ES}$ *form a coreflection.*

Proof. Immediate consequence of Theorem 4.14. □

Explicitly, for any event structure E the corresponding connected ES $\mathcal{C}(E) = \langle E', \vdash', \#' \rangle$ is defined as follows. The set of events is $E' = \{[\langle C, e\rangle]_\sim \mid C \vdash_0 e\}$, where $\sim$ is the least equivalence such that $\langle C, e\rangle \sim \langle C', e\rangle$ if $\langle C, e\rangle$ and $\langle C', e\rangle$ are consistent. Moreover $[\langle C, e\rangle]_\sim \#' [\langle C', e'\rangle]_\sim$ if for all $\langle C_1, e\rangle \sim \langle C, e\rangle$ and $\langle C_1', e'\rangle \sim \langle C', e'\rangle$ the minimal enablings $\langle C_1, e\rangle$ and $\langle C_1', e_1'\rangle$ are not consistent. Finally, for $X \subseteq E'$, $X \vdash' [\langle C, e\rangle]_\sim$ if there exists $\langle C', e\rangle \sim \langle C, e\rangle$ such that $C' \subseteq \{e'' \mid [\langle C'', e''\rangle]_\sim \in X\}$.

An overall picture of the results discussed up to now can be found in Fig. 9. The arrows from classes of event structures to domains are restrictions of the functor $\mathcal{D}(\cdot)$, while the converse arrows are restrictions of the functor $\mathcal{E}(\cdot)$. The Venn diagram stresses the fact that prime ES are exactly the ES which are stable and connected (see Lemma 4.6) showing how the notion of connectedness naturally emerges in the framework.

5. Related characterisations

In this section we present a characterisation of our proposal in terms of a formalism reminiscent of the prime event structures with equivalence [23,24]. We also discuss and formalise the relation of our work with alternative characterisations of the domains of (prime) event structures proposed in the literature, based on intervals and on asynchronous graphs.

5.1. Prime event structures with equivalence

The previous sections showed that the domains of configurations of unstable ES are weak prime domains, i.e., they satisfy the same conditions as those of prime domains but only up to the equivalence induced by interchangeability. Symmetrically, this suggests the possibility of viewing unstable ES as stable ones up to some equivalence on events. In this section we consider a formalisation for such a view, leading to a set up that is closely related to the framework devised in [23,24], which we also call *prime event structures with equivalence* for the space of this article, since no confusion can arise.

In Section 2.1 we mentioned that in prime ES a global notion of causality can be used in place of the enabling. We next recall the formal definition. We also introduce a notation for direct (i.e., non-inherited) conflict that will play a role later.

Definition 5.1 *(Causality/direct conflict in prime event structures).* Let $P = \langle E, \vdash, \# \rangle$ be a prime ES. Given an event $e \in E$, the unique $C \in Conf(P)$ such that $C \vdash_0 e$ is called the set of *strict causes* of e and denoted by $\downarrow e$, while the set of causes is $\downarrow e = \downarrow e \cup \{e\}$. The *strict causality relation* $<$ is defined by $e' < e$ if $e' \in \downarrow e$, and, as usual, we denote by $\leq$ the reflexive closure of $<$. We say that $e, e' \in E$ are in *direct conflict*, written $e \#_d e'$, when $e \# e'$ and $\downarrow e \cup \{e'\}$, $\downarrow e' \cup \{e\}$ are consistent.

We next introduce our notion of prime ES with equivalence. Given a prime ES P with an equivalence over the set of events $\sim \subseteq E \times E$, we say that a subset $X \subseteq E$ is $\sim$*-saturated* if for all $e \in X$ and $e' \in E$, if $e \sim e'$ and $\downarrow e' \subseteq X$ then $e' \in X$. Since the intersection of saturated sets is saturated, given a set X we can always consider the smallest saturated superset of X, called the *saturation* of X and denoted $\tilde{X}$.

Definition 5.2 *(Prime event structures with equivalence).* A *prime* ES *with equivalence* (EPES for short) is a pair $\langle P, \sim \rangle$ where $P = \langle E, \vdash, \# \rangle$ is a prime ES and $\sim$ is an equivalence on E such that for all $e, e', e_1, e'_1 \in E$

1. if $[\downarrow e]_\sim \subseteq [\downarrow e']_\sim$ then $e \leq e'$; if in addition $e \sim e'$ then $e = e'$;
2. if $e \sim e'$ and $\downarrow e \cup \downarrow e'$ consistent then $\neg(e \# e')$.
3. if $e \sim e'$, $e_1 \sim e'_1$, and $e \#_d e_1$ then $e' \#_d e'_1$.

We say that $\langle P, \sim \rangle$ is *connected* if $\sim = (\sim \setminus \#)^*$. A morphism of EPES $f : \langle P_1, \sim_1 \rangle \to \langle P_2, \sim_2 \rangle$ is an ES morphism $f : P_1 \to P_2$ such that for all $e_1, e'_1 \in P_1$, $e_1 \sim_1 e'_1$ iff $f(e_1) \sim_2 f(e'_1)$. We denote by epES the corresponding category.

An ES with equivalence is thus just an ES equipped with an equivalence on events. Condition (1) essentially says that an event is determined by the equivalence classes of events in its causal history. In particular, as a consequence, if $\downarrow e \subseteq \downarrow e'$ and $e \sim e'$ then $e = e'$, which intuitively means that distinct equivalent events must correspond to different enablings of the same event. Moreover, it implies that the set $\downarrow e$ is $\sim$-saturated and thus it is a configuration (see Definition 5.3 and Lemma 5.4). Conditions (2) and (3) essentially say that equivalent events can have different conflicts only for the fact that their minimal enablings have different conflicts. Connectedness amounts to the fact that equivalent events must be connected by a chain of equivalences going through consistent events. We next introduce a notion of configuration.

Definition 5.3 *(Configurations).* Let $\langle P, \sim \rangle$ be an EPES. Then $Conf(\langle P, \sim \rangle) = \{C \mid C \in Conf(P) \ \wedge \ C \ \sim\text{-saturated}\}$.

In words, a configuration of a prime ES with equivalence is a configuration C of the underlying event structure, where all events enabled in C that are equivalent to some event already in C are also in C. Thus equivalent events may have different minimal enablings, but whenever a configuration contains the causes of two equivalent events, their executions cannot be taken apart.

Lemma 5.4 *(Histories are configurations). Let $\langle P, \sim \rangle$ be a* EPES *and $e \in E$. Then $\downarrow e$ is a configuration.*

As an example, the connected ES of our running example (see Fig. 2), corresponds to the prime ES with equivalence in Fig. 10a, where we have two distinct copies of event c, namely $c_a \sim c_b$, corresponding to the possible minimal enablings. Graphically, causality is represented by a straight directed line. The corresponding domain of configurations is depicted in Fig. 10b. Note that $C = \{a, b, c_a\}$ is not a configuration despite the fact that it is downward closed, since it is not $\sim$-saturated: event c_b is missing, but its causes $\{b\}$ are in C.

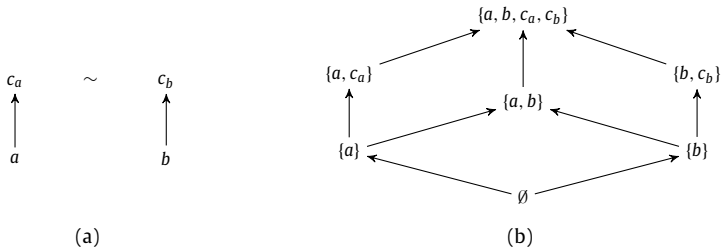


Fig. 10. A prime ES with equivalence and its domain of configurations.

Our definition of EPES is similar to that in [23,24]. Concerning configurations, while [23,24] identifies unambiguous configurations where there is a unique representative for each equivalence class, here instead we saturate including *all* equivalent events that are not in conflict.

We finally observe that the constructions above can be "translated" into constructions that relate directly EPES and weak prime domains.

Proposition 5.5 *(Weak prime domain for* EPES*). Let $\langle P, \sim\rangle$ be a* EPES*. Then $\mathcal{D}_{eq}(\langle P, \sim\rangle) = \langle Conf(\langle P, \sim\rangle), \subseteq\rangle$ is a weak prime domain. Conversely, if D is a weak prime domain then $\mathcal{E}_{eq}(D) = \langle\langle ir(D), \#, \vdash\rangle, \leftrightarrow^*\rangle$ is an* EPES *with conflict and enabling defined by*

- $i_1 \# i_2$ *if* $\neg(i_1 \frown i_2)$*;*
- $X \vdash i$ *if* $X \supseteq ir(i) \setminus \{i\}$*.*

The correspondence above can be translated to an analogous correspondence between EPES and unstable ES. It is however impossible to make such correspondence functorial essentially for the same reason why [23,24] resorts to a pseudo-adjunction. We try to enucleate the problem by showing a correspondence between (unstable) event structures and EPES.

Definition 5.6 *(From* ES *to* EPES *and back).* Let $\langle P, \sim\rangle$ be an EPES, where $P = \langle E, \vdash, \#\rangle$. The corresponding ES is $\mathcal{M}(\langle P, \sim\rangle) = \langle E_\sim, \vdash_\sim, \#_\sim\rangle$, with $\vdash_\sim$ and $\#_\sim$ defined by

- $[X]_\sim \vdash_\sim [e]_\sim$ when $X \vdash e$;
- $[e]_\sim \#_\sim [e']_\sim$ when $e_1 \# e'_1$ for all $e_1 \in [e]_\sim$ and $e'_1 \in [e']_\sim$.

Conversely, given an ES $P = \langle E, \vdash, \#\rangle$ the corresponding EPES is $\mathcal{U}(P) = \langle Q, \sim\rangle$, with $Q = \langle E', \vdash', \#'\rangle$ defined by

- $E' = \{\langle C, e\rangle \mid C \in Conf(E) \ \wedge \ e \in E \ \wedge \ C \vdash_0 e\}$;
- $X \vdash' \langle C, e\rangle$ if $C \subseteq \bigcup\{C' \cup \{e'\} \mid \langle C', e'\rangle \in X\}$;
- $\langle C, e\rangle \#' \langle C', e'\rangle$ if $C \cup C' \cup \{e, e'\}$ is not consistent

and the equivalence is defined by $\langle C, e\rangle \sim \langle C', e\rangle$ for all C, C' such that $C \vdash_0 e$ and $C' \vdash_0 e$.

We can easily show, exploiting Proposition 5.5, that the constructions above produce well-defined structures and map connected structures to connected structures. Moreover, the two constructions are inverse of each other.

Proposition 5.7. *Let $\langle P, \sim\rangle$ be an* EPES*. Then $\langle P, \sim\rangle$ and $\mathcal{U}(\mathcal{M}(\langle P, \sim\rangle))$ are isomorphic. Dually, let $P = \langle E, \vdash, \#\rangle$ be an* ES*. Then $\mathcal{M}(\mathcal{U}(P))$ and P are isomorphic.*

Observe that the construction from EPES to ES can be easily turned into a functor $\mathcal{M} : \mathsf{epES} \to \mathsf{ES}$. In fact, given a morphism $f : \langle P_1, \sim_1\rangle \to \langle P_2, \sim_2\rangle$ we can let $\mathcal{M}(f)([e_1]_{\sim_1}) = [f(e_1)]_{\sim_2}$.

Instead, making the converse construction from ES to EPES functorial is problematic. In fact, consider the ES of the running example $E = \{a, b, c\}$, with $\emptyset \vdash_0 a$, $\emptyset \vdash_0 b$ and $\{a, b\} \vdash_0 c$ and the ES with events $E' = \{a', b', c'\}$ with $\emptyset \vdash_0 a'$, $\emptyset \vdash_0 b'$ and $\{a'\} \vdash_0 c'$ and $\{b'\} \vdash_0 c'$ and the morphism $f : E \to E'$ with $f(x) = x'$ for $x \in \{a, b, c\}$. Then $\mathcal{U}(E) = \{\langle\emptyset, a\rangle, \langle\emptyset, b\rangle, \langle\{a, b\}, c\rangle\}$ and $\mathcal{U}(E') = \{\langle\emptyset, a'\rangle, \langle\emptyset, b'\rangle, \langle\{a'\}, c'\rangle, \langle\{b'\}, c'\rangle\}$. Observe that, while clearly $\mathcal{U}(f)(\langle\emptyset, a\rangle) = \langle\emptyset, a'\rangle$ and $\mathcal{U}(f)(\langle\emptyset, b\rangle) = \langle\emptyset, b'\rangle$, when we come to $\mathcal{U}(f)(\langle\{a, b\}, c\rangle)$ we can define it as one of the two equivalent events $\langle\{a'\}, c'\rangle$ and $\langle\{b'\}, c'\rangle$.

The solution offered by [23,24] is to move towards pseudo-functors, i.e., considering two EPES morphisms $g, g' : P_1 \to P_2$ equivalent if $g(e_1) \sim_2 g'(e_1)$ for all $e_1 \in P_1$ and requiring that functors are defined only up-to morphism equivalence. Indeed, it is easy to see that the two possible choices for f above lead to equivalent morphisms.

5.2. Relation with interval based characterisations

The correspondence between event structures and domains has been often studied in the literature by relying on the notion of interval [21,1,16,22].

Definition 5.8 *(Interval).* Let D be a domain. An *interval* is a pair $[d, d']$ of elements of D such that $d \prec d'$. The set of intervals of D is denoted by $Int(D)$. Given two intervals $[c, c'], [d, d'] \in Int(D)$ we define

$$[c, c'] \leq [d, d'] \quad \text{if } (c = c' \sqcap d) \ \wedge \ (c' \sqcup d = d'),$$

and we let $\sim$ be the equivalence obtained as the symmetric and transitive closure of $\leq$.

It can be shown that $\leq$ is a partial order on intervals and thus $\sim$ is indeed an equivalence. An interval represents a pair of elements differing only for a "quantum" of information, intuitively the execution of an event. The equivalence $\sim$ is intended to identify intervals corresponding to the execution of the same event in different states.

Indeed, in [1] it is shown that for prime domains there is a bijective correspondence between $\sim$-classes of intervals and complete primes. In weak prime domains we can establish a similar correspondence, with $\leftrightarrow^*$-classes of irreducibles playing the role of the primes.

Lemma 5.9 *(Intervals vs. irreducibles). Let D be a weak prime domain. Define $\zeta : Int(D)_\sim \to ir(D)_{\leftrightarrow^*}$ by*

$$\zeta([d, d']_\sim) = [i]_{\leftrightarrow^*},$$

where i is any element in $\delta(d', d)$. Then ζ is a bijection, whose inverse $\iota : ir(D)_{\leftrightarrow^*} \to Int(D)_\sim$ is defined by

$$\iota([i]_{\leftrightarrow^*}) = [p(i), i]_\sim.$$

In [21,22] the domain of configurations of general event structures with binary conflict is characterised in terms of intervals. It is shown (see, e.g., [21, Theorem 3.3.3]) that given an event structure with binary conflict, the domain of configurations is an algebraic complete partial order where the following axioms hold

(F) for all $d \in \mathsf{K}(D)$ the set $\downarrow d$ is finite;
(C) for all $x, y, z \in \mathsf{K}(D)$, if $x \prec y$, $x \prec z$, $y \frown z$, and $y \neq z$ then there exists $y \sqcup z$ and $y \prec y \sqcup z$ and $z \prec y \sqcup z$;
(R) for all intervals $[x, y]$, $[x, z]$ if $[x, y] \sim [x, z]$ then $y = z$;
(V) for all $x, x', y, y', x'', y'' \in \mathsf{K}(D)$ if $[x, x'] \sim [y, y']$, $[x, x''] \sim [y, y'']$, and $x' \frown x''$ then $y' \frown y''$.

In [22] a construction of the ES corresponding to a domain is provided. Given $d \in \mathsf{K}(D)$, let $s(d) = \{[c, c']_\sim \mid c' \sqsubseteq d\}$.

Definition 5.10 *(Event structure from a domain [22]).* Given a domain D satisfying the axioms (F), (C), (R), (V), the corresponding ES with binary conflict is defined as $\mathcal{E}_{wd}(D) = (E, \#, \vdash)$ where

- $E = Int(D)_\sim$;
- $[c, c']_\sim \# [d, d']_\sim$ if for all $[c_1, c_1']$, $[d_1, d_1']$ such that $[c_1, c_1'] \sim [c, c']$ and $[d_1, d_1'] \sim [d, d']$ it holds $\neg(c_1' \frown d_1')$;
- for $X \subseteq E$, $X \vdash [c, c']_\sim$ if $s(c_1) \subseteq X$ for some interval $[c_1, c_1'] \sim [c, c']$.

The above construction produces an event structure with binary conflict that is mapped back to the original domain (see, e.g., [22, Corollary 2.10]).

Theorem 5.11. *Let D be a domain satisfying axioms (F), (C), (R), (V). Then $\mathcal{D}(\mathcal{E}_{wd}(D))$ is isomorphic to D.*

We build on the above results to show that the domains satisfying axioms (F), (C), (R) and (V) are exactly the weak prime domains.

Proposition 5.12 *(Weak prime domains and intervals). Let D be a domain. Then D is a weak prime domain iff it satisfies axioms (F), (C), (R) and (V).*

Proof. Let D be a domain satisfying axioms (F), (C), (R) and (V). By Theorem 5.11, $\mathcal{D}(\mathcal{E}_{wd}(D)) \simeq D$. Since, by Proposition 4.4, the set of configurations of any event structure forms a weak prime domain, we conclude that D is weak prime.

For the converse, let D be a weak prime domain. By Theorem 4.14, we have that $\mathcal{D}(\mathcal{E}(D)) \simeq D$ and thus, since by [21,22], the domain of configuration of an event structure with binary conflict satisfies axioms (F), (C), (R) and (V), we conclude. □

Moreover, relying on Lemma 5.9, we can show that the event structures associated with a domain in [22] (Definition 5.10) and in our work (Definition 4.10) coincide.

Proposition 5.13. *Let D be a weak prime domain. Then $\mathcal{E}(D)$ and $\mathcal{E}_{wd}(D)$ are isomorphic.*

Proof. By Lemma 5.9, the function $\zeta : Int(D)_{\sim} \to ir(D)_{\leftrightarrow^*}$ is a bijection. Note that $Int(D)_{\sim}$ and $ir(D)_{\leftrightarrow^*}$ are the sets of events respectively of $\mathcal{E}(D)$ and $\mathcal{E}_{wd}(D)$. We next show that ζ is an isomorphism of event structures.

Let e_1, e_2 be events in $\mathcal{E}_{wd}(D)$. We show that $e_1 \# e_2$ iff $\zeta(e_1) \# \zeta(e_2)$.

If $\neg(e_1 \# e_2)$, from Definition 5.10 we get that there exist $[c_1, c_1'] \in e_1$ and $[c_2, c_2'] \in e_2$ such that $c_1' \frown c_2'$. Let $d \in D$ be an upper bound, i.e., $c_1', c_2' \sqsubseteq d$. Now, $\zeta(e_j) = [i_j]_{\leftrightarrow^*}$ for $i_j \in \delta(c_j, c_j')$, for $j \in \{1, 2\}$. Clearly, $i_1, i_2 \in ir(d)$ whence $[i_1]_{\leftrightarrow^*}, [i_2]_{\leftrightarrow^*} \subseteq [ir(d)]_{\leftrightarrow^*}$ and thus, according to Definition 4.10, we have $\neg([i_1]_{\leftrightarrow^*} \# [i_2]_{\leftrightarrow^*})$, as desired. The argument can be reversed to prove that if $\neg(\zeta(e_1) \# \zeta(e_2))$ then $\neg(e_1 \# e_2)$.

Concerning the enabling relation, we show that $X \vdash e$ in $\mathcal{E}_{wd}(D)$ iff $\zeta(X) \vdash \zeta(e)$ in $\mathcal{E}(D)$. Assume that $X \vdash e$ in $\mathcal{E}_{wd}(D)$. This means that there exists $[c, c'] \in e$ such that $s(c) = \{[d, d']_{\sim} \mid d' \sqsubseteq c\} \subseteq X$. Now, recall that $\zeta(e) = [i]_{\leftrightarrow^*}$ with $i \in \delta(c', c)$. In order to show that $\zeta(X) \vdash \zeta(e)$, according to Definition 4.10, we prove that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} \subseteq \zeta(X)$. Let $j \in ir(i) \setminus \{i\}$. Clearly $j \in \delta(j, p(j))$ and thus $[j]_{\leftrightarrow^*} = \zeta([p(j), j]_{\sim})$. Moreover, by Lemma 4.8(2) the set $\delta(c', c)$ is flat and thus, since $j \sqsubset i$ necessarily $j \notin \delta(c', c)$. Since $j \in ir(c')$ we conclude that $j \in ir(c)$, namely $j \sqsubseteq c$. This implies that $[p(j), j]_{\sim} \in s(c)$ and thus

$$\begin{aligned} [j]_{\leftrightarrow^*} &= \zeta([p(j), j]_{\sim}) \\ &\subseteq \zeta(s(c)) \\ &\subseteq \zeta(X) \qquad [\text{since } s(c) \subseteq X] \end{aligned}$$

We thus conclude that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} \subseteq \zeta(X)$ as desired.

Also in this case, the argument can be easily reversed to prove the converse implication. □

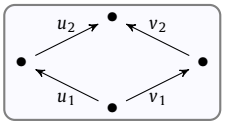
The paper by Droste [22] considers also the case of event structures that are equipped with a general consistency relation (rather than just binary conflict). The correspondence with our approach can be extended also to this setting, as we have detailed in [32].

5.3. Relation with asynchronous graphs

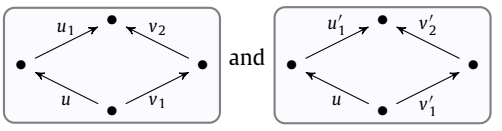
As mentioned before, a characterisation of the transition graph of prime event structures in terms of local axioms has been given in [28]. A slightly different, yet equivalent formalisation has been rediscovered in [29]. Here we show that an analogous characterisation can be obtained for (connected) event structures. For our development we refer to the formalisation in [29]. Given a graph $G = \langle N, U, s, t \rangle$, a sequence of edges $w = u_1; \ldots; u_n \in U^*$ is a path whenever each edge has a target that coincide with the source of the subsequent edge, i.e., for all $i \in [1, n-1]$, $t(u_i) = s(u_{i+1})$. Let us denote by $P_2(G)$ the set of paths of length 2, i.e., $P_2(G) = \{u_1; u_2 \mid u_1, u_2 \in E\}$. Note that two paths of length 2 with the same source and target can be seen as a "square" in the graph. An asynchronous graph is then a transition system where some squares are declared to commute.

Definition 5.14 *(Asynchronous graph).* An *asynchronous graph* is a tuple $A = \langle G, n_0, \simeq \rangle$ where $G = \langle N, U, s, t \rangle$ is a directed graph, $n_0 \in N$ is the origin and $\simeq \subseteq P_2(G) \times P_2(G)$ is an equivalence relation on coinitial and cofinal paths of length 2 (i.e., if $u_1; u_2 \simeq v_1; v_2$ then $s(u_1) = s(v_1)$ and $t(u_2) = t(v_2)$) such that the following axioms hold (in pictures, all squares depicted are assumed to commute)

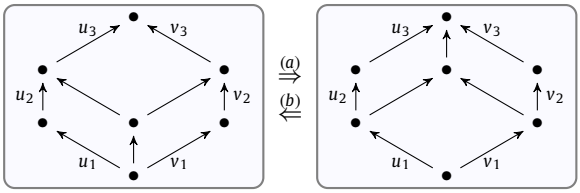
1. if $u_1; u_2 \simeq v_1; v_2$ and $u_2 \neq v_2$ then $u_1 \neq v_1$;



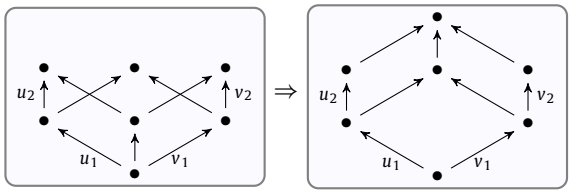
2. if $u; u_1 \simeq v_1; v_2$ and $u; u_1' \simeq v_1'; v_2'$ then $(u_1 = u_1'$ iff $v_1 = v_1')$;



3. Cube



4. Coherence axiom



Given an asynchronous graph, in the following we denote by the same symbol $\simeq$ the extension of the equivalence to all paths by contextual closure, i.e., $w_1; w; w_2 \simeq w_1; w'; w_2$ for all $w_1, w_2, w, w' \in U^*$ with $w \simeq w'$. The equivalence classes of paths from the origin can be ordered by prefix, thus leading to a partial order $P(A)$. Then it can be shown that the partial orders of finite configurations of prime ES exactly correspond to asynchronous graphs such that all cofinal paths from the origin are equivalent.

Definition 5.15 *(Prime asynchronous graph).* An asynchronous graph $A = \langle G, n_0, \simeq \rangle$ is called *prime* if all cofinal paths from the origin n_0 are equivalent.

It can be seen that the requirement of having all cofinal paths equivalent amounts to that of having all coinitial and cofinal paths of length 2 (squares) equivalent. This is indeed how the condition is formalised in [28].

Theorem 5.16 *(Asynchronous graphs/prime* ES *[29]). Let A be a prime asynchronous graph. The ideal completion* $\mathsf{Idl}(P(A))$ *is a prime domain. Conversely, each prime domain is isomorphic to* $\mathsf{Idl}(P(A))$ *for some prime asynchronous graph A.*

With respect to [29], we added the coherence axiom (4) in the definition of asynchronous graph, which is going to be pivotal in our later characterisation of weak prime domains (Proposition 5.18). This is actually necessary already for having a correspondence with prime domains and ES.[1]

The correspondence established by Theorem 5.16 generalises to connected ES and what we call weak asynchronous graphs, i.e., asynchronous graphs where only the forward part of the cube axiom holds, while the converse implication (indeed sometimes referred to as stability axiom) may fail.

Definition 5.17 *(Weak asynchronous graphs).* A *weak asynchronous graph* is defined as in Definition 5.14, but omitting the stability axiom (3b). It is called *weak prime* if additionally all cofinal paths from the origin are equivalent.

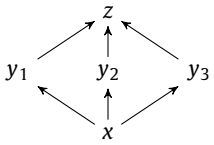
Then we can prove that weak prime domains are exactly the partial orders generated by weak prime asynchronous graphs (which in turn correspond to connected ES).

Proposition 5.18 *(Weak asynchronous graphs and domains). Let A be a weak prime asynchronous graph. The ideal completion* $\mathsf{Idl}(P(A))$ *is a weak prime domain. Conversely, each weak prime domain is isomorphic to* $\mathsf{Idl}(P(A))$ *for some weak prime asynchronous graph A.*

Proof. First observe that in a weak asynchronous graph $A = \langle G, n_0, \simeq \rangle$ with $G = \langle N, U, s, t \rangle$ such that all the cofinal paths from the origin are equivalent we have that all the squares are commuting. Thus axioms (1) and (2) imply that the graph is simple, that there are at most two different paths of length 2 with the same source and target, and that there is at most one way of closing a square.

[1] In a personal communication, Paul Andrée Melliès agreed that condition (4) is necessary for the correctness of Theorem 3 of Section 2.6 of [29], rephrased here as Theorem 5.16.

Now, let D be a weak prime domain and consider the subset of compact elements $\mathsf{K}(D)$. It can be seen as an (acyclic) graph by taking compact elements as nodes and intervals as edges, with source and target functions being the obvious ones $s([c,c']) = c$ and $t([c,c']) = c'$. Then taking $\emptyset$ as origin and letting all the squares commute, we get a weak asynchronous graph where all the paths are equivalent. In detail, as observed above, axiom (1) follows from the fact that the graph is simple. Axiom (2) says that there are at most two paths of length 2 between the same source and target. Assume that this is not the case, i.e., $\mathsf{K}(D)$ contains a substructure as below, with y_1, y_2, y_3 pairwise distinct.

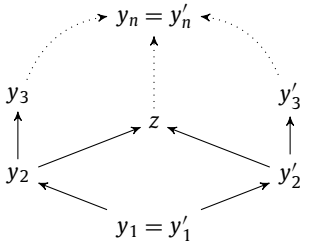


Then we would have that y_1 is an irreducible which is not a weak prime. In fact $y_1 \sqsubseteq y_2 \sqcup y_3$, but it is not the case that either $y_1 \leftrightarrow y_2$ or $y_1 \leftrightarrow y_3$.

Axiom (3a) follows from bounded completeness and the fact that if $x \prec y_1$ and $x \prec y_2$, with $y_1 \neq y_2$ then $y_1 \prec y_1 \sqcup y_2$ and $y_2 \prec y_1 \sqcup y_2$.

Axiom (4) is an immediate consequence of coherence.

Finally, we have to prove that all the paths from $\emptyset$ to the same target are equivalent. We prove more generally that all coinitial and cofinal paths are equivalent. First notice that given two paths $w = y_1 \dots y_n$ and $w' = y'_1 \dots y'_m$ with $y_1 = y'_1$ and $y_n = y'_m$ then $n = m = |[ir(y_n)]_{\leftrightarrow^*} \setminus [ir(y_1)]_{\leftrightarrow^*}|$, by Lemma A.3. We prove by induction on $n = m$ that the two paths are equivalent. The base cases $n = 1$ and $n = 2$ are obvious. In the inductive case, consider $z = y_2 \sqcup y'_2$.



Then, as already observed, $y_2 \prec z$ and $y'_2 \prec z$. Then

$$y_1 y_2 z \simeq y'_1 y'_2 z \tag{1}$$

Moreover, since $z \sqsubseteq y_n = y'_n$ there is a path $y_2 z \dots y_n$ of length $n-1$ in a way that we can apply the inductive hypothesis to prove that $y_2 y_3 \dots y_n \simeq y_2 z \dots y_n$. Similarly, on the left side, we get $y'_2 y'_3 \dots y_n \simeq y'_2 z \dots y'_n$. Therefore, together with (1), we conclude that $w = y_1 y_2 y_3 \dots y_n \simeq y_1 y_2 z \dots y_n \simeq y'_1 y'_2 z \dots y'_n \simeq y'_1 y'_2 y'_3 \dots y'_n = w'$.

Conversely, let $A = \langle G, n_0, \simeq \rangle$ where $G = \langle N, U, s, t \rangle$ is a weak asynchronous graph such that all the paths from the origin are equivalent. Then, in particular, all the squares are commuting and, by axiom (1), the graph is simple, i.e., we can think of edges as a relation on nodes. This allows us to view A as a concurrent automata $(Q, \Sigma, T, (\|_q)_{q \in Q})$ in the sense of [25] as follows. Define an equivalence on edges by $u \equiv u'$ if there are $v, v' \in U$ such that $uv \sim v'u'$ (namely, u, u' are the opposite edges of a square). Then take nodes as states $Q = N$, equivalence classes of edges as labels $\Sigma = U_{\equiv}$, transition relation $T = \{(s(u), u, t(u)) \mid u \in U\}$ and local concurrency given by $[u]_{\equiv} \|_n [v]_{\equiv}$ when u, v are such that $s(u) = s(v) = n$ and there are $u', v' \in E$ such that $uu' \sim vv'$. The fact that $\|_n$ is well-defined uses in an essential way axioms (3a) and (4). Then an immediate adaptation of [25, Theorem 10] to asynchronous graphs shows that $P(A)$ is a domain that satisfies axioms (F), (C), and (R) in Subsection 5.2. Finally, observe that axiom (V) is a "global" version of the axiom (1). The fact that the latter implies the former can be proved by exploiting the fact that each bounded subset of $P(A)$ is a semimodular lattice [33, Theorem 3.1]. Hence D is a weak prime domain. □

6. Conclusions and related work

In the paper we provided a characterisation of a class of domains, referred to as weak prime algebraic domains, which is appropriate for describing the concurrent semantics of those formalisms where a computational step can merge parts of the state. We show a categorical equivalence between weak prime algebraic domains and a class of connected event structures. We also prove that the category of general event structures coreflects into the category of weak prime algebraic domains. The appropriateness of such structure is witnessed also by the fact that the characterisations of prime domains and event structures in terms of intervals and asynchronous graphs naturally extend to this setting. The characterisation of weak prime domains in terms of the interchangeability equivalence on irreducibles naturally suggest a presentation in terms of prime event structures endowed with an equivalence relation, allowing us to establish a link with the work in [23,24].

Technically, the starting point for our proposal is the relaxation of the stability condition for event structures. As already noted by Winskel in [5] "[t]he stability axiom would go if one wished to model processes which had an event which could be caused in several compatible ways [...]; then I expect complete irreducibles would play a similar role to complete primes here". Indeed, the correspondence between irreducibles and weak primes, which exploits the notion of interchangeability, is the ingenious step that allows us to obtain a smooth extension of the classical duality between prime event structures and prime algebraic domains.

The coreflection between the category of general event structures (with binary conflict) and the one of weak prime algebraic domains says that the latter are exactly the partial orders of configurations of the former. Such class of domains has been studied originally in [21] where, generalising the work on concrete domains and sequentiality [34], a characterisation is given in terms of a set of axioms expressing properties of prime intervals. In our paper we also provide an in depth comparison with these results, based on the observation that, roughly speaking, weak primes correspond to executions of events with their minimal enablings, while intervals can be seen as executions of events in a generic configuration. A comparison is also drawn with the recent notion of asynchronous graph [29], an alternative representation of prime algebraic domains based on the notion of path equivalence, which we generalise in order to account for weak prime ones. We mentioned that, in a related line of work [27], the transition graphs of prime event structures are characterised as a class of transition systems with independence. More specifically, occurrence transition systems with independence, an acyclic variant satisfying an extra axiom, are exactly the transition systems induced by prime event structures. An interesting question is whether the correspondence can be extended to general event structures. Due to the non-local nature of independence, this is not a consequence of what we proved in the paper. Surely one should omit from the definition of occurrence transition systems with independence [27, Definition 4.1] the axiom stating that a pair of cofinal transitions can always be completed backward to a commuting square (this is related to what we called stability axiom), but further non-trivial modifications are required. Without getting into technical details, this is due to the fact that independence is no longer a property of events but rather of events in contexts. For instance, in our running example in Fig. 2c, events a and c are independent in configuration $\{b\}$ while they are not in the empty configuration. This is an interesting direction of further research.

The need of resorting to unstable ES for modelling the concurrent computations of name passing process calculi has been observed by several authors. In particular, in [17] an ES semantics for the π-calculus is defined by relying on ES *with names*, namely labelled ES tailored for modelling parallel extrusions. An event can have various minimal enablings but with the constraint that distinct minimal enablings can differ only for one event (intuitively, the extruder). ES with names are not connected ES since they can have non-connected minimal enablings (roughly, because identical events in disconnected minimal enablings can be identified via the labelling). Nevertheless, a connected ES semantics could be obtained by transforming ES with names through the coreflection in the paper. More details are reported in [32].

Concerning the appropriateness of the class of weak prime domains, length and readability concerns suggested us to move to a different paper the results preliminarily reported in [35] and further refined in [32] showing that weak prime algebraic domains are precisely those arising from graph rewriting systems where rules can also merge graph items, a paradigmatic example of formalism with fusion. Indeed, we prove there that, in the same way as prime algebraic domains/prime event structures are exactly what is needed for Petri nets/linear graph rewriting systems, weak prime algebraic domains/connected event structures are exactly what is needed for left-linear graph rewriting systems: each rewriting system maps to a connected event structure and conversely each connected event structure arises as the semantics of some rewriting system. This supports the adequateness of weak prime algebraic domains and connected event structures as semantics structures for formalisms with fusions, hinted at in the present paper with a few examples.

We believe that our results cover a long road in establishing weak prime domains and connected event structures as a foundational concept in the event-based semantics for concurrent computational systems. Our next step will be to look at possible more general formalisms. In particular, the paper [36] studies a characterisation of the partial order of configurations for a variety of classes of event structures in terms of axiomatisability of the associated propositional theories. Even if the focus in the present paper is on event structures that generalise Winskel's ones, we believe that our work can provide interesting suggestions for further development.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proofs and technicalities

A.1. Proofs for § 3. Weak prime algebraic domains

Lemma 3.2 *(Irreducibility and compactness). Let D be a domain and $d \in \mathsf{K}(D)$. Then d is a complete irreducible iff for all $x, y \in D$, $x \frown y$ and $d = x \sqcup y$ implies $d = x$ or $d = y$.*

Proof. Let $d \in \mathsf{K}(D)$. Assume that for all $x, y \in D$, $x \frown y$ and $d = x \sqcup y$ implies $d = x$ or $d = y$. Assume that $d = \bigsqcup X$ for some pairwise consistent X. It is easy to see that $X' = \{\bigsqcup Y \mid Y \in \mathbf{2}^X_{fin}\}$ is directed and moreover $\bigsqcup X' = \bigsqcup X = d$. Since d is compact, there is $x' \in X'$ such that $d \sqsubseteq x'$, hence $d = x'$. By definition of X', this means that there exists $Y \in \mathbf{2}^X_{fin}$ such that $d = \bigsqcup Y$. Now, using the hypothesis, an inductive reasoning allows us to conclude that $d \in Y \subseteq X$, as desired.

The converse implication is trivial: if $d \in ir(D)$ and $d \sqsubseteq x \sqcup y$ then, by definition of complete irreducible, $d \in \{x, y\}$, i.e., either $d = x$ or $d = y$, as desired. □

Lemma 3.3 *(Unique predecessor for irreducibles). Let D be a domain and $i \in D$. Then $i \in ir(D)$ iff it has a unique immediate predecessor.*

Proof. Assume that $i \in D$ has a unique immediate predecessor $d \prec i$, and let $X \subseteq D$ be pairwise consistent and such that $i = \bigsqcup X$. Hence for any $x \in X$ we have $x \sqsubseteq i$. Assume by contradiction that $i \notin X$. This implies that for all elements $x \in X$ we have $x \sqsubseteq d$, and therefore $i = \bigsqcup X \sqsubseteq d \prec i$, which is a contradiction. Hence it must be $i \in X$, which means that i is irreducible.

Vice versa, let i be irreducible and let $d_1, d_2 \prec i$ be immediate predecessors. Since D is a domain and $d_1 \frown d_2$, we can take $d = d_1 \sqcup d_2$ and we know $d_1 \sqsubseteq d \sqsubseteq i$. Since i is irreducible it cannot be $d = i$, therefore $d = d_1$ and thus $d_1 = d_2$. Thus we conclude that i has a unique immediate predecessor. □

Proposition 3.5 *(Domains are irreducible algebraic). Let D be a domain and $d \in D$. Then it holds $d = \bigsqcup ir(d)$.*

Proof. We first prove that for any compact element $d \in \mathsf{K}(D)$ it holds that $d = \bigsqcup(\downarrow d \cap ir(D))$. The thesis then immediately follows from algebraicity of D. Since D is a domain, $\downarrow d$ is finite, hence we can proceed by induction on $|\downarrow d|$. When $|\downarrow d| = 1$, we have that $d = \bot$, hence $\downarrow d \cap ir(D) = \emptyset$ and indeed $\bot = \bigsqcup \emptyset$. When $|\downarrow d| = k > 1$ consider the immediate predecessors of d and denote them $d_1, \ldots, d_n \prec d$. Since D is a domain and $\{d_1, \ldots, d_n\}$ is consistent, there exists $\bigsqcup\{d_1, \ldots, d_n\} = d'$ and $d_i \sqsubseteq d' \sqsubseteq d$. There are two cases

- $d' = d_i$, for all $i \in [1, n]$, i.e., d has a unique immediate predecessor, hence it is an irreducible and thus clearly $d = \bigsqcup(\downarrow d \cap ir(D))$ or
- $d = d' = \bigsqcup\{d_1, \ldots, d_n\}$. Since, in turn, by inductive hypothesis $d_i = \bigsqcup(\downarrow d_i \cap ir(D))$ and $\downarrow d \cap ir(D) = \bigcup_{i=1}^{n}(\downarrow d_i \cap ir(D))$, we immediately get the thesis. □

Proposition 3.6 *(Irreducibles vs. primes). Let D be a domain. Then $pr(D) \subseteq ir(D)$. Moreover, D is a prime domain iff $pr(D) = ir(D)$.*

Proof. Let D be a domain. We show that $pr(D) \subseteq ir(D)$. Let $d \in pr(D)$. Assume that $d = \bigsqcup X$ for some pairwise consistent set X. By primality, since $d \sqsubseteq \bigsqcup X$ there must be $x \in X$ such that $d \sqsubseteq x$. We have also $x \sqsubseteq \bigsqcup X = d$ and thus $d = x \in X$.

For the second part, let us assume that D is a prime domain. We have to prove that $pr(D) = ir(D)$. We already know that $pr(D) \subseteq ir(D)$. For the converse inclusion, let $i \in ir(D)$. By prime algebraicity $i = \bigsqcup(\downarrow i \cap pr(D))$. Since i is irreducible, there exists $p \in \downarrow i \cap pr(D)$ such that $i = p$, hence i is a prime.

Vice versa, if D is a domain, by Proposition 3.5 we know that D is irreducible algebraic. Hence, if $pr(D) = ir(D)$, we immediately conclude that D is prime. □

Lemma 3.7 *(Unique decomposition in prime domains). Let D be a prime domain and $X, X' \subseteq ir(D)$ downward closed sets of irreducibles. If $\bigsqcup X = \bigsqcup X'$ then $X = X'$.*

Proof. Let $X, X' \subseteq ir(D)$ be downward closed sets of irreducibles such that $\bigsqcup X = \bigsqcup X'$. Take any $i' \in X'$. Then $i' \sqsubseteq \bigsqcup X$. Since the domain is prime algebraic, and thus i' is prime, there must exist $i \in X$ such that $i' \sqsubseteq i$ and thus $i' \in X$. Therefore $X' \subseteq X$. By symmetry also the converse inclusion holds, whence equality. □

We first observe that distinct irreducibles related by the interchangeability relation are necessarily incomparable.

Lemma A.1 *($\leftrightarrow$ vs. $\sqsubseteq$). Let D be a domain and $i, i' \in ir(D)$. If $i \leftrightarrow i'$ and $i \sqsubseteq i'$ then $i = i'$.*

Proof. Let $i \leftrightarrow i'$ and $i \sqsubseteq i'$. If $i \neq i'$ and we let $X = ir(p(i'))$, it turns out that $X \cup \{i\} = X$ and $X \cup \{i'\}$ are consistent and downward closed. Moreover $\bigsqcup X \cup \{i\} = \bigsqcup X = p(i') \neq \bigsqcup X \cup \{i'\} = i'$, contradicting $i \leftrightarrow i'$. □

Lemma 3.9 *(Characterising $\leftrightarrow$). Let D be a domain and $i, i' \in ir(D)$. Then the following are equivalent*

1. $i \leftrightarrow i'$;
2. $i \frown i'$ *and for all* $d \in \mathsf{K}(D)$ *if* $p(i), p(i') \sqsubseteq d$ *then* $d \sqcup i = d \sqcup i'$;
3. $i \frown i'$ *and* $i \sqcup p(i') = p(i) \sqcup i'$.

Proof. (1 → 2) Assume that $i \leftrightarrow i'$. By definition, $i \frown i'$. Let $d \in \mathsf{K}(D)$ be such that $p(i), p(i') \sqsubseteq d$. If we let $X = ir(d)$ we have that $ir(i) \setminus \{i\} \subseteq X$ and similarly $ir(i') \setminus \{i'\} \subseteq X$. This implies that $X \cup \{i\}$ and $X \cup \{i'\}$ are downward closed and consistent. Hence $d \sqcup i = \bigsqcup X \sqcup i = \bigsqcup(X \cup \{i\}) = \bigsqcup(X \cup \{i'\}) = \bigsqcup X \sqcup i' = d \sqcup i'$.

(2 → 3) Assume (2). Let $p = p(i) \sqcup p(i')$, which is in $\mathsf{K}(D)$ since $p(i), p(i') \in \mathsf{K}(D)$. Clearly, $p(i), p(i') \sqsubseteq p$. Therefore $i \sqcup p(i') = i \sqcup p(i) \sqcup p(i') = i \sqcup p = p \sqcup i' = p(i) \sqcup p(i') \sqcup i' = p(i) \sqcup i'$.

(3 → 1) Assume (3). Let $X \subseteq ir(D)$ be such that $X \cup \{i\}$ and $X \cup \{i'\}$ are downward closed and consistent sets of irreducibles. This implies that $ir(p(i)) \subseteq X$ and similarly $ir(p(i')) \subseteq X$. Hence, if we let $P = ir(p(i)) \cup ir(p(i'))$, we have

$$P \subseteq X \quad \text{and} \quad \bigsqcup P = p(i) \sqcup p(i').$$

Therefore

$$\begin{aligned}
&\bigsqcup(X \cup \{i\}) = \\
&= (\bigsqcup X \setminus P) \sqcup \bigsqcup P \sqcup i = \\
&= (\bigsqcup X \setminus P) \sqcup p(i) \sqcup p(i') \sqcup i = \\
&= (\bigsqcup X \setminus P) \sqcup i \sqcup p(i') = \\
&= (\bigsqcup X \setminus P) \sqcup p(i) \sqcup i' = \\
&= (\bigsqcup X \setminus P) \sqcup p(i) \sqcup p(i') \sqcup i' = \\
&= (\bigsqcup X \setminus P) \sqcup \bigsqcup P \sqcup i' = \\
&= \bigsqcup(X \cup \{i'\}) \quad \square
\end{aligned}$$

Lemma 3.12 *(Weak primes in prime domains). Let D be a prime domain. Then $\leftrightarrow$ is the identity and $wpr(D) = pr(D)$.*

Proof. Let $i, i' \in ir(D)$ be such that $i \leftrightarrow i'$.

If i and i' are comparable, i.e., $i \sqsubseteq i'$ or $i' \sqsubseteq i$, by Lemma A.1 we deduce $i = i'$ and we are done.

Otherwise, let $X = (ir(i) \setminus \{i\}) \cup (ir(i') \setminus \{i'\})$. Note that $X \cup \{i\}$ and $X \cup \{i'\}$ are consistent, since, by definition of $\leftrightarrow$, i and i' are so. Moreover $X \cup \{i\}$ and $X \cup \{i'\}$ are downward closed, and thus, from $i \leftrightarrow i'$, we deduce $\bigsqcup(X \cup \{i\}) = \bigsqcup(X \cup \{i'\})$. Since D is prime, by Lemma 3.7, this implies that $X \cup \{i\} = X \cup \{i'\}$. Since i and i' are incomparable, $i, i' \notin X$ and we conclude $i = i'$. □

Theorem 3.16 (pDom *as a subcategory of* wDom*). The category of prime domains* pDom *is the full subcategory of* wDom *having prime domains as objects.*

Proof. We just need to show that weak prime domain morphisms preserve meets on prime domains, i.e., that if D_1, D_2 are prime domains and $f : D_1 \to D_2$ is a weak prime domain morphism then $f(\bigsqcap X_1) = \bigsqcap f(X_1)$ for all $X_1 \subseteq D_1$ pairwise consistent.

We first show that for $d_1, d_1' \in \mathsf{K}(D_1)$, with $d_1 \frown d_1'$, it holds that $f(d_1 \sqcap d_1') = f(d_1) \sqcap f(d_1')$. We proceed by induction on $k = |{\downarrow} d_1 \cap pr(D)|$.

When $k = 0$ we have $d_1 = \bot$. Since f preserves joins, we have that $f(\bot) = f(\bigsqcup \emptyset) = \bigsqcup f(\emptyset) = \bigsqcup \emptyset = \bot$. Hence

$$f(d_1 \sqcap d_1') = f(\bot \sqcap d_1') = f(\bot) = \bot = \bot \sqcap f(d_1') = f(\bot) \sqcap f(d_1') = f(d_1) \sqcap f(d_1').$$

Suppose now $k > 0$. We distinguish two subcases. If d_1 is not prime then, recalling that in a prime domain, primes and irreducibles coincide, d_1 is not irreducible and thus $d_1 = e_1 \sqcup f_1$ with $d_1 \neq e_1, f_1 \neq \bot$. It is immediate to see that $|{\downarrow} e_1 \cap pr(D)| < k$ and $|{\downarrow} f_1 \cap pr(D)| < k$. Moreover, since any prime algebraic domain is distributive we have $d_1 \sqcap d_1' = (e_1 \sqcup f_1) \sqcap d_1' = (e_1 \sqcap d_1') \sqcup (f_1 \sqcap d_1')$.

Summing up

$$
\begin{aligned}
& f(d_1 \sqcap d_1') = \\
& \quad = f((e_1 \sqcap d_1') \sqcup (f_1 \sqcap d_1')) && \text{[Preservation of } \sqcup\text{]} \\
& \quad = f(e_1 \sqcap d_1') \sqcup f(f_1 \sqcap d_1') && \text{[Inductive hypothesis]} \\
& \quad = (f(e_1) \sqcap f(d_1')) \sqcup (f(f_1) \sqcap f(d_1')) && \text{[Distributivity]} \\
& \quad = (f(e_1) \sqcup f(f_1)) \sqcap f(d_1') && \text{[Preservation of } \sqcup\text{]} \\
& \quad = f(e_1 \sqcup f_1) \sqcap f(d_1') = \\
& \quad = f(d_1) \sqcap f(d_1')
\end{aligned}
$$

If instead d_1 is prime then note that if $d_1 \sqsubseteq d_1'$ the thesis is immediate: by monotonicity $f(d_1) \sqsubseteq f(d_1')$, thus $f(d_1 \sqcap d_1') = f(d_1) = f(d_1) \sqcap f(d_1')$, as desired. Therefore, let us assume that $d_1 \not\sqsubseteq d_1'$. In this case $d_1 \sqcap d_1' = p(d_1) \sqcap d_1'$, since the set of lower bounds of $\{d_1, d_1'\}$ and of $\{p(d_1), d_1'\}$ is the same. Observe that

$$p(d_1) = d_1 \sqcap (p(d_1) \sqcup d_1') \tag{A.1}$$

In fact, the join exists since $d_1 \frown d_1'$. Moreover, by distributivity, $d_1 \sqcap (p(d_1) \sqcup d_1') = (d_1 \sqcap p(d_1)) \sqcup (d_1 \sqcap d_1') = p(d_1) \sqcup (p(d_1) \sqcap d_1') = p(d_1)$.

Therefore

$$
\begin{aligned}
& f(d_1 \sqcap d_1') = \\
& \quad = f(p(d_1) \sqcap d_1') && \text{[Inductive hypothesis]} \\
& \quad = f(p(d_1)) \sqcap f(d_1') && \text{[Using (A.1)]} \\
& \quad = f(d_1 \sqcap (p(d_1) \sqcup d_1')) \sqcap f(d_1') && \text{[By Definition 3.15(3)]} \\
& \quad = f(d_1) \sqcap f(p(d_1) \sqcup d_1')) \sqcap f(d_1') && \text{[Preservation of } \sqcup\text{]} \\
& \quad = f(d_1) \sqcap f(d_1')
\end{aligned}
$$

as desired. This extends to the meet of finite sets of compact elements, by associativity of $\sqcap$, and to infinite sets of compacts by observing that, given an infinite set X, by finitariness we can identify a finite subset $F \subseteq X$ such that $\bigsqcap X = \bigsqcap F$. The last assertion can be proved by induction on $k = \min\{|\downarrow d| : d \in X\}$. In fact, let $d \in X$ be an element such that $|\downarrow d| = k$. If $k = 1$ then $d = \bot$ and thus $\bigsqcap X = \bot = \bigsqcap\{d\}$, as desired. If $k > 1$, then we distinguish two possibilities. If for all $d' \in X$ it holds $d \sqcap d' = d$ then $\bigsqcap X = d = \bigsqcap\{d\}$. If instead, there is $d' \in X$ such that $d \sqcap d' \sqsubset d$ then recall that the meet of compact elements is compact and consider $X' = X \cup \{d \sqcap d'\}$. We have that $\bigsqcap X = \bigsqcap X'$. Moreover $|\downarrow d \sqcap d'| < k$, hence we can apply the inductive hypothesis to X' and get a finite subset $F' \subseteq X'$ such that $\bigsqcap X' = \bigsqcap F'$. We conclude observing that $\bigsqcap X = \bigsqcap X' = \bigsqcap F' = \bigsqcap((F' \setminus \{d \sqcap d'\}) \cup \{d, d'\})$. Therefore we can take $F = (F' \setminus \{d \sqcap d'\}) \cup \{d, d'\}$ and we conclude. □

A.2. Proofs for § 4.1. From event structures to weak prime domains

Lemma 4.3 *(Properties of the domain of configurations). Let $\langle E, \vdash, Con\rangle$ be an* ES*. Then*

1. *$\mathcal{D}(E)$ is a domain, $\mathsf{K}(\mathcal{D}(E)) = Conf_{fin}(E)$, join is union, and $C \prec C'$ iff $C' = C \cup \{e\}$ for some $e \in E \setminus C$;*
2. *$C \in Conf(E)$ is irreducible iff $C = C' \cup \{e\}$ and $C' \vdash_0 e$; in this case we denote C as $\langle C', e\rangle$;*
3. *for $C \in Conf(E)$, we have $ir(C) = \{\langle C', e'\rangle \mid e' \in C \ \wedge \ C' \subseteq C \ \wedge \ C' \vdash_0 e'\}$; moreover $p(\langle C', e'\rangle) = C'$;*
4. *for $\langle C_1, e_1\rangle, \langle C_2, e_2\rangle \in ir(\mathcal{D}(E))$, we have $\langle C_1, e_1\rangle \leftrightarrow \langle C_2, e_2\rangle$ iff $e = e_1 = e_2$ and $C_1 \overset{e}{\frown} C_2$;*
5. *$\mathcal{D}(E)$ is interchangeable.*

Proof. 1. We first observe that, given a pairwise consistent set of configurations $X \subseteq Conf(E)$, the join is the union $\bigsqcup X = \bigcup X$. The fact that $\bigcup X$ is a configuration, i.e., that it is consistent and secured immediately follows from the fact that each $C \in X$ is.

Let $C \in Conf(E)$ be a configuration. For every event $e \in E$, since C is secured, we can consider a set $C_e = \{e_1, \ldots, e_n\} \subseteq C$ such that $e_n = e$ and $\{e_1, \ldots, e_{k-1}\} \vdash e_k$ for all $k \in [1, n]$. It is immediate to see that $C_e \in Conf_{fin}(E)$ and clearly $C = \bigsqcup_{e \in C} C_e$.

From the above it is almost immediate to conclude that the compact elements of $\mathcal{D}(E)$ are the finite configurations $\mathsf{K}(\mathcal{D}(E)) = Conf_{fin}(E)$ and that $\mathcal{D}(E)$ is algebraic. Moreover, $\mathcal{D}(E)$ is finitary, since the number of subsets of a finite configurations is clearly finite. Hence $\mathcal{D}(E)$ is a domain.

Concerning immediate precedence, let $C, C' \in Conf_{fin}(E)$. If $C' = C \cup \{e\}$ with $e \notin C$ then clearly $C \prec C'$, since the order is subset inclusion. Conversely, if $C \prec C'$ by definition $C \subseteq C'$ and it must be $|C' \setminus C| = 1$. In fact, $C \subseteq C'$ and $C \neq C'$, hence $C' \setminus C \neq \emptyset$. Let $e, e' \in C' \setminus C$. Let us prove that $e = e'$. Since C' is secured there is a set of events $D = \{e_1, \ldots, e_n\} \subseteq C'$, such that $e_n = e$ and $\{e_1, \ldots, e_{k-1}\} \vdash e_k$ for all $k \in [1, n]$. Now, if $e' \notin D$, observe that $C \cup D$ is a configuration and $C \subset C \cup D \subset C'$, contradicting $C \prec C'$. Assume that, instead, $e' \in D$. If $e' = e_k$ for $k < n$ we would have that $D' = \{e_1, \ldots, e_k\}$ is a configuration and we could replace D by D' in the contradiction above. Hence it must be $e = e'$, as desired.

2. Let $C \in Conf(E)$ be a configuration and assume that $C = C' \cup \{e\}$ with $C' \vdash_0 e$. Then C is a finite configuration, and thus a compact element. Moreover, if $C = C_1 \cup C_2$ for $C_1, C_2 \in Conf(E)$, then e must occur either in C_1 or in C_2. If $e \in C_1$, since C_1 is secured, there exists $C_1' \subseteq C_1 \setminus \{e\}$ such that $C_1' \vdash e$. Hence, by monotonicity of enabling, $C_1 \setminus \{e\} \vdash e$. Since $C' \vdash_0 e$ and $C_1 \setminus \{e\} \subseteq C'$ it follows that $C_1 \setminus \{e\} = C'$ and thus $C_1 = C$. Therefore, by Lemma 3.2, C is an irreducible.
Vice versa, let $C \in Conf(E)$ be an irreducible. It is compact, hence finite. Hence we can consider a secured execution $\langle e_1, \ldots, e_n \rangle$ of configuration C. Note that for any $k \in [1, n-1]$ it must be $\{e_1, \ldots, e_{k-1}\} \not\vdash e_n$. Otherwise, if it were $\{e_1, \ldots, e_{k-1}\} \vdash e_n$ for some $k \in [1, n-1]$, we would have that $C' = \{e_1, \ldots, e_k, e_n\}$ and $C'' = \{e_1, \ldots, e_{n-1}\}$ are two proper subconfigurations of C such that $C = C' \cup C''$, violating the fact that C is irreducible. But this means exactly that $\{e_1, \ldots, e_{n-1}\} \vdash_0 e_n$, as desired.

3. Immediate.

4. Let $I_j = \langle C_j, e_j \rangle \in ir(\mathcal{D}(E))$ for $j \in \{1, 2\}$ be irreducibles. Assume $I_1 \leftrightarrow I_2$. By Lemma 3.9(3), observing that $p(I_j) = C_j$, we must have $I_1 \cup C_2 = C_1 \cup I_2$, namely $C_1 \cup \{e_1\} \cup C_2 = C_1 \cup C_2 \cup \{e_2\}$, from which we conclude that it must be $e_1 = e_2$, i.e., as desired $I_j = \langle C_j, e \rangle$, where $e = e_1 = e_2$ for $j \in \{1, 2\}$. Additionally, $I_1 \frown I_2$, by definition of $\leftrightarrow$, meaning that $C_1 \overset{e}{\frown} C_2$.
For the converse, consider two irreducibles $I_1 = \langle C_1, e \rangle$ and $I_2 = \langle C_2, e \rangle$, such that $C_1 \overset{e}{\frown} C_2$. Hence $C_1 \vdash_0 e$, $C_2 \vdash_0 e$ and $C = C_1 \cup C_2 \cup \{e\}$ is consistent. Since $I_1, I_2 \subseteq C$, they are consistent in $\mathcal{D}(E)$. Moreover, $p(I_1) = C_1$, $p(I_2) = C_2$ and $I_1 \cup C_2 = I_2 \cup C_1 = C$. Hence by Lemma 3.9(3) we have $I_1 \leftrightarrow I_2$, as desired.

5. We have to show that $\mathcal{D}(E)$ satisfies the conditions of Definition 3.10. Concerning condition (1), let $I_1 = \langle C_1, e_1 \rangle$ and $I_2 = \langle C_2, e_2 \rangle$ such that $I_1 \leftrightarrow^* I_2$ and $p(I_1) = C_1, p(I_2) = C_2$ consistent. From $I_1 \leftrightarrow^* I_2$, by the above result, we deduce $e_1 = e_2$. Since, C_1, C_2 consistent, we deduce $C_1 \overset{e}{\frown} C_2$ and thus, again by the same result, $I_1 \leftrightarrow I_2$.
As for Condition (2), consider the irreducibles I, I', J and J' such that $I \leftrightarrow^* I'$, $J \leftrightarrow^* J'$, and $I' \frown J'$, $p(I) \frown J$ and $p(J) \frown I$. From $I \leftrightarrow^* I'$ and $J \leftrightarrow^* J'$ we deduce that $I = \langle C, e \rangle$, $I' = \langle C', e \rangle$, $J = \langle D, f \rangle$ and $J' = \langle D', f \rangle$. Moreover, we have $p(I) = C$ and $p(J) = D$, hence the hypotheses say $C \frown J$ and $D \frown I$. From $I' \frown J'$ we deduce that $\{e, f\}$ consistent. Therefore we have that $I = \langle C, e \rangle$ and $J = \langle D, f \rangle$ are consistent, as desired. □

A.3. Proofs for § 4.2. From weak prime domains to connected event structures

In order to prove Lemma 4.8, we need a preliminary technical result.

Lemma A.2 *(Immediate precedence and irreducibles/1). Let D be a weak prime domain, $d \in \mathsf{K}(D)$, and $i \in ir(D)$ such that $d \frown i$ and $p(i) \sqsubseteq d$. Then*

1. *for all $i' \in \delta(d \sqcup i, d)$ minimal, it holds $i \leftrightarrow i'$;*
2. *$d \preceq d \sqcup i$.*

Proof. 1. Clearly, if $d = d \sqcup i$ then $\delta(d \sqcup i, d) = \emptyset$ and the property trivially holds. Assume $d \neq d \sqcup i$ and take $i' \in \delta(d \sqcup i, d)$ minimal. Note that minimality implies that $p(i') \sqsubseteq d$. In fact, for all $i_1' \in ir(p(i'))$ we have $i_1' \sqsubset i' \sqsubseteq d \sqcup i$. Hence $i_1' \sqsubseteq d$, otherwise $i_1' \in \delta(d \sqcup i, d)$, violating minimality of i'. Therefore $p(i') = \bigsqcup ir(p(i')) \sqsubseteq d$.
Now, from $i' \sqsubseteq d \sqcup i$, since D is a weak prime domain and thus irreducibles are weak primes, there must be $i'' \in ir(D)$, $i'' \leftrightarrow i'$ such that $i'' \sqsubseteq d$ or $i'' \sqsubseteq i$. We first note that it cannot be $i'' \sqsubseteq d$, otherwise $d = d \sqcup i'' = d \sqcup i'$, the last equality motivated by Lemma 3.9(2), which implies that $i' \sqsubseteq d$, contradicting the hypothesis. Hence it must be $i'' \sqsubseteq i$, which by Lemma 3.3 means that either $i'' = i$ or $i'' \sqsubseteq p(i)$. Since $p(i) \sqsubseteq d$ by hypothesis, the latter case would contradict $i'' \not\sqsubseteq d$, hence $i'' = i$ which means that $i' \leftrightarrow i$, as desired.

2. Let us assume that $d \neq d \sqcup i$ (otherwise the property is trivial), and consider d' such that $d \prec d' \sqsubseteq d \sqcup i$: we prove that $d' = d \sqcup i$. Since $d \prec d'$, hence $d \neq d'$, we know that $\delta(d', d)$ is not empty. Take a minimal $i' \in \delta(d', d)$. Thus i' is minimal also in $\delta(d \sqcup i, d)$, and thus, by point (1), $i \leftrightarrow i'$. By minimality of i' we deduce also that $p(i') \sqsubseteq d$. Since also $p(i) \sqsubseteq d$ by hypothesis, using Lemma 3.9(2), we have $d \sqcup i = d \sqcup i'$. Observing that $d \sqcup i' \sqsubseteq d' \sqsubseteq d \sqcup i$ we conclude that $d' = d \sqcup i$, as desired. □

Lemma 4.8 *(Immediate precedence and irreducibles/2). Let D be a weak prime domain and $d, d' \in D$ such that $d \preceq d'$. Then for all $i, i' \in \delta(d', d)$*

1. *$d' = d \sqcup i$;*
2. *if $i \sqsubseteq i'$ then $i = i'$;*
3. *$i \leftrightarrow i'$.*

Proof. If $d = d'$ all properties hold trivially.

1) Let $i \in \delta(d', d)$. Then $d \sqsubseteq d \sqcup i \sqsubseteq d'$. It follows that either $d \sqcup i = d$ or $d \sqcup i = d'$. The first possibility can be excluded for the fact that it would imply $i \sqsubseteq d$, while we know that $i \notin ir(d)$. Hence we get the thesis.
2) Let $i, i' \in \delta(d', d)$, with $i \sqsubseteq i'$. Let us first assume i minimal in $\delta(d', d)$, hence $p(i) \sqsubseteq d$. Then $i' \sqsubseteq d' = d \sqcup i$. Since i' is a weak prime, there exists $i'' \in ir(D)$ such that $i' \leftrightarrow i''$ and either $i'' \sqsubseteq i$ or $i'' \sqsubseteq d$. The second possibility is excluded. In fact, if $i'' \sqsubseteq d$, then we would have $p(i), p(i'') \sqsubseteq d$ and thus, by Lemma 3.9(2), $d' = d \sqcup i = d \sqcup i'' = d$, contradicting $d \neq d'$. Hence it must be $i'' \sqsubseteq i$. Since $i \sqsubseteq i'$, by transitivity $i'' \sqsubseteq i'$ and since $i' \leftrightarrow i''$, by Lemma A.1, $i'' = i'$ and thus $i'' = i = i'$. If instead, i is not minimal in $\delta(d', d)$, take $i_1 \sqsubseteq i$ minimal. By the argument above, we have that $i_1 \leftrightarrow i'$, and thus, by Lemma A.1, $i_1 = i'$. Recalling that $i_1 \sqsubseteq i \sqsubseteq i'$ we conclude $i = i'$, as desired.
3) Let $i, i' \in \delta(d', d)$ be irreducibles. By (1) we have $d' = d \sqcup i$, hence $i' \in \delta(d \sqcup i, d)$. By (2) i' is minimal in $\delta(d \sqcup i, d)$. Therefore, by Lemma A.2(1), we conclude $i \leftrightarrow i'$. □

We next show another technical result, i.e., that chains of immediate precedence are generated in essentially a unique way by sequences of irreducibles. Given a domain D and an irreducible $i \in ir(D)$, we denote by $[i]_{\leftrightarrow^*}$ the corresponding equivalence class. For $X \subseteq ir(D)$ we define $[X]_{\leftrightarrow^*} = \{[i]_{\leftrightarrow^*} \mid i \in X\}$.

Lemma A.3 *(Chains of immediate precedence). Let D be a weak prime domain, $d \in \mathsf{K}(D)$, and $ir(d) = \{i_1, \ldots, i_n\}$ such that the sequence $i_1, \ldots, i_n$ is compatible with the order (i.e., for all h, k if $i_h \sqsubseteq i_k$ then $h \leq k$). If we let $d_k = \bigsqcup_{h=1}^{k} i_h$ for $k \in \{1, \ldots, n\}$ we have*

$$\bot = d_0 \preceq d_1 \preceq \ldots \preceq d_n = d$$

Vice versa, given a chain $\bot = d_0 \prec d_1 \prec \ldots \prec d_n$ and taking $i_h \in \delta(d_h, d_{h-1})$ for $h \in \{1, \ldots, n\}$ we have

$$d_n = \bigsqcup_{h=1}^{n} i_h \;\text{ and }\; \forall i \in ir(d_n)\,.\,\exists h \in [1, n].\; i \leftrightarrow i_h.$$

Therefore $[\{i_1, \ldots, i_n\}]_{\leftrightarrow^*} = [ir(d_n)]_{\leftrightarrow^*}$.

Proof. For the first part, observe that for $k \in \{1, \ldots, n\}$ we have that

$$p(i_k) \sqsubseteq d_{k-1}$$

In fact, recalling that $ir(i_k) \subseteq ir(d)$, we have that irreducibles in $ir(p(i_k)) = ir(i_k) \setminus \{i_k\}$, which are smaller than i_k, must occur before in the list hence

$$ir(p(i_k)) = ir(i_k) \setminus \{i_k\} \subseteq \{i_1, \ldots, i_{k-1}\}.$$

Therefore $p(i_k) = \bigsqcup ir(p(i_k)) \sqsubseteq \bigsqcup\{i_1, \ldots, i_{k-1}\} = d_{k-1}$. Thus we use Lemma A.2(2) to infer $d_{k-1} \preceq d_{k-1} \sqcup i_k = d_k$.

For the second part, we proceed by induction on n.

- $(n = 0)$ Note that $d_0 = \bigsqcup \emptyset = \bot$ and $ir(\bot) = \emptyset$, hence the thesis trivially holds.
- $(n > 0)$ By induction hypothesis

$$d_{n-1} = \bigsqcup_{h=1}^{n-1} i_h \;\text{ and }\; \forall i \in ir(d_{n-1})\,.\,\exists h \in [1, n-1].\; i \leftrightarrow i_h.$$

Since by construction $i_n \in \delta(d_n, d_{n-1})$, by Lemma 4.8(1) we deduce

$$d_n = i_n \sqcup d_{n-1} = \bigsqcup_{h=1}^{n} i_h$$

Moreover, for all $i \in \delta(d_n, d_{n-1})$, we have $i \sqsubseteq d_n = i_n \sqcup d_{n-1}$. By definition of weak prime domain, there exists $i' \leftrightarrow i$ such that $i' \sqsubseteq d_{n-1}$ or $i' \sqsubseteq i_n$. In the first case, since $i' \sqsubseteq d_{n-1}$, by the inductive hypothesis there is $h \in [1, n-1]$ such that $i' \leftrightarrow i_h$. Since $i \leftrightarrow i' \leftrightarrow i_h$, and $i, i_h \sqsubseteq d_n$ are consistent, by using the fact that D is interchangeable we deduce $i \leftrightarrow i_h$, as desired. If, instead, we are in the second case, namely $i' \sqsubseteq i_n$, by Lemma 4.8(2) it follows that $i_n = i' \leftrightarrow i$, as desired. □

Proposition 4.9 *(Unique decomposition up to $\leftrightarrow$). Let D be a weak prime domain, $d \in \mathsf{K}(D)$, and $X \subseteq D$ a downward closed and consistent set such that $[X]_{\leftrightarrow^*} \subseteq [ir(d)]_{\leftrightarrow^*}$. Then $d = \bigsqcup X$ iff $[X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$.*

Proof. ($\Rightarrow$) Let $d = \bigsqcup X$. By hypothesis $[X]_{\leftrightarrow^*} \subseteq [ir(d)]_{\leftrightarrow^*}$. Hence we only need to prove that $[ir(d)]_{\leftrightarrow^*} \subseteq [X]_{\leftrightarrow^*}$. Let $i \in ir(d)$. Hence $i \sqsubseteq d = \bigsqcup X$. By definition of weak prime domain, this implies that there exists $i' \leftrightarrow i$ and $x \in X$ such that $i' \sqsubseteq x$. Since X is downward closed, necessarily $i' \in X$ and thus $[i]_{\leftrightarrow^*} \in [X]_{\leftrightarrow^*}$, as desired.

($\Leftarrow$) Let $[X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$. We can prove that $\bigsqcup X = d$ by induction on $k(X) = |(ir(d) \setminus X) \cup (X \setminus ir(d)|$. If $k(X) = 0$ then $X = ir(d)$ and thus, by Proposition 3.5, we conclude that $d = \bigsqcup X$. If $k(X) > 0$ we distinguish two subcases.

- First assume $ir(d) \setminus X \neq \emptyset$. Then take a minimal element $i \in ir(d) \setminus X$. Therefore $X' = X \cup \{i\}$ is downward closed and, by minimality of i, we have $p(i) \sqsubseteq \bigsqcup X$. Since $[X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$, there is $i' \in X$ such that $i \leftrightarrow^* i'$ and thus, since $p(i), p(i') \sqsubseteq \bigsqcup X$ are consistent and D is interchangeable, $i \leftrightarrow i'$. Therefore
$$\bigsqcup X' = \bigsqcup X \cup \{i\} = \bigsqcup X \cup \{i'\} = \bigsqcup X. \tag{A.2}$$
Since $[X']_{\leftrightarrow^*} = [X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$ and $|ir(d) \setminus X'| = |ir(d) \setminus X| - 1$, we have $k(X') < k(X)$, and thus by inductive hypothesis $\bigsqcup X' = d$. Hence, using (A.2), we get $\bigsqcup X = d$, as desired.
- If instead $ir(d) \setminus X = \emptyset$, i.e., $ir(d) \subseteq X$, since $k(X) > 0$, it must be $X \setminus ir(d) \neq \emptyset$. Consider a maximal element $i \in X \setminus ir(d)$, and let $X' = X \setminus \{i\}$. Clearly, X' is downward closed because so are X and $ir(d)$. Since $[X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$, there is $i' \in ir(d) \subseteq X$ such that $i \leftrightarrow^* i'$. Since X is consistent and D is interchangeable, $i \leftrightarrow i'$. Therefore
$$\bigsqcup X = \bigsqcup X' \cup \{i\} = \bigsqcup X' \cup \{i'\} = \bigsqcup X'. \tag{A.3}$$
Since by construction $k(X') = k(X) - 1$, the inductive hypothesis gives us $\bigsqcup X' = d$. Hence, using (A.3), we get $\bigsqcup X = d$, as desired. □

In the following we often use a technical lemma that holds in any domain.

Lemma A.4. *Let D be a domain and $a, b, c \in D$ such that $c \sqsubseteq a$ and $c \prec b$. Then either $b \sqsubseteq a$ or $c = a \sqcap b$.*

Proof. Recall that in a domain the meet of non-empty sets exists. Since c is a lower bound for a and b, necessarily $c \sqsubseteq a \sqcap b \sqsubseteq b$. If it were $c \neq a \sqcap b$ then we would have $a \sqcap b = b$, hence $b \sqsubseteq a$, as desired. □

Lemma 4.11 *(From weak prime domains to event structures). Let D be a weak prime domain. Then $\mathcal{E}(D)$ is an* ES*. Moreover, given two weak prime domains D_1, D_2 and a morphism $f : D_1 \to D_2$, its image $\mathcal{E}(f) : \mathcal{E}(D_1) \to \mathcal{E}(D_2)$ is an* ES *morphism.*

Proof. We first show that $\mathcal{E}(D)$ is a live ES. In fact, it is an ES: if $X \vdash e$ and $X \subseteq Y$ then $Y \vdash e$. In fact, by definition, if $X \vdash e$ then there exists $i \in e$ such that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} \subseteq X$. Hence if $X \subseteq Y$ it immediately follows that $Y \vdash e$. Moreover $\mathcal{E}(D)$ is live. The fact that conflict is saturated follows immediately by the definition of conflict and the characterisation of configurations provided later in Lemma 4.12. Conflict is irreflexive since for any $e \in \mathcal{E}(D)$, if $e = [i]_{\leftrightarrow^*}$ then $e \in [ir(i)]_{\leftrightarrow^*}$, which is a configuration again by Lemma 4.12.[2]

Given a morphism $f : D_1 \to D_2$, its image $\mathcal{E}(f) : \mathcal{E}(D_1) \to \mathcal{E}(D_2)$ is defined for $[i_1]_{\leftrightarrow^*} \in E_1$ as $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*}$, where $i_2 \in \delta(f(i_1), f(p(i_1)))$, and $\mathcal{E}(f)([i_1]_{\leftrightarrow^*})$ is undefined if $f(p(i_1)) = f(i_1)$. First observe that $\mathcal{E}(f)([i_1]_{\leftrightarrow^*})$ does not depend on the choice of the representative. In fact, let $i_2, i_2' \in \delta(f(i_1), f(p(i_1)))$. Since $p(i_1) \prec i_1$, by definition of domain morphism, $f(p(i_1)) \prec f(i_1)$. Thus, by Lemma 4.8(3), $i_2 \leftrightarrow i_2'$.

We next show that $\mathcal{E}(f)$ is an ES morphism.

- If $\mathcal{E}(f)(e_1) \# \mathcal{E}(f)(e_1')$ then $e_1 \# e_1'$.
We prove the contronominal, namely if e_1, e_1' consistent then $\mathcal{E}(f)(e_1), \mathcal{E}(f)(e_1')$ consistent.
The fact that e_1, e_1' consistent means that there exists $d_1 \in \mathsf{K}(D_1)$ such that $e_1, e_1' \in [ir(d_1)]_{\leftrightarrow^*}$. We show that $\mathcal{E}(f)(e_1), \mathcal{E}(f)(e_1') \in [ir(f(d_1))]_{\leftrightarrow^*}$ (note that $f(d_1)$ is a compact, since f is a domain morphism).
Let us show, for instance, that $\mathcal{E}(f)(e_1) \in [ir(f(d_1))]_{\leftrightarrow^*}$. The fact that $e_1 \in [ir(d_1)]_{\leftrightarrow^*}$ means that $e_1 = [i_1]_{\leftrightarrow^*}$ for some $i_1 \sqsubseteq d_1$. By definition $\mathcal{E}(f)(e_1) = [i_2]_{\leftrightarrow^*}$, where $i_2 \in \delta(f(i_1), f(p(i_1)))$ (since $\mathcal{E}(f)(e_1)$ is defined the irreducible difference cannot be empty). Now, since $i_1 \sqsubseteq d_1$ we have that $f(i_1) \sqsubseteq f(d_1)$, whence $i_2 \sqsubseteq f(i_1) \sqsubseteq f(d_1)$ and $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*} \in [ir(f(d_1))]_{\leftrightarrow^*}$, as desired.
- If $\mathcal{E}(f)(e_1) = \mathcal{E}(f)(e_1')$ and $e_1 \neq e_1'$ then $e_1 \# e_1'$.
We prove the contronominal, namely if e_1, e_1' consistent and $\mathcal{E}(f)(e_1) = \mathcal{E}(f)(e_1')$ then $e_1 = e_1'$.
Assume e_1, e_1' consistent and $\mathcal{E}(f)(e_1) = \mathcal{E}(f)(e_1')$. By the first condition and the definition of conflict, there must be $d_1 \in \mathsf{K}(D_1)$ such that $e_1, e_1' \in [ir(d_1)]_{\leftrightarrow^*}$, namely $e_1 = [i_1]_{\leftrightarrow^*}$ and $e_1' = [i_1']_{\leftrightarrow^*}$ with $i_1, i_1' \sqsubseteq d_1$.

[2] This forward reference is only useful to simplify the structure of the presentation and to avoid breaking the statement in two parts, but it introduces no cyclic dependency.

Moreover, $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*}$ and $\mathcal{E}(f)([i'_1]_{\leftrightarrow^*}) = [i'_2]_{\leftrightarrow^*}$ where i_2 and i'_2 are in $\delta(f(i_1), f(p(i_1)))$ and $\delta(f(i'_1), f(p(i'_1)))$, respectively, and $[i_2]_{\leftrightarrow^*} = [i'_2]_{\leftrightarrow^*}$, which means $i_2 \leftrightarrow^* i'_2$, and in turn, being i_2 and i'_2 consistent, by the fact that D is interchangeable, implies $i_2 \leftrightarrow i'_2$.
We distinguish two cases.

A. If i_1 and i'_1 are comparable, e.g., if $i_1 \sqsubseteq i'_1$, then $i_1 = i'_1$ and we are done. In fact, otherwise, if $i_1 \neq i'_1$ we have $p(i_1) \prec i_1 \sqsubseteq p(i'_1) \prec i'_1$. By monotonicity of f we have $f(p(i_1)) \prec f(i_1) \sqsubseteq f(p(i'_1)) \prec f(i'_1)$ (where strict inequalities $\prec$ are motivated by the definition of $\mathcal{E}(f)$, since both $\mathcal{E}(f)([i_1]_{\leftrightarrow^*})$ and $\mathcal{E}(f)([i'_1]_{\leftrightarrow^*})$ are defined). Now notice that $p(i_2) \sqsubseteq i_2 \sqsubseteq f(i_1) \sqsubseteq f(p(i'_1))$. Moreover, $i'_2 \in \delta(f(i'_1), f(p(i'_1)))$, therefore $p(i'_2) \sqsubseteq i'_2 \sqsubseteq f(p(i'_1))$. Hence, using the fact that $i_2 \leftrightarrow i'_2$, by Lemma 3.9(2) we have

$$f(p(i'_1)) = f(p(i'_1)) \sqcup i_2 = f(p(i'_1)) \sqcup i'_2 = f(i'_1)$$

contradicting the fact that $f(p(i'_1)) \prec f(i'_1)$.

B. Assume now that i_1 and i'_1 are incomparable: we show that this leads to a contradiction. Let $p = p(i_1) \sqcup p(i'_1)$. We can also assume $i_1, i'_1 \not\sqsubseteq p$. In fact, otherwise, e.g., if $i_1 \sqsubseteq p$, then, by the defining property of weak prime domains, we derive the existence of $i''_1 \leftrightarrow i_1$ such that $i''_1 \sqsubseteq p(i_1)$ or $i''_1 \sqsubseteq p(i'_1)$. The first possibility can be excluded because it would imply $i''_1 \sqsubseteq i_1$. Hence, since $i''_1 \leftrightarrow i_1$, by Lemma A.1, we would get $i_1 = i''_1$, contradicting $i''_1 \sqsubseteq p(i_1)$. Then it should be $i''_1 \sqsubseteq p(i'_1) \sqsubseteq i'_1$. Therefore, if we take i''_1 as representative of the equivalence class we are back to case A above.
Using the fact that $i_1, i'_1 \not\sqsubseteq p$ and $p(i_1), p(i'_1) \sqsubseteq p$, by Lemma A.2(2) we deduce that $p \prec p \sqcup i_1$ and $p \prec p \sqcup i'_1$. Hence $f(p) \prec f(p \sqcup i_1)$ with strict inequality again motivated by the definition of $\mathcal{E}(f)$, since $\mathcal{E}(f)([i_1]_{\leftrightarrow^*})$ is defined.
By Lemma 4.8(1), since $i_2 \in \delta(f(i_1), f(p(i_1)))$ and $i'_2 \in \delta(f(i'_1), f(p(i'_1)))$, we have

$$f(p(i_1)) \sqcup i_2 = f(i_1) \qquad f(p(i'_1)) \sqcup i'_2 = f(i'_1) \tag{A.4}$$

Now, observe that

$$\begin{aligned}
&f(p \sqcup i_1) = \\
&\quad = f(p(i_1) \sqcup p(i'_1) \sqcup i_1) \\
&\quad = f(p(i'_1) \sqcup i_1) \\
&\quad = f(p(i'_1)) \sqcup f(i_1) && \text{[preservation of } \sqcup\text{]} \\
&\quad = f(p(i'_1)) \sqcup f(p(i_1)) \sqcup i_2 && \text{[by (A.4)]} \\
&\quad = f(p(i'_1)) \sqcup f(p(i_1)) \sqcup i'_2 && \text{[by Lemma 3.9(2), since } i_2 \leftrightarrow i'_2\text{]} \\
&\quad = f(i'_1) \sqcup f(p(i_1)) && \text{[by (A.4)]} \\
&\quad = f(p(i_1) \sqcup i'_1) && \text{[preservation of } \sqcup\text{]} \\
&\quad = f(p(i_1) \sqcup p(i'_1) \sqcup i'_1) \\
&\quad = f(p \sqcup i'_1)
\end{aligned}$$

Since $p \prec p \sqcup i_1$ and $p \prec p \sqcup i'_1$, by Lemma A.4, we have $(p \sqcup i_1) \sqcap (p \sqcup i'_1) = p$. Therefore, on the one hand $f((p \sqcup i_1) \sqcap (p \sqcup i'_1)) = f(p)$. On the other hand, since the meet is an immediate predecessor, by definition of weak domain morphism (Definition 3.15), it is preserved: $f((p \sqcup i_1) \sqcap (p \sqcup i'_1)) = f(p \sqcup i_1) \sqcap f(p \sqcup i'_1) = f(p \sqcup i_1) = f(p \sqcup i'_1)$. Putting things together, $f(p) = f(p \sqcup i_1) = f(p \sqcup i'_1)$, contradicting the fact that $f(p) \prec f(p \sqcup i_1)$.

- if $C_1 \vdash_1 [i_1]_{\leftrightarrow^*}$ and $\mathcal{E}(f)([i_1]_{\leftrightarrow^*})$ is defined then $\mathcal{E}(f)(C_1) \vdash_2 \mathcal{E}(f)([i_1]_{\leftrightarrow^*})$.
Recall that $C_1 \vdash_1 [i_1]_{\leftrightarrow^*}$ means that $[ir(i'_1) \setminus \{i'_1\}]_{\leftrightarrow^*} = [ir(p(i'_1))]_{\leftrightarrow^*} \subseteq C_1$ for some $i'_1 \leftrightarrow i_1$.
By definition, $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*}$ where $i_2 \in \delta(f(i'_1), f(p(i'_1)))$. We show that $\mathcal{E}(f)(C_1) \vdash_2 [i_2]_{\leftrightarrow^*}$, namely that

$$[ir(i_2) \setminus \{i_2\}]_{\leftrightarrow^*} = [ir(p(i_2))]_{\leftrightarrow^*} \subseteq \mathcal{E}(f)(C_1) \tag{A.5}$$

Observe that since $i_2 \in \delta(f(i'_1), f(p(i'_1)))$ and distinct elements in $\delta(f(i'_1), f(p(i'_1)))$ are incomparable by Lemma 4.8(2), it holds $p(i_2) \sqsubseteq f(p(i'_1))$. Therefore, we have

$$ir(p(i_2)) \subseteq ir(f(p(i'_1)))$$

Hence, in order to conclude (A.5), it suffices to show that

$$[ir(f(p(i'_1)))]_{\leftrightarrow^*} \subseteq \mathcal{E}(f)(C_1) \tag{A.6}$$

In order to reach this result, first note that, by Lemma A.3, if $ir(p(i'_1)) = \{j^1_1, \ldots, j^n_1\}$ is a sequence of irreducibles compatible with the order, we can obtain a $\preceq$-chain

$$\bot = d^0_1 \preceq d^1_1 \preceq \ldots \preceq d^n_1 = p(i'_1) \prec i'_1$$

We can extract a strictly increasing subsequence

$$\bot = d'^0_1 \prec d'^1_1 \prec \ldots \prec d'^m_1 = p(i'_1) \prec i'_1$$

and, if we take irreducibles $j'^1_1, \ldots, j'^m_1$ in $\delta(d'^i_1, d'^{i-1}_1)$, again by Lemma A.3 we know that

$$[ir(p(i'_1))]_{\leftrightarrow^*} = [\{j'^1_1, \ldots, j'^m_1\}]_{\leftrightarrow^*} \tag{A.7}$$

Since f is a domain morphism, it preserves $\preceq$, namely

$$\bot = f(d'^0_1) \preceq f(d'^1_1) \preceq \ldots \preceq f(d'^m_1) = f(p(i'_1)) \prec f(i'_1)$$

where the last inequality is strict since $\mathcal{E}(f)([i'_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*}$ is defined. Moreover, whenever $f(d'^{h-1}_1) \prec f(d'^h_1)$, then $\mathcal{E}(f)([j'^h_1]_{\leftrightarrow^*}) = [\ell^h_2]_{\leftrightarrow^*}$ where ℓ^h_2 is any irreducible in $\delta(f(d'^h_1), f(d'^{h-1}_1))$, otherwise $\mathcal{E}(f)([j'^h_1]_{\leftrightarrow^*})$ is undefined.
Once more by Lemma A.3 we know that

$$[ir(f(p(i'_1)))]_{\leftrightarrow^*} = [\{\ell^1_2, \ldots, \ell^m_2\}]_{\leftrightarrow^*} = \mathcal{E}(f)([\{j'^1_1, \ldots, j'^m_1\}]_{\leftrightarrow^*}),$$

thus, using (A.7)

$$[ir(f(p(i'_1)))]_{\leftrightarrow^*} = \mathcal{E}(f)([ir(p(i'_1))]_{\leftrightarrow^*}). \tag{A.8}$$

Hence, recalling that, by hypothesis, $[ir(p(i'_1))]_{\leftrightarrow^*} \subseteq C_1$, we conclude the desired inclusion (A.6). □

Lemma 4.12 *(Compacts vs. configurations). Let D be a weak prime domain and $C \subseteq \mathcal{E}(D)$ a finite set of events. Then C is a configuration in the* ES *$\mathcal{E}(D)$ iff there exists a unique $d \in \mathsf{K}(D)$ such that $C = [ir(d)]_{\leftrightarrow^*}$. Moreover, for all $e \in \mathcal{E}(D)$ we have that $C \vdash_0 e$ iff $C = [ir(i) \setminus \{i\}]_{\leftrightarrow^*}$ for some $i \in e$.*

Proof. The left to right implication of the first part follows by proving that, given a configuration $C \in Conf_{fin}(\mathcal{E}(D))$, there exists $X \subseteq ir(D)$ downward closed and consistent such that $[X]_{\leftrightarrow^*} = C$. Hence, if we let $d = \bigsqcup X$, by Proposition 4.9, we have that $C = [X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$. Moreover, d is uniquely determined, since, by the same proposition we have that for any other X' such that $[X']_{\leftrightarrow^*} = C$, since $[X']_{\leftrightarrow^*} = C = [X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$, necessarily $d = \bigsqcup X'$.

Let us thus prove the existence of $X \subseteq ir(D)$ consistent and downward closed such that $[X]_{\leftrightarrow^*} = C$. We proceed by induction on the cardinality of C.

- if $|C| = 0$, namely $C = \emptyset$ then we can take $X = \emptyset$, and trivially conclude.
- if $|C| > 0$, since C is secured, there is $[i]_{\leftrightarrow^*} \in C$ such that $C' = C \setminus \{[i]_{\leftrightarrow^*}\} \vdash [i]_{\leftrightarrow^*}$. By inductive hypothesis there is $X' \subseteq ir(D)$, downward closed and consistent such that $[X']_{\leftrightarrow^*} = C'$.
  The fact that $C' = C \setminus \{[i]_{\leftrightarrow^*}\} \vdash [i]_{\leftrightarrow^*}$ means that for some $i' \in ir(D)$ such that $i' \leftrightarrow^* i$, it holds $[ir(i') \setminus \{i'\}]_{\leftrightarrow^*} = [ir(p(i'))]_{\leftrightarrow^*} \subseteq C'$. Therefore, there is $X'' \subseteq X'$ such that $[X'']_{\leftrightarrow^*} = [ir(p(i))]_{\leftrightarrow^*}$ and thus, by Proposition 4.9, $p(i') \sqsubseteq \bigsqcup X'$. We can assume, without loss of generality that $ir(p(i')) \subseteq X'$. If not, we can replace X' by $X' \cup ir(p(i'))$. By the consideration above, it is consistent and it has the same join of X'.
  Now, an induction on the cardinality k of $X' \setminus ir(p(i'))$ allows us to show that $i' \frown j$ for all $j \in X'$. If $k = 0$ then $X' \setminus ir(p(i')) = \emptyset$ and the thesis is trivial. Otherwise, consider $j' \in X' \setminus ir(p(i'))$ maximal and $X'' = X' \setminus \{j'\}$. Since $|X' \setminus ir(p(i'))| = k - 1$, by inductive hypothesis, for all $j \in X''$, we have $j \frown i'$. Now, since $j, p(i) \sqsubseteq \bigsqcup X'$, we have that $j \frown p(i)$. Moreover, since $ir(j') \setminus \{j'\} = ir(p(j')) \subseteq X''$, we have that $i \frown p(j')$.
  Finally, recalling that, since C is consistent, we have that $\neg([j']_{\leftrightarrow^*} \# [i']_{\leftrightarrow^*})$, i.e., there is $d \in \mathsf{K}(D)$ such that $\{[j']_{\leftrightarrow^*}, [i']_{\leftrightarrow^*}\} \subseteq [ir(d)]_{\leftrightarrow^*}$. More explicitly, this means that there are $j'', i'' \in ir(D)$ such that $j'' \leftrightarrow^* j'$, $i'' \leftrightarrow^* i'$ and $j' \frown i''$. Since D is interchangeable, by condition (2) of Definition 3.10, we conclude $j' \frown i'$.
  We can thus conclude that $X = X' \cup \{i'\}$ is consistent, and downward closed since $ir(p(i')) \subseteq X'$. Hence we conclude.

For the converse, let $C = [ir(d)]_{\leftrightarrow^*}$. Let $\bot = d_0 \prec d_1 \prec \ldots d_{n-1} \prec d_n = d$ be a chain of immediate precedence and for each $h \in \{1, \ldots, n\}$ take $i_h \in \delta(d_h, d_{h-1})$. By Lemma A.3, $d = \bigsqcup\{i_1, \ldots, i_n\}$ and $[ir(d)]_{\leftrightarrow^*} = [\{i_1, \ldots, i_n\}]_{\leftrightarrow^*}$. Moreover, for all $h \in \{1, \ldots, n\}$, we have $[ir(i_h) \setminus \{i_h\}]_{\leftrightarrow^*} \subseteq [ir(d_{h-1})]_{\leftrightarrow^*}$, hence $[ir(d_{h-1})]_{\leftrightarrow^*} \vdash [i_h]_{\leftrightarrow^*}$. Therefore C is secured. Moreover, it is clearly consistent and thus $C \in Conf(\mathcal{E}(D))$.

The second part follows immediately by Definition 4.10. □

A.4. Proofs for § 4.3. Relating categories of models

Theorem 4.14 *(Coreflection of* ES *and* wDom*). The functors* $\mathcal{D} : \mathsf{ES} \to \mathsf{wDom}$ *and* $\mathcal{E} : \mathsf{wDom} \to \mathsf{ES}$ *form a coreflection* $\mathcal{E} \dashv \mathcal{D}$*. It restricts to an equivalence between* wDom *and* cES*.*

Proof. Let E be an ES. Recall that the corresponding domain of configurations is $\mathcal{D}(E) = \langle Conf(E), \subseteq \rangle$. Then, $\mathcal{E}(\mathcal{D}(E)) = \langle E', \#', \vdash' \rangle$, where the set of events E' is defined as

$$E' = [ir(\mathcal{D}(E))]_{\leftrightarrow^*} = \{[\langle C, e \rangle]_{\leftrightarrow^*} \mid C \vdash_0 e\}$$

By Lemma 4.3(4), the equivalence class of an irreducible $\langle C, e \rangle$ consists of all minimal enablings of event e which are connected. Therefore we can define a morphism, which is the counit of the adjunction, as follows

$$\begin{aligned} \theta_E : \quad & \mathcal{E}(\mathcal{D}(E)) \to E \\ & [\langle C, e \rangle]_{\leftrightarrow^*} \mapsto e \end{aligned}$$

Observe that θ_E is surjective. In fact E is live and thus any event $e \in E$ has a minimal enabling $C \vdash_0 e$. If we let $I = \langle C, e \rangle$, then $[I]_{\leftrightarrow^*} \in \mathcal{E}(\mathcal{D}(E))$ and $\theta_E([I]_{\leftrightarrow^*}) = e$. The mapping θ_E is clearly a morphism of event structures. In fact, observe that

- For $I_1, I_2 \in ir(\mathcal{D}(E))$, if $\theta_E([I_1]_{\leftrightarrow^*}) \# \theta_E([I_2]_{\leftrightarrow^*})$ then $[I_1]_{\leftrightarrow^*} \#' [I_2]_{\leftrightarrow^*}$.
  Let $I_1 = \langle C_1, e_1 \rangle$ and $I_2 = \langle C_2, e_2 \rangle$. If $\theta_E([I_1]_{\leftrightarrow^*}) = e_1 \# e_2 = \theta_E([I_2]_{\leftrightarrow^*})$, then there cannot be any configuration $C \in Conf(E)$ such that $I_1, I_2 \subseteq C$. Hence, by definition of conflict in $\mathcal{E}(\mathcal{D}(E))$, we have $[I_1]_{\leftrightarrow^*} \#' [I_2]_{\leftrightarrow^*}$.
- For $I_1, I_2 \in ir(\mathcal{D}(E))$, with $[I_1]_{\leftrightarrow^*} \neq [I_2]_{\leftrightarrow^*}$, we have that $\theta_E([I_1]_{\leftrightarrow^*}) = \theta_E([I_2]_{\leftrightarrow^*})$ implies $[I_1]_{\leftrightarrow^*} \#' [I_2]_{\leftrightarrow^*}$.
  In fact, by Lemma 4.3(2), the irreducibles I_1 and I_2 are of the kind $I_1 = \langle C_1, e_1 \rangle$ and $I_2 = \langle C_2, e_2 \rangle$. We show that if $[I_1]_{\leftrightarrow^*}$ and $[I_2]_{\leftrightarrow^*}$ are consistent and $\theta_E([I_1]_{\leftrightarrow^*}) = \theta_E([I_2]_{\leftrightarrow^*})$ then $[I_1]_{\leftrightarrow^*} = [I_2]_{\leftrightarrow^*}$.
  Assume $\theta_E([I_1]_{\leftrightarrow^*}) = \theta_E([I_2]_{\leftrightarrow^*})$, hence $e_1 = e_2$. Since $[I_1]_{\leftrightarrow^*}$ and $[I_2]_{\leftrightarrow^*}$ are consistent, there exists $k \in \mathsf{K}(\mathcal{D}(E))$ such that $[I_1]_{\leftrightarrow^*}, [I_2]_{\leftrightarrow^*} \in [ir(k)]_{\leftrightarrow^*}$. Compacts in $\mathcal{D}(E)$ are finite configurations, hence the condition amounts to the existence of $C \in Conf_{fin}(E)$ such that $[I_1]_{\leftrightarrow^*}, [I_2]_{\leftrightarrow^*} \in [ir(C)]_{\leftrightarrow^*}$, i.e., there are I_1', I_2' with $I_i \leftrightarrow^* I_i'$ for $i \in \{1, 2\}$, such that $I_1', I_2' \subseteq C$. Since the choice of the representatives is irrelevant, we can assume that $I_1 = I_1'$ and $I_2 = I_2'$. Summing up, I_1 and I_2 are consistent minimal enablings of the same event, hence by Lemma 4.3(4), $I_1 \leftrightarrow I_2$, i.e., $[I_1]_{\leftrightarrow^*} = [I_2]_{\leftrightarrow^*}$, as desired.
- For the enabling relation, we have to show that if $X \vdash' [\langle C, e \rangle]_{\leftrightarrow^*}$ then $\theta_E(X) \vdash \theta([\langle C, e \rangle]_{\leftrightarrow^*}) = e$. Assume $X \vdash' [\langle C, e \rangle]_{\leftrightarrow^*}$. According to the definition of the functor $\mathcal{E}$, this means that there exists $i \in [\langle C, e \rangle]_{\leftrightarrow^*}$ such that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} \subseteq X$. Let such $i \in [\langle C, e \rangle]_{\leftrightarrow^*}$ be $i = \langle C', e \rangle$ with $C' \vdash_0 e$. We have

  $$ir(\langle C', e \rangle) \setminus \{\langle C', e \rangle\} = ir(C') = \{[\langle C'', e'' \rangle]_{\leftrightarrow^*} \mid \langle C'', e'' \rangle \subseteq C'\}.$$

  Therefore from $[ir(\langle C', e' \rangle) \setminus \{\langle C', e' \rangle\}]_{\leftrightarrow^*} \subseteq X$ we deduce

  $$\theta_E([ir(\langle C', e' \rangle) \setminus \{\langle C', e' \rangle\}]_{\leftrightarrow^*}) = C' \subseteq \theta_E(X).$$

  Since $C' \vdash_0 e$, by monotonicity of enabling, we conclude $\theta_E(X) \vdash e$, as desired.

We prove the naturality of θ by showing that the diagram below commutes.

$$\begin{array}{ccc} \mathcal{E}(\mathcal{D}(E_1)) & \xrightarrow{\theta_{E_1}} & E_1 \\ {\scriptstyle \mathcal{E}(\mathcal{D}(f))}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{E}(\mathcal{D}(E_2)) & \xrightarrow[\theta_{E_2}]{} & E_2 \end{array}$$

Consider $[\langle C_1, e_1 \rangle]_{\leftrightarrow^*} \in \mathcal{E}(\mathcal{D}(E_1))$. Recall that $\mathcal{E}(\mathcal{D}(f))([\langle C_1, e_1 \rangle]_{\leftrightarrow^*})$ is computed by considering the image of the irreducible $\langle C_1, e_1 \rangle$ and of its predecessor, namely

$$\mathcal{D}(f)(C_1) = f(C_1) \quad \text{and} \quad \mathcal{D}(f)(\langle C_1, e_1 \rangle) = f(C_1 \cup \{e_1\})$$

If $f(e_1)$ is defined, then $f(C_1) \prec f(C_1 \cup \{e_1\})$ and $\mathcal{E}(\mathcal{D}(f))([\langle C_1, e_1 \rangle]_{\leftrightarrow^*}) = f(e_1)$, otherwise $\mathcal{E}(\mathcal{D}(f))([\langle C_1, e_1 \rangle]_{\leftrightarrow^*})$ is undefined. This means that in all cases, as desired

$$\mathcal{E}(\mathcal{D}(f))([\langle C_1, e_1 \rangle]_{\leftrightarrow^*}) = f(e_1) = f(\theta_{E_1}([\langle C_1, e_1 \rangle]_{\leftrightarrow^*})).$$

Vice versa, let D be a weak prime domain. Recall from Definition 4.10 that $\mathcal{E}(D) = \langle E, \#, \vdash \rangle$ is defined as

- $E = [ir(D)]_{\leftrightarrow^*}$;
- $e \# e'$ if there is no $d \in \mathsf{K}(D)$ such that $e, e' \in [ir(d)]_{\leftrightarrow^*}$;
- $X \vdash e$ if there exists $i \in e$ such that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} \subseteq X$.

Consider now $\mathcal{D}(\mathcal{E}(D))$. Elements of $\mathsf{K}(\mathcal{D}(\mathcal{E}(D)))$ are configurations of $C \in Conf_{fin}(\mathcal{E}(D))$. We can define the unit of the adjunction as

$$\begin{aligned}\eta_D : \quad & \mathsf{K}(D) \to \mathsf{K}(\mathcal{D}(\mathcal{E}(D))) \\ & d \mapsto [ir(d)]_{\leftrightarrow^*}\end{aligned}$$

Observe that it is well defined, since by Lemma 4.12, $[ir(d)]_{\leftrightarrow^*}$ is a finite configuration of $\mathcal{E}(D)$ and thus a compact element in $\mathsf{K}(\mathcal{D}(\mathcal{E}(D)))$. The function is clearly monotone and bijective with inverse $\eta_D^{-1} : \mathsf{K}(\mathcal{D}(\mathcal{E}(D))) \to \mathsf{K}(D)$ defined, for $C \in \mathsf{K}(\mathcal{D}(\mathcal{E}(D))) = Conf_{fin}(\mathcal{E}(D))$ by letting $\eta_D^{-1}(C) = d$, where d is the unique element, given by Lemma 4.12, such that $C = [ir(d)]_{\leftrightarrow^*}$. By algebraicity of the domains, this function thus uniquely extends to an isomorphism $\eta_D : D \to \mathcal{D}(\mathcal{E}(D))$.

Finally, we prove the naturality of η_D. It is convenient to prove the naturality of the inverse, i.e., to show that the diagram below commutes.

$$\begin{array}{ccc} \mathcal{D}(\mathcal{E}(D_1)) & \xrightarrow{\eta_{D_1}^{-1}} & D_1 \\ {\scriptstyle \mathcal{D}(\mathcal{E}(f))}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{D}(\mathcal{E}(D_2)) & \xrightarrow{\eta_{D_2}^{-1}} & D_2 \end{array}$$

Let $C_1 \in \mathsf{K}(\mathcal{D}(\mathcal{E}(D_1)))$, namely $C_1 \in Conf_{fin}(\mathcal{E}(D_1))$, and let $\eta_{D_1}^{-1}(C_1) = d_1$ be the element such that $C_1 = [ir(d_1)]_{\leftrightarrow^*}$.

The construction offered by Lemma A.3 provides a chain

$$d_1^0 = \bot \prec d_1^1 \prec d_1^2 \prec \ldots \prec d_1^n = d_1$$

and, by the same lemma, if we take an irreducible $i_1^h \in \delta(d_1^h, d_1^{h-1})$ for $1 \leq h \leq n$ we have that $C_1 = [ir(d_1)]_{\leftrightarrow^*} = [\{i_1^1, \ldots, i_1^n\}]_{\leftrightarrow^*}$. Therefore the image

$$\mathcal{D}(\mathcal{E}(f))(C_1) = \{\mathcal{E}(f)([j_1]_{\leftrightarrow^*}) \mid [j_1]_{\leftrightarrow^*} \in C_1\} = \{\mathcal{E}(f)([i_1^h]_{\leftrightarrow^*}) \mid h \in [1, n]\}$$

is the set of equivalence classes of irreducibles $i_2^1, \ldots, i_2^k$ corresponding to

$$f(d_1^0) = \bot \prec f(d_1^1) \prec f(d_1^2) \prec \ldots \prec f(d_1^n) = f(d_1)$$

namely $i_2^j \in \delta(f(d_1^j), f(d_1^{j-1}))$, and, again, by Lemma A.3, $[\{i_2^1, \ldots, i_2^k\}]_{\leftrightarrow^*} = [ir(f(d_1))]_{\leftrightarrow^*}$. Summing up

$$\eta_{D_2}^{-1}(\mathcal{D}(\mathcal{E}(f))(C_1)) = \eta_{D_2}^{-1}(\{[i_2^h]_{\leftrightarrow^*} \mid 1 \leq h \leq k\}) = f(d_1) = f(\eta_{D_1}^{-1}(C_1))$$

as desired. □

We finally show that the above coreflection restricts to an equivalence between wDom and cES. For this, just observe that, in the proof above, when E is a connected ES, then the morphism θ_E defined as

$$\begin{aligned}\theta_E : \quad & \mathcal{E}(\mathcal{D}(E)) \to E \\ & [\langle C, e\rangle]_{\leftrightarrow^*} \mapsto e\end{aligned}$$

is an isomorphism. We already know that it is surjective. We next show that it is also injective. In fact, if $\theta_E([I]_{\leftrightarrow^*}) = \theta_E([I']_{\leftrightarrow^*})$ then I and I' are minimal enablings of the same event, i.e., $I = [\langle C, e\rangle]_{\leftrightarrow^*}$ and $I' = [\langle C', e\rangle]_{\leftrightarrow^*}$. Since E is a weak prime domain, $C \overset{e}{\frown}{}^* C'$ and thus, by Lemma 4.3(4), $I \leftrightarrow^* I'$, i.e., $[I]_{\leftrightarrow^*} = [I']_{\leftrightarrow^*}$. Proving that also the inverse is an ES morphism is immediate, by exploiting the fact that the ES is live. □

A.5. Proofs for § 5.1. Prime event structures with equivalence

Lemma 5.4 *(Histories are configurations). Let $\langle P, \sim\rangle$ be a* EPES *and $e \in E$. Then $\downarrow e$ is a configuration.*

Proof. Let $e \in E$ be any event. We have to show that $\downarrow e$ is saturated. If there are $e' \in \downarrow e$ and $e'' \sim e'$ such that $\downarrow e'' \subseteq \downarrow e$ then $[\downarrow e'']_\sim \subseteq [\downarrow e]_\sim$ and hence, by condition (1) in Definition 5.2, $e'' \preceq e$ which means $e'' \in \downarrow e$. □

Proposition 5.5 *(Weak prime domain for* EPES*). Let $\langle P, \sim\rangle$ be a* EPES*. Then $\mathcal{D}_{eq}(\langle P, \sim\rangle) = \langle Conf(\langle P, \sim\rangle), \subseteq\rangle$ is a weak prime domain. Conversely, if D is a weak prime domain then $\mathcal{E}_{eq}(D) = \langle\langle ir(D), \#, \vdash\rangle, \leftrightarrow^*\rangle$ is an* EPES *with conflict and enabling defined by*

- $i_1 \# i_2$ *if* $\neg(i_1 \frown i_2)$;
- $X \vdash i$ *if* $X \supseteq ir(i) \setminus \{i\}$.

Proof. Let $\langle P, \sim\rangle$ be a EPES. Then it is easy to see that the irreducibles of $\mathcal{D}_{eq}(\langle P, \sim\rangle)$ are the minimal enablings $\downarrow e$ for $e \in E$. Moreover, given a set of pairwise consistent configurations $X \subseteq Conf(\langle P, \sim\rangle)$, the join $\bigsqcup X$ is the saturation of their union. Interchangeability is given by $\downarrow e \leftrightarrow \downarrow e'$ if $e \sim e'$ and $\neg(e \# e')$. Using these facts it is now almost immediate to conclude that $\mathcal{D}_{eq}(\langle P, \sim\rangle)$ is a weak prime domain. As a first step, we observe that $\mathcal{D}_{eq}(\langle P, \sim\rangle)$ is interchangeable (Definition 3.10)

- Condition (1) requires that for all $e, e' \in P$ if $\downarrow e \leftrightarrow^* \downarrow e'$ and $\Downarrow e \cup \Downarrow e'$ consistent then $\downarrow e \leftrightarrow \downarrow e'$. Observe that $\downarrow e \leftrightarrow^* \downarrow e'$ implies $e \sim e'$. Moreover, by condition (2) in Definition 5.2, $\Downarrow e \cup \Downarrow e'$ consistent implies $\neg(e \# e')$. Hence we conclude $\downarrow e \leftrightarrow \downarrow e'$.
- Condition (2) is an easy consequence of condition (3) of Definition 5.2. In fact, let $e, e', e_1, e'_1 \in E$ such that $\downarrow e \leftrightarrow^* \downarrow e'$, $\downarrow e_1 \leftrightarrow^* \downarrow e'_1$, i.e., $e \sim e'$ and $e_1 \sim e'_1$. Assume moreover that $\downarrow e' \frown \downarrow e'_1$, $\downarrow e \frown \Downarrow e_1$, $\Downarrow e \frown \downarrow e_1$, meaning that the sets $\downarrow e' \cup \downarrow e'_1$, $\downarrow e \cup \Downarrow e_1$, $\Downarrow e \cup \downarrow e_1$ are consistent.
  From the consistency of $\downarrow e' \cup \downarrow e'_1$ we have $\neg(e' \# e'_1)$. Moreover, the consistency of $\downarrow e \cup \Downarrow e_1$, $\Downarrow e \cup \downarrow e_1$ implies that if $e \# e_1$ then the conflict would be direct and this would violate condition (3) of Definition 5.2. Hence we must have $\neg(e \# e_1)$, i.e., $\downarrow e \frown \downarrow e_1$, as desired.

Finally, we show that all irreducibles are weak prime. Let $e \in P$, consider the irreducible $\downarrow e$ and a consistent set of configurations $X \subseteq Conf(\langle P, \sim\rangle)$. Assume that $\downarrow e \subseteq \bigsqcup X$. This means that e is in the saturation of $\bigcup X$, which in turn means that there is $C \in X$ and $e' \in C$, whence $\downarrow e' \subseteq C$, such that $e' \sim e$. Since $e, e' \in \bigsqcup X$, they are consistent, hence $\downarrow e \leftrightarrow \downarrow e'$. Summing up $\downarrow e' \subseteq C$ and $e \sim e'$, as desired.

Conversely, let D be a weak prime domain. Observe that the causal order in $\mathcal{E}_{eq}(D)$ is the restriction of the domain order to irreducibles. Condition (1) in Definition 5.2 is an immediate consequence of Proposition 4.9.

Condition (2) is immediately implied by condition (1) in the definition of interchangeable domain (Definition 3.10).

Concerning condition (3), observe that it becomes for $i, i', i_1, i'_1 \in ir(D)$, if $i \leftrightarrow^* i'$, $i_1 \leftrightarrow^* i'_1$, i, i_1 not consistent and i', i'_1 consistent then either $ir(p(i)) \cup \{i_1\}$ or $ir(p(i_1)) \cup \{i\}$ not consistent. In turn this is easily seen to be equivalent to condition (2) in the definition of interchangeable domain (Definition 3.10). □

Proposition 5.7. *Let $\langle P, \sim\rangle$ be an* EPES. *Then $\langle P, \sim\rangle$ and $\mathcal{U}(\mathcal{M}(\langle P, \sim\rangle))$ are isomorphic. Dually, let $P = \langle E, \vdash, \#\rangle$ be an* ES. *Then $\mathcal{M}(\mathcal{U}(P))$ and P are isomorphic.*

Proof. Let $\langle P, \sim\rangle$ be an EPES. Recall that the events in $\mathcal{U}(\mathcal{M}(\langle P, \sim\rangle))$ are minimal enablings in $\mathcal{M}(\langle P, \sim\rangle)$. By definitions of $\mathcal{M}(\langle P, \sim\rangle)$, for all $e \in P$ we have that $[X]_\sim \vdash_{\mathcal{M}(\langle P, \sim\rangle)} [e]_\sim$ when $X \vdash e$. Therefore $[\Downarrow e]_\sim \vdash_{\mathcal{M}(\langle P, \sim\rangle)} [e]_\sim$, and this enabling is minimal since, by Definition 5.2(1), whenever $e' \sim e$ and $[\Downarrow e']_{\leftrightarrow^*} \subseteq [\Downarrow e]_{\leftrightarrow^*}$ we have that $e = e'$. And, again relying on the definition of enabling, one sees that all minimal enablings are of this shape. Therefore we can define $c : \langle P, \sim\rangle \to \mathcal{U}(\mathcal{M}(\langle P, \sim\rangle))$ by $c(e) = \langle [\Downarrow e]_\sim, [e]_\sim\rangle$. By the previous arguments it is a bijection and c can be shown to be an isomorphism of EPES.

Conversely, let $\langle E, \vdash, \#\rangle$ be an ES. According to the definition, events in $\mathcal{U}(E)$ are minimal enablings $\langle C, e\rangle$ in E, and they are equivalent when they are minimal enablings of the same event. Then events in $\mathcal{M}(\mathcal{U}(E))$ are just equivalence classes of events in $\mathcal{U}(E)$. Therefore we can define $u : E \to \mathcal{M}(\mathcal{U}(E))$ by $u(e) = \{\langle C, e\rangle \mid C \in Conf(E) \ \wedge \ C \vdash_0 e\}$. It is immediate to see that it is a bijection and an isomorphism of ES. □

A.6. Proofs for § 5.2. Relation with interval based characterisations

Lemma 5.9 *(Intervals vs. irreducibles). Let D be a weak prime domain. Define $\zeta : Int(D)_\sim \to ir(D)_{\leftrightarrow^*}$ by*

$$\zeta([d, d']_\sim) = [i]_{\leftrightarrow^*},$$

where i is any element in $\delta(d', d)$. Then ζ is a bijection, whose inverse $\iota : ir(D)_{\leftrightarrow^*} \to Int(D)_\sim$ is defined by

$$\iota([i]_{\leftrightarrow^*}) = [p(i), i]_\sim.$$

Proof. We first observe that ζ is well-defined, i.e., if $[c, c'] \sim [d, d']$ are equivalent intervals then for all $i \in \delta(c', c)$, $i' \in \delta(d', d)$ it holds $i \leftrightarrow i'$. This follows by noting that if $[c, c'] \leq [d, d']$, $i \in \delta(c', c)$ and $i' \in \delta(d', d)$ then $i \leftrightarrow i'$. In order to prove the last assertion, observe that since $i \in ir(c')$ we have $i \sqsubseteq c' \sqsubseteq d'$, thus $i \in ir(d')$. Moreover, $i \notin ir(d)$, otherwise, by $i \sqsubseteq d$, $i \sqsubseteq c'$ and $c = d \sqcap c'$, we would get $i \sqsubseteq c$, contradicting the assumption that $i \in \delta(c', c)$. Hence $i \in \delta(d', d)$ and by Lemma 4.8(3) we conclude.

Also the converse map ι is well-defined. This follows from the observation that for all irreducibles $i, i' \in ir(D)$ if $i \leftrightarrow i'$ then $[p(i), i], [p(i'), i'] \leq [p(i) \sqcup p(i'), i \sqcup i']$ and thus $[p(i), i] \sim [p(i'), i']$. Let us prove, for instance, that

$$[p(i), i] \leq [p(i) \sqcup p(i'), i \sqcup i'].$$

Since $i \leftrightarrow i'$, surely $p(i) \sqsubseteq p(i) \sqcup p(i')$ and $p(i) \prec i$, hence by Lemma A.4, we deduce $i \sqsubseteq p(i) \sqcup p(i')$ or $p(i) = i \sqcap (p(i) \sqcup p(i'))$. The first possibility, $i \sqsubseteq p(i) \sqcup p(i')$, by the fact that i is irreducible leads to $i \sqsubseteq p(i')$ (since $i \sqsubseteq p(i)$ is clearly false). Thus $i \sqcup p(i') = p(i') \prec i' \sqsubseteq p(i) \sqcup i'$, that, by Lemma 3.9(3), contradicts $i \leftrightarrow i'$. Hence the second possibility must hold, i.e., $p(i) = i \sqcap (p(i) \sqcup p(i'))$. Moreover, again by Lemma 3.9(3), we have $i \sqcup (p(i) \sqcup p(i')) = i \sqcup i'$. Hence $[p(i), i] \leq [p(i) \sqcup p(i'), i \sqcup i']$ as desired.

The two maps are inverse each other.

- If $[d, d'] \in Int(D)$ and $i \in \delta(d', d)$ then $[d, d'] \sim [p(i), i]$.
  Observe that $d \sqcup i = d'$ by Lemma 4.8(1). Moreover, in order to show that $d \sqcap i = p(i)$, note that, since $i \in \delta(d', d)$ and, by Lemma 4.8(2), the set $\delta(d', d)$ is flat, we have that $p(i) \sqsubseteq d$. Moreover $p(i) \prec i$, therefore by Lemma A.4, $p(i) = d \sqcap i$, as desired.
- If $i \in ir(D)$ and $i' \in [p(i), i]$ then $i \leftrightarrow i'$.
  Just observe that $i \in [p(i), i]$ and then use Lemma 4.8(3). □

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