A Lattice-Theoretical Perspective on Adhesive Categories

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Abstract

It is a known fact that the subobjects of an object in an adhesive category form a distributive lattice. Building on this observation, in the paper we show how the representation theorem for finite distributive lattices applies to subobject lattices. In particular, we introduce a notion of irreducible object in an adhesive category, and we prove that any finite object of an adhesive category can be obtained as the colimit of its irreducible subobjects. Furthermore we show that every arrow between finite objects in an adhesive category can be interpreted as a lattice homomorphism between subobject lattices and, conversely, we characterize those homomorphisms between subobject lattices which can be seen as arrows.

Key words: adhesive categories, Van-Kampen-colimits, lattice theory

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1. Introduction

Adhesive categories (Lack and Sobociński, 2005) have been shown to provide a general categorical setting in which double-pushout rewriting (Corradini et al., 1997) can be defined in a way that some fundamental properties of rewriting (e.g., the Local Church-Rosser theorem based on the notions of parallel and sequential independence) hold without the need of further assumptions. As a notable example, standard directed graphs and graph morphisms form an adhesive category, but by the closure properties of adhesive categories many other graphical structures which are useful in the modelling of concurrent and distributed systems, including hypergraphs, hierarchical graphs, Petri nets (Sassone and Sobociński, 2005; Ehrig et al., 2006) can be captured in the realm of adhesive categories. As a consequence, the interest for the theory of adhesive categories goes beyond the mathematical aspects and it is motivated also by its potential applications to concurrency and to the theory of distributed systems.

It is known that the subobjects of an object in an adhesive category form a distributive lattice. Exploiting this fact, several proofs in the theory of adhesive categories can be carried out at a lattice-theoretical level. Still, while the theory of distributive lattices has been extensively studied and literature on the subject abounds (Davey and Priestley, 2002), to the best of our knowledge the possibility of applying such theory to adhesive categories has not been systematically investigated.

In general terms, this paper aims at strengthening the connection between the theory of adhesive categories and of distributive lattices, by showing that some relevant lattice-theoretic concepts and results find a natural counterpart in the setting of adhesive categories.

The long-term goal of this research is the development of a representation theorem for adhesive categories, at least for the finite case. We are interested in having a characterization theorem for objects of adhesive categories that would allow us to view them as instances of "graph-like structures". Several techniques for rewriting can be formulated in the abstract setting of adhesive grammars (Ehrig et al., 2006; Baldan et al., 2009) and a representation theorem would allow us to obtain a much more intuitive characterization of structures which are instances of this general framework. Since most of the insights provided by a representation theorem arises from the representation of the morphisms, we specifically study arrows in adhesive categories and show how they can be viewed as lattice homomorphisms. Conversely we characterize those lattice homomorphisms between subobject lattices which correspond to arrows. For this we build upon the classical representation theory of lattices (Birkhoff, 1967; Davey and Priestley, 2002) and known results for adhesive categories (Lack and Sobociński, 2005). In particular, we will exploit the notion of Van Kampen colimit introduced in (Cockett and Guo, 2007). Adhesive categories are a relatively new concept and we are not aware of any work in this direction, especially concerning the duality between lattice homomorphisms and arrows in adhesive categories.

Since in adhesive categories only finite colimits are sufficiently well-behaved, we will restrict ourselves to the theory of finite distributive lattices, which is much simpler than the infinite case. Hence, in several places, we will consider only (subobject-)finite objects, i.e., objects that have only finitely many subobjects.

Summary We first show that the concept of irreducible element from lattice theory can be used to identify the basic building blocks for objects in adhesive categories. More specifically, after defining the notion of irreducible object in an adhesive category, we prove that, as any element of a finite distributive lattice can be obtained as the join of its irreducibles, similarly, any (finite) object of an adhesive category can be obtained as the colimit of its irreducible subobjects. Additionally, such colimit turns out to be Van Kampen (Cockett and Guo, 2007; Heindel and Sobociński, 2009), i.e., roughly speaking, it is well-behaved w.r.t. pullbacks.

The representation theory for distributive lattices in (Birkhoff, 1933; Davey and Priestley, 2002) is then exploited in order to establish a correspondence between arrows in adhesive categories and lattice homomorphisms between the corresponding subobject lattices. We first show that any arrow $\varphi : A \to B$ in an adhesive category induces a lattice homomorphism from $Sub(B)$ to $Sub(A)$, which is essentially given by the inverse image functor (or pullback functor) φ^{-1}. While preservation of meets holds for any category with pullbacks, preservation of joins is strictly related to adhesivity.

Vice versa, not any lattice homomorphism of $\gamma : Sub(B) \to Sub(A)$ corresponds to some arrow $\varphi : A \to B$. We discuss two possible characterizations of classes of lattice homomorphism which ensure that they correspond to φ^{-1} for some arrow $\varphi : A \to B$. Such characterization relies on the fact that an object can be obtained as Van Kampen colimit of sets of subobjects which enjoy a suitable closure property (weak meet-closedness). The second characterization is based on the notion of cover and on the existence of cover-mono factorizations in adhesive categories, which leads to a characterization of functor $\exists_\varphi$, the left adjoint to the inverse image functor φ^{-1}.

Finally the special case of adhesive categories with a strict initial object, which are thus extensive (Carboni et al., 1993), is discussed, where one can obtain stronger results.

2. Background

2.1. Van Kampen Colimits and Adhesive Categories

Adhesive categories have been introduced in (Lack and Sobociński, 2005), as categories where pushouts along monomorphisms are so-called *Van Kampen squares*. In (Cockett and Guo, 2007; Heindel and Sobociński, 2009), the notion of Van Kampen square has been generalized to that of *Van Kampen colimit*. Since these notions will often recur in the rest of the paper, we start by introducing Van Kampen colimits, then we zoom on Van Kampen squares and finally we introduce adhesive categories.

We start by recalling the definition of a cartesian natural transformation, which will be fundamental for the definition of Van Kampen colimits.

Definition 1 (Cartesian natural transformation)**.** Let $F, G : \mathbb{C} \to \mathbb{D}$ be two functors. A *natural transformation* $\beta : F \Rightarrow G$ associates to every object x in $\mathbb{C}$ an arrow $\beta_x : F(x) \to G(x)$ in $\mathbb{D}$, such that, for every morphism $d : x \to y$ in $\mathbb{C}$, the following diagram commutes.

$$\begin{tikzcd}
F(x) \arrow[r, "F(d)"] \arrow[d, "\beta_x"'] & F(y) \arrow[d, "\beta_y"] \\
G(x) \arrow[r, "G(d)"'] & G(y)
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] y & & z \\ & x \arrow[ul, "g"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] D(y) & & D(z) \\ & D(x) \arrow[ul, "D(g)"] \arrow[ur, "D(f)"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & A & \\ D(y) \arrow[ur, "\varphi_y^A"] & & D(z) \arrow[ul, "\varphi_z^A"'] \\ & D(x) \arrow[ul, "D(g)"] \arrow[ur, "D(f)"'] \arrow[uu, "\varphi_x^A"] & \end{tikzcd}$$

$\mathbb{I}_{\mathsf{po}}$ $\qquad$ D $\qquad$ φ^A

Fig. 1. Pushout as an instance of colimit.

We say that β is a *cartesian* natural transformation if the above diagram is a pullback for every arrow $d : x \to y$ in $\mathbb{C}$.

Given two natural transformations $\beta : F \Rightarrow G$ and $\gamma : G \Rightarrow H$, we use $\beta;\gamma : F \Rightarrow H$ to denote their composition defined as $(\beta;\gamma)_x = \beta_x;\gamma_x$ for each object x of $\mathbb{C}$.

In the following, overloading the notation, given two categories $\mathbb{I}$ and $\mathbb{C}$ and an object A of $\mathbb{C}$, we will denote simply by A the constant functor $\Delta_{\mathbb{I}}(A) : \mathbb{I} \to \mathbb{C}$, which maps each object to A and each arrow to the identity id_A. Similarly, for an arrow $d : B \to A$ we will write simply $d : B \to A$ to denote the natural transformation $\delta : \Delta_{\mathbb{I}}(B) \Rightarrow \Delta_{\mathbb{I}}(A)$ with $\delta_x = d$ for all x in $\mathbb{I}$. The category $\mathbb{I}$ will always be clear from the context.

Definition 2 (Colimit)**.** Let $D : \mathbb{I} \to \mathbb{C}$ be a *diagram*, i.e., a functor from a given *scheme category* $\mathbb{I}$ into $\mathbb{C}$. A *cocone* for D is a natural transformation $\varphi^A : D \Rightarrow A$ to a constant functor $A : \mathbb{I} \to \mathbb{C}$. A *colimit* for D is a cocone φ^A such that for every other cocone φ^B there exists a unique arrow $\psi : A \to B$ with $\varphi^A;\psi = \varphi^B$.

It is instructive, for later use, to spell out how this definition subsumes the one of pushout. Consider the scheme category $\mathbb{I}_{\mathsf{po}}$ and the diagram $D : \mathbb{I}_{\mathsf{po}} \to \mathbb{C}$, depicted in Fig. 1. A cocone is a natural transformation $\varphi^A : D \Rightarrow A$ for the constant functor $A : \mathbb{I}_{\mathsf{po}} \to \mathbb{C}$. Hence it consists of an object A and three arrows φ_x^A, φ_y^A, φ_z^A of $\mathbb{C}$ such that $\varphi_x^A; id_A = D(f);\varphi_z^A$ and $\varphi_x^A; id_A = D(g);\varphi_y^A$, i.e., such that the diagram in the right part of Fig. 1 commutes. Now the cocone φ^A is a colimit if for every other cocone φ^B there exists a unique arrow ψ such that $\varphi_x^A;\psi = \varphi_x^B$, $\varphi_y^A;\psi = \varphi_y^B$ and $\varphi_z^A;\psi = \varphi_z^B$. But the first equality can be omitted, as it is implied by the other two: therefore the definition of colimit for a diagram over $\mathbb{I}_{\mathsf{po}}$ coincides with the usual definition of pushout.

We now define the notion of a Van Kampen colimit, first introduced in (Cockett and Guo, 2007; Heindel and Sobociński, 2009).

Definition 3 (Van Kampen colimit)**.** Let $D : \mathbb{I} \to \mathbb{C}$ be a diagram with colimit $\varphi^A : D \Rightarrow A$. We say that φ^A is a Van Kampen colimit (VK-colimit) if—given another diagram $D' : \mathbb{I} \to \mathbb{C}$, a cocone $\varphi^B : D' \Rightarrow B$, a cartesian natural transformation $\beta : D' \Rightarrow D$ and an arrow $d : B \to A$ in $\mathbb{C}$, such that $\varphi^B; d = \beta;\varphi^A$ (see the diagram on the left)—it holds that φ^B is a colimit if and only if the square on the right below is a pullback for every object x of $\mathbb{I}$.

$$\begin{tikzcd} D' \arrow[r, Rightarrow, "\varphi^B"] \arrow[d, Rightarrow, "\beta"'] & B \arrow[d, "d"] \\ D \arrow[r, Rightarrow, "\varphi^A"] & A \end{tikzcd} \qquad \begin{tikzcd} D'(x) \arrow[r, "\varphi_x^B"] \arrow[d, "\beta_x"'] & B \arrow[d, "d"] \\ D(x) \arrow[r, "\varphi_x^A"] & A \end{tikzcd}$$

The above definition coincides, when $\mathbb{I}$ is the scheme category $\mathbb{I}_{\mathsf{po}}$ of Fig. 1, with the one of *Van Kampen squares*, that are defined as follows in (Lack and Sobociński, 2005):

A Van Kampen square is a pushout $\varphi^A : D \Rightarrow A$ as in Fig. 1, satisfying the following condition: given a commuting cube (as shown on the right) with $\varphi^A : D \Rightarrow A$ as bottom face and such that the back and the left faces are pullbacks—it holds that the top face is a pushout if and only if the front and the right face are pullbacks.

$$\begin{tikzcd}
D'(x) \arrow[rr, "D'(f)"] \arrow[dr, "D'(g)"] \arrow[dd, "\beta_x"'] & & D'(z) \arrow[dr, "\varphi_z^B"] \arrow[dd, "\beta_z"] & \\
& D'(y) \arrow[rr, "\varphi_y^B"] \arrow[dd, "\beta_y"] & & B \arrow[dd, "d"] \\
D(x) \arrow[rr, "D(f)"] \arrow[dr, "D(g)"'] & & D(z) \arrow[dr, "\varphi_z^A"] & \\
& D(y) \arrow[rr, "\varphi_y^A"] & & A
\end{tikzcd}$$

In fact, the bottom face of the cube is the colimit φ^A, and the top face is the cocone φ^B. The left and the back faces are induced by the natural transformation β, hence they commute and they are pullbacks since β is cartesian. The front and the right faces are induced by the arrow d. The condition $\varphi^B; d = \beta; \varphi^A$ (in Definition 3) means that the front and the right faces of the cube commute (i.e., $\varphi_y^B; d = \beta_y; \varphi_y^A$ and $\varphi_z^B; d = \beta_z; \varphi_z^A$).

Van Kampen squares will be called *Van Kampen pushouts* in the following. We are now ready to introduce adhesive categories (Lack and Sobociński, 2005).

Definition 4 (Adhesive category)**.** A category $\mathbb{C}$ is called *adhesive* if
(1) $\mathbb{C}$ has pullbacks;
(2) $\mathbb{C}$ has pushouts along monos;
(3) pushouts along monos are VK-pushouts.

2.2. The Representation Theorem for Finite Distributive Lattices

In this section we review the duality theory for finite distributive lattices and finite partially ordered sets, following mainly the presentation in (Davey and Priestley, 2002).

We first recall some basic definitions. A *partially ordered set* (or poset) is a set P equipped with a reflexive, transitive and antisymmetric relation $\sqsubseteq$. The maximum and the minimum in P, when they exist, are called *top* and *bottom* and denoted by $\top_P$ and $\bot_P$, respectively (the subscript will be omitted when clear from the context). A poset with a bottom element will be called *pointed poset*.

Given two posets $(P_i, \sqsubseteq_i)$, $i \in \{1, 2\}$, a mapping $f : P_1 \to P_2$ is called *monotone* if $a \sqsubseteq_1 b$ for $a, b \in P_1$ implies $f(a) \sqsubseteq_2 f(b)$. If P_1 and P_2 are pointed, the mapping $f : P_1 \to P_2$ is called *strict* if it preserves bottom, i.e., if $f(\bot_{P_1}) = \bot_{P_2}$.

In this paper, we will focus on *finite pointed posets*, i.e., pointed posets where the underlying set is finite. Finite pointed posets and strict monotone maps form the category $\mathbb{FPOS}$.

A *lattice* is a poset $(L, \sqsubseteq)$ where any pair of elements $a, b \in L$ admits meet and join, denoted $a \sqcap b$ and $a \sqcup b$ respectively. The lattice is called *distributive* if for any $a, b, c \in L$, $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$ (or, equivalently $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$).

Note that a non-empty finite lattice always has a bottom element $\bot$ and a top element $\top$ (which are the meet and the join, respectively, of all elements).

Given two lattices $(L_i, \sqsubseteq_i)$, $i \in \{1, 2\}$, a mapping $\alpha : L_1 \to L_2$ is called *lattice homomorphism* if it preserves binary (hence, finite non-empty) meets and joins. The mapping α is called $\{\bot, \top\}$-homomorphism if in addition it preserves $\bot$ and $\top$. Similarly,

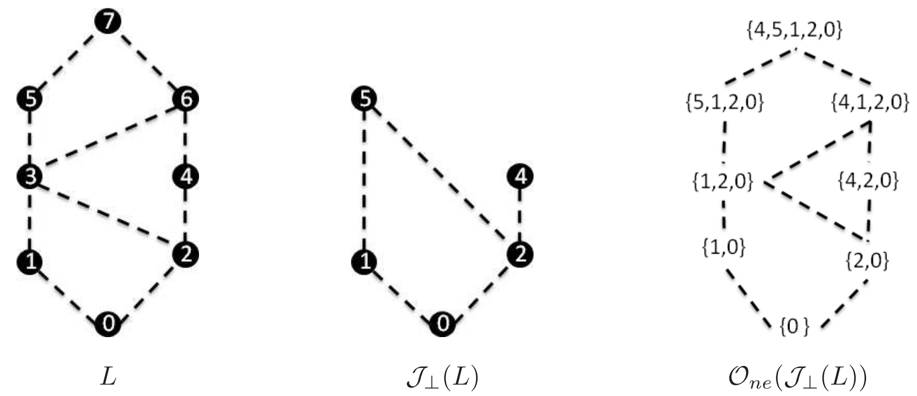


Fig. 2. A distributive lattice, the corresponding poset of weak irreducibles and the isomorphic lattice of subsets.

it is called a $\top$-homomorphism ($\perp$-homomorphism) if it preserves $\top$ ($\perp$). Note that lattice homomorphisms are always monotone. Non-empty finite distributive lattices and $\top$-homomorphisms form the category $\mathbb{FDL}$.

The duality between finite distributive lattices and finite partially ordered sets essentially states that $\mathbb{FPOS}$ and $\mathbb{FDL}$ are equivalent via a contravariant functor. The original theory slightly differs from ours in the fact that $\mathbb{FDL}$ has $\{\perp, \top\}$-homomorphisms, instead of $\top$-homomorphism as arrows and, dually, in $\mathbb{FPOS}$ objects are general, possibly non-pointed, posets and arrows are monotone mappings. We need to consider these variations since lattice homomorphisms which arise naturally as pullback functors in adhesive categories do not necessarily preserve the bottom element (but they always preserve top). The situation is different if we assume that the adhesive category under consideration has a strict initial object (for the treatment of this special case see Section 6).

A pivotal definition is the following, which identifies those elements of a lattice which cannot be decomposed as the join of other elements and thus, intuitively, can be seen as the basic building blocks of the lattice.

Definition 5 (Irreducible)**.** A lattice element $a \in L$ is said to be *weak (join)-irreducible* whenever $a = b \sqcup c$ for some $b, c \in L$ implies $a = b$ or $a = c$. An *irreducible* is any weak irreducible $a \in L$ different from the bottom element. The (pointed) poset of weak irreducibles of L with the ordering induced by L is denoted by $\mathcal{J}_\perp(L)$, while $\mathcal{J}(L)$ denotes the poset of irreducibles.

A lattice L and the poset $\mathcal{J}_\perp(L)$ of its weak irreducibles, represented as Hasse diagrams, can be found in Fig. 2 (left and center). Note, in particular, that the element 3 is not a weak irreducible since $3 = 1 \sqcup 2$.

For a finite lattice it can be easily seen that an element a is weak irreducible if and only if it has at most one direct predecessor with respect to $\sqsubset$ (where $a \sqsubset b \iff a \sqsubseteq b \ \wedge\ a \neq b$). Furthermore in a finite lattice every element b can be written as the join of weak irreducibles, namely $b = \bigsqcup\{a \in \mathcal{J}_\perp(L) \mid a \sqsubseteq b\}$. Note however that this representation is not unique since some irreducibles could be subsumed by others.

This leads us directly to (a slight variation of) Birkhoff's representation theorem for finite distributive lattices (Birkhoff, 1933). We first need some notation: let $(P, \sqsubseteq)$ be any poset. Then we denote by $\mathcal{O}_{ne}(P)$ the lattice consisting of the non-empty downward-closed subsets of P, ordered by subset inclusion $\subseteq$.

Theorem 1 (Birkhoff's representation theorem)**.** Let $(L, \sqsubseteq)$ be a finite distributive lattice and let $\mathcal{J}_\perp(L)$ be the poset of its weak irreducibles. Then the mapping $\alpha_L : L \to \mathcal{O}_{ne}(\mathcal{J}_\perp(L))$ given by

$$\alpha_L(a) = \{x \in \mathcal{J}_\perp(L) \mid x \sqsubseteq a\}$$

is a lattice isomorphism.

This means that every finite distributive lattice is isomorphic to a lattice of sets: the isomorphism maps every element a to the maximal set of weak irreducibles which generates a. An example of this construction can be found in Fig. 2 (right).

Analogously, any finite pointed poset P is isomorphic to $\mathcal{J}_\perp(\mathcal{O}_{ne}(P))$. The isomorphism $\iota_P : P \to \mathcal{J}_\perp(\mathcal{O}_{ne}(P))$ maps every $x \in P$ to $\{y \mid y \sqsubseteq x\}$ (which is a weak irreducible of $\mathcal{O}_{ne}(P)$). This defines a close relation between the objects of $\mathbb{FPOS}$ and $\mathbb{FDL}$. The duality theorem extends this relation also to their arrows.

Theorem 2 (Duality)**.** Given two finite distributive lattices L and K, a $\top$-homomorphism $\gamma : K \to L$ induces a strict monotone map $f_\gamma : \mathcal{J}_\perp(L) \to \mathcal{J}_\perp(K)$ defined as

$$f_\gamma(y) = \min\{x \in \mathcal{J}_\perp(K) \mid y \sqsubseteq \gamma(x)\}$$

for all $y \in \mathcal{J}_\perp(L)$.

Conversely, given two finite pointed posets P and Q, a strict monotone map $f : P \to Q$ induces a $\top$-homomorphism $\gamma_f : \mathcal{O}_{ne}(Q) \to \mathcal{O}_{ne}(P)$, defined as

$$\gamma_f(Y) = \{x \in P \mid f(x) \in Y\}$$

for all $Y \in \mathcal{O}_{ne}(Q)$.

Furthermore, for any $\top$-homomorphism $\gamma : K \to L$ it holds that $\gamma_{f_\gamma} = \alpha_K^{-1}; \gamma; \alpha_L$, where α_K and α_L are the isomorphisms in Theorem 1. Conversely, for any monotone map $f : P \to Q$ it holds that $f_{\gamma_f} = \iota_P^{-1}; f; \iota_Q$.

This leads to an equivalence between the categories $\mathbb{FPOS}$ and $\mathbb{FDL}$, via a contravariant functor. For an example consider the mappings γ and f_γ in Fig. 3. (The mapping α is explained later.)

2.3. Galois Connections

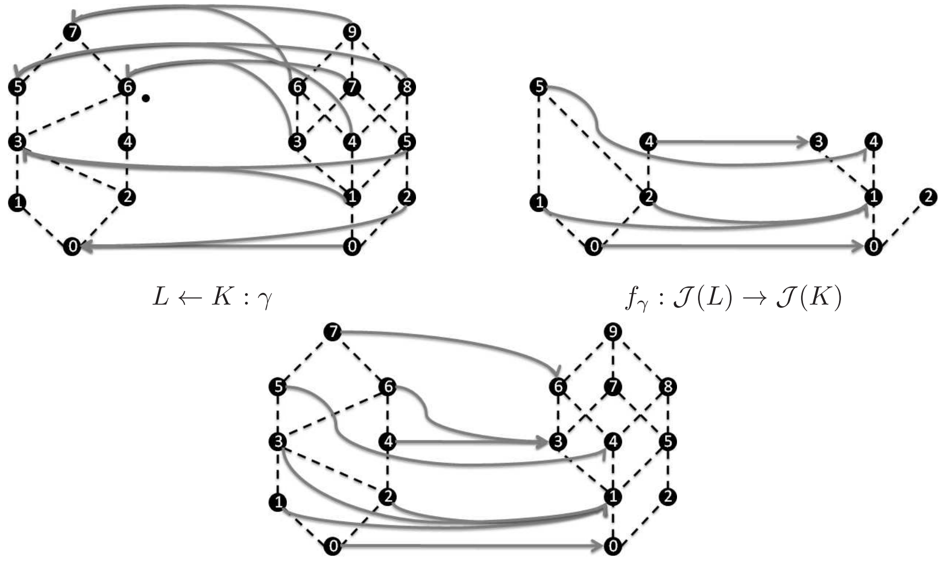
We now review a closely related notion of duality: Galois connections on lattices.

Definition 6 (Galois connection)**.** Let L, K be two lattices and let $\alpha : L \to K$, $\gamma : K \to L$ be monotone maps. Then $\langle \alpha, \gamma \rangle$ is a *Galois connection* if
- for every $\ell \in L$ we have $\gamma(\alpha(\ell)) \sqsupseteq \ell$;
- for every $k \in K$ we have $\alpha(\gamma(k)) \sqsubseteq k$.

In this case α and γ are called the left and right adjoint, respectively.

If we consider the lattices themselves as categories, Galois connections exactly correspond to adjoint pairs. The following properties will be relevant.

Lemma 1. (1) Every monotone map has at most one right (resp. left) adjoint.



$L \leftarrow K : \gamma$ $\qquad$ $f_\gamma : \mathcal{J}(L) \to \mathcal{J}(K)$

$\alpha : L \to K$

Fig. 3. A $\top$-homomorphism γ, the dual map f_γ on irreducibles and its left adjoint α.

(2) Left adjoints preserve arbitrary joins (and thus $\perp$) and right adjoints preserve arbitrary meets (and thus $\top$).
(3) Every mapping $\alpha : L \to K$ that preserves arbitrary joins has a right adjoint $\gamma : K \to L$ defined as $\gamma(k) = \max\{\ell \in L \mid \alpha(\ell) \sqsubseteq k\}$. Dually, every mapping $\gamma : K \to L$ that preserves arbitrary meets has a left adjoint $\alpha : L \to K$, defined as $\alpha(\ell) = \min\{k \in K \mid \ell \sqsubseteq \gamma(k)\}$.

The use of max (resp. min) in the statement is justified by the observation that the indicated set has a maximal (resp. minimal) element. An example of a Galois connection consisting of mappings α and γ is given in Fig. 3.

When dealing with finite lattices, every $\top$-lattice homomorphism γ has a left adjoint α (preservation of top is needed as it corresponds to preservation of the meet of the empty set), and similarly every $\perp$-lattice homomorphism α has a right adjoint γ. The left adjoint of a lattice homomorphism enjoys interesting properties with respect to irreducibles, establishing a clear link with Theorem 2.

Proposition 1. Let K, L be finite lattices, let $\gamma : K \to L$ be a $\top$-lattice homomorphism and let $\alpha : L \to K$ be its left adjoint. Then α preserves weak irreducibles, i.e., for any $l \in \mathcal{J}_\perp(L)$ we have that $\alpha(l) \in \mathcal{J}_\perp(K)$.

Hence the restriction $\alpha_{|\mathcal{J}_\perp(L)} : \mathcal{J}_\perp(L) \to \mathcal{J}_\perp(K)$ of α to the weak irreducibles can be defined as

$$\alpha_{|\mathcal{J}_\perp(L)}(l) = \min\{k \in \mathcal{J}_\perp(K) \mid l \sqsubseteq \gamma(k)\},$$

i.e., $\alpha_{|\mathcal{J}_\perp(L)} = f_\gamma$.

Proof. For the proof we use an equivalent alternative definition of weak irreducibles in distributive lattices: b is a weak irreducible whenever $b \sqsubseteq b_1 \sqcup b_2$ implies that $b \sqsubseteq b_1$ or $b \sqsubseteq b_2$, for all b_1, b_2.

Assume that l is an irreducible and $\alpha(l) \sqsubseteq k_1 \sqcup k_2$. Then we have

$$l \sqsubseteq \gamma(\alpha(l)) \sqsubseteq \gamma(k_1 \sqcup k_2) = \gamma(k_1) \sqcup \gamma(k_2).$$

where the last equality is motivated by the fact that γ is a lattice homomorphism and thus it preserves joins. Since l is an irreducible, we deduce that $l \sqsubseteq \gamma(k_1)$ or $l \sqsubseteq \gamma(k_2)$. By applying α on both sides we obtain

$$\alpha(l) \sqsubseteq \alpha(\gamma(k_1)) \sqsubseteq k_1 \quad \text{or} \quad \alpha(l) \sqsubseteq \alpha(\gamma(k_2)) \sqsubseteq k_2$$

which proves that $\alpha(l)$ is irreducible. $\square$

Since any element can be expressed as the join of weak irreducibles, $\alpha_{|\mathcal{J}_\perp(L)}$ determines the left adjoint $\alpha : L \to K$ as $\alpha(x) = \bigsqcup\{\alpha_{|\mathcal{J}_\perp(L)}(l) \mid l \in \mathcal{J}_\perp(L) \ \wedge \ l \sqsubseteq x\}$.

3. Subobject Lattices and Irreducibles in Adhesive Categories

As a first step, in this section, we study the subobject lattices arising in adhesive categories. More specifically, we show that a notion of irreducible object can be defined, which is independent of the specific subobject lattice. Moreover we show that irreducible objects can be seen as the building blocks for (finite) objects of adhesive categories in the sense that any object can be obtained as a colimit of irreducibles and such a colimit is VK.

Hereafter, $\mathbb{C}$ denotes an adhesive category, and $\rightarrowtail$ and $\xrightarrow{\sim}$ denote monos and isos respectively.

3.1. Irreducibles in Adhesive Categories

Definition 7 (Subobject)**.** Let A be an object in $\mathbb{C}$. Two monos $b : B \rightarrowtail A$, $c : C \rightarrowtail A$ are called isomorphic if there is an iso $\psi : B \xrightarrow{\sim} C$ such that $\psi; c = b$. A *subobject* of A is an isomorphism class of monos into A. It is denoted $[b : B \rightarrowtail A]$ or simply $[b]$, where $b : B \rightarrowtail A$ is any representative.

It is known that the subobjects of an object in an adhesive category form a *distributive lattice*, where the order $\sqsubseteq$ is given by $[b : B \rightarrowtail A] \sqsubseteq [c : C \rightarrowtail A]$ whenever there is a mono $\varphi : B \rightarrowtail C$ with $\varphi; c = b$ (note that the mono φ is unique, if it exists). The meet of two subobjects $[b : B \rightarrowtail A], [c : C \rightarrowtail A]$ is realized by taking their pullback, whereas the join can be obtained by taking a pushout over their meet (Lack and Sobociński, 2005).

$$\begin{tikzcd} & B \arrow[dr, tail, "b"] & \\ B \sqcap C \arrow[ur, tail] \arrow[dr, tail] \arrow[r, phantom, "\text{PB}"] & {} & A \\ & C \arrow[ur, tail, "c"'] & \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dr, tail] \arrow[drr, tail, "b"] & & \\ B \sqcap C \arrow[ur, tail] \arrow[dr, tail] \arrow[r, phantom, "\text{PO}"] & {} & B \sqcup C \arrow[r, tail, dotted] & A \\ & C \arrow[ur, tail] \arrow[urr, tail, "c"'] & & \end{tikzcd}$$

Furthermore the top element is represented by $[id_A : A \xrightarrow{\sim} A]$. In the sequel, given an object A in an adhesive category, we will write $Sub(A)$ to denote the subobject lattice of A.

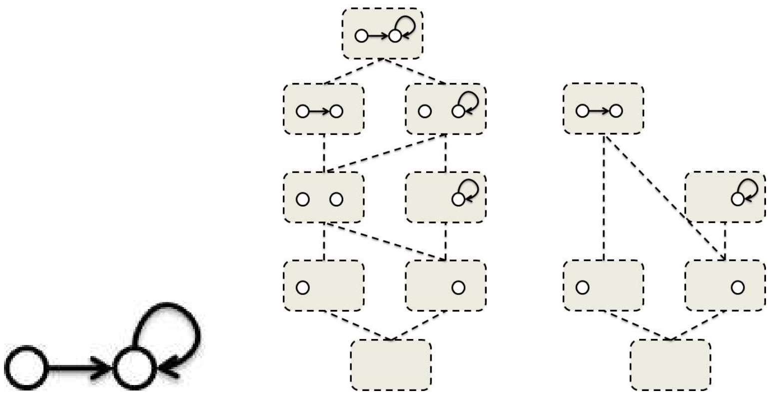


Fig. 4. A graph, its subobject lattice and its weak irreducibles.

Definition 8 (Forgetful functor from a subobject lattice)**.** Viewing the subobject lattice as a category, we denote by $|\cdot|_A : Sub(A) \to \mathbb{C}$ a functor that maps any $[b : B \rightarrowtail A]$ to the domain B of a chosen representative and, similarly, any arrow $[b : B \rightarrowtail A] \sqsubseteq [b' : B' \rightarrowtail A]$ to the corresponding mono $|[b]|_A \rightarrowtail |[b']|_A$.

In the sequel, we will often write $|\cdot|$ instead of $|\cdot|_A$ as the object A will be clear from the context.

In the next example, which is then developed throughout the paper, we will consider as adhesive category the category of directed graphs and graph morphisms (Lack and Sobociński, 2005; Ehrig et al., 2006). Fig. 4 shows a graph A and the corresponding lattice of subobjects $Sub(A)$. Here each element of $Sub(A)$, i.e., each $[b]$ such that $b : B \rightarrowtail A$, is represented just by the source graph B. The monic arrow $b : B \rightarrowtail A$ is implicitly expressed by the position of nodes and edges. For example, the two graphs in $Sub(A)$ consisting of one node and no edges are isomorphic, but they represent different subobjects of A. Indeed, the leftmost subgraph implicitly describes a monic arrow that maps the unique node into the leftmost node of A, while the rightmost subobject maps the node into the rightmost node of A.

The notion of (weak) irreducible in subobject lattices will play a fundamental role in the rest of the paper. The following proposition shows that, in adhesive category, this notion is "global", i.e., whenever an object is irreducible in some subobject lattice then it is irreducible in any subobject lattice of the category.

Proposition 2. Let A, B, I be objects of $\mathbb{C}$ and let $i_A : I \rightarrowtail A$, $i_B : I \rightarrowtail B$ be two monos. Then $[i_A]$ is a weak irreducible if and only if $[i_B]$ is a weak irreducible.

Proof. Let us assume that $[i_A]$ is a weak irreducible.

Let $[i_B]$ be the join of two subobjects $[c : C \rightarrowtail B], [d : D \rightarrowtail B]$ of B. That is, there exists a pushout of the following form with all arrows mono and $c'; i_B = c$, $d'; i_B = d$.

Furthermore $C \sqcap D$ is the pullback of c, d.

$$\begin{tikzcd}
C \sqcap D \arrow[r, tail] \arrow[d, tail] & C \arrow[d, tail, "c'"] \arrow[ddr, tail, bend left, "c"] & \\
D \arrow[r, tail, "d'"'] \arrow[drr, tail, bend right, "d"] & I \arrow[dr, tail, "i_B"] & \\
& & B
\end{tikzcd}$$

If we post-compose this pullback with $i_A : I \rightarrowtail A$ (instead of $i_B : I \rightarrowtail B$), we obtain that $[i_A]$ is the join of two subobjects $[d'; i_A]$, $[c'; i_A]$. Since $[i_A]$ is irreducible we have that $[i_A] = [c'; i_A]$ or $[i_A] = [d'; i_A]$. Let us assume without loss of generality that the former is the case. Then, since i_A is a mono and can be eliminated, c' is an iso and thus $[i_B] = [c'; i_B] = [c]$. $\square$

As an example, in the category of directed (unlabeled) graphs, the irreducibles are the single node, the single edge and the loop (see Fig. 4, right). If graphs are labeled, the irreducibles are essentially the same, but we must take a copy of the single node, edge and loop for any distinct label. If we consider graphs with higher-order edges where an edge of order k may connect edges of order $\ell < k$, then all types of edges, with their connections fused in arbitrary ways, are irreducibles.

3.2. Finite Objects as VK-colimits of Irreducibles

In adhesive categories, the union of two subobjects is realised as a pushout (along their intersection). Such a pushout is particularly well-behaved, i.e., it is a VK-square. In this section we show that, more generally, we can relate the notion of join in lattices with the notion of colimit (that also represents some form of union). More specifically we will show that, like lattice-theoretically an element of a finite lattice can be obtained as the join of its irreducibles, similarly, any finite object of an adhesive category can be obtained as the colimit of the diagram consisting of its irreducibles, and such colimit is VK.

This is stated by Corollary 1 at the end of this section. In order to prove it we employ the following proof strategy: we start with a diagram that has a "largest" object (for instance the diagram consisting of all subobjects of a given object) and show that it is a VK-colimit. Then, step by step, we remove all objects which are not weak irreducibles.

Lemma 2 (Partial-Order diagrams with $\top$ are VK-colimits)**.** Let $\mathbb{I}$ be a partial order with a largest element $\top$ and let $D : \mathbb{I} \to \mathbb{C}$ be a diagram. We set $T = D(\top)$ and define the cocone $\varphi^T : D \Rightarrow T$ by setting $\varphi^T_x = D(x \sqsubseteq \top)$ for every object x of $\mathbb{I}$.

Then φ^T is a VK-colimit.

Proof. We first show that φ^T is well-defined, i.e., it is indeed a natural transformation. Let x, y with $x \sqsubseteq y$ be two objects of $\mathbb{I}$. Then it holds that $D(x \sqsubseteq y); \varphi^T_y = D(x \sqsubseteq y); D(y \sqsubseteq \top) = D(x \sqsubseteq \top) = D(x \sqsubseteq \top); id_T = \varphi^T_x; id_T = \varphi^T_x$.

In the next step we show that φ^T is a colimit. Take any other natural transformation $\varphi^A : D \Rightarrow A$. Then define $\psi = \varphi^A_\top : T \to A$. Commutativity follows from $\varphi^T_x; \psi = D(x \sqsubseteq \top); \varphi^A_\top = \varphi^A_x; id_T = \varphi^A_x$. In order to show that the arrow ψ is unique, take any other arrow $\psi' : T \to A$ with $\varphi_x^T; \psi' = \varphi_x^A$, for all x. Then we have that $\psi = \varphi_\top^A = \varphi_\top^T; \psi' = id_T; \psi' = \psi'$.

Finally we prove that φ^T is VK: let $D' : \mathbb{I} \to \mathbb{C}$ be another diagram with cocone $\varphi^{T'} : D' \Rightarrow T'$ and let $\beta : D' \Rightarrow D$ where β is cartesian and $d : T' \Rightarrow T$ with $\varphi^B; d = \beta; \varphi^A$. We show both implications in the definition of VK-pushout. First, let the squares consisting of morphisms $\varphi_x^{T'}, \beta_x, \varphi_x^T, d$ be pullbacks. Observe that for $x = \top$ we have $\varphi_\top^T = id_T$ and hence $\varphi_\top^{T'}$ must be an iso. It now holds that $\varphi_x^{T'} = \varphi_x^{T'}; id_{T'} = D'(x \sqsubseteq \top); \varphi_\top^{T'}$ and hence—by the first part of this proof—it is a colimit. In order to show the other direction, assume that $\varphi^{T'}$ is a colimit and, without loss of generality, assume $T' = D'(\top)$ and $\varphi_x^{T'} = D'(x \sqsubseteq \top)$. Since β is a cartesian natural transformation we have that for each arrow $x \sqsubseteq y$ the following squares are pullbacks:

$$\begin{tikzcd} D'(x) \arrow[r, "D'(x\sqsubseteq y)"] \arrow[d, "\beta_x"'] & D'(y) \arrow[d, "\beta_y"] \\ D(x) \arrow[r, "D(x\sqsubseteq y)"] & D(y) \end{tikzcd}$$

If we set $y = \top$ we obtain $D(y) = D(\top) = T$, $D'(y) = D'(\top) = T'$, $D(x \sqsubseteq y) = D(x \sqsubseteq \top) = \varphi_x^T$ and $D'(x \sqsubseteq y) = D'(x \sqsubseteq \top) = \varphi_x^{T'}$, thus obtaining the required pullback squares. $\square$

Note that this proof holds for any category $\mathbb{C}$ (not necessarily adhesive). Now, in order to prove our main result, we will show that if we remove objects that are not weak-irreducible from a VK-colimit, then the result is still a VK-colimit. First, we have to formally define what we mean by *removing objects* from a diagram.

Definition 9 (Removing objects from diagrams)**.** Let $\mathbb{I}$ be a scheme category and let x be an arbitrary object of $\mathbb{I}$. By $\mathbb{I} - x$ we denote the scheme that is obtained by removing x—and all arrows which have x as source or target—from $\mathbb{I}$. This gives rise to an obvious embedding functor $E_{\mathbb{I}}^x : (\mathbb{I} - x) \to \mathbb{I}$.

For a diagram $D : \mathbb{I} \to \mathbb{C}$ we denote by D^x the diagram $D^x = E_{\mathbb{I}}^x; D : (\mathbb{I} - x) \to \mathbb{C}$.

For a natural transformation $\varphi : D \Rightarrow D'$ let $\varphi^x : D^x \Rightarrow D'^x$ be the natural transformation with $\varphi_y^x = \varphi_y$ for every object y of $(\mathbb{I} - x)$.

Lemma 3 (Removing unions from VK-colimits)**.** Let $D : \mathbb{I} \to \mathbb{C}$ be a diagram, where $\mathbb{I}$ is a partial order, and let $\varphi^A : D \Rightarrow A$ be a VK-colimit.

Let $x \in \mathbb{I}$ be an object such that

(i) the full subcategory of $\mathbb{I}$ with objects $\{y \in \mathbb{I} \mid y \sqsubseteq x\}$ is $\mathit{Sub}(D(x))$;

(ii) the diagram D chooses representatives for these (sub-)objects, i.e. the equation $y = [D(y \sqsubseteq x)]$ holds for all arrows $y \sqsubseteq x$ into x;

(iii) x is not a weak irreducible, i.e. $x = u \sqcup v$ for some $u, v \in \mathbb{I}$ satisfying $u \neq x \neq v$.

Then the restricted cocone $(\varphi^A)^x : D^x \Rightarrow A$ is still a VK-colimit.

Proof. Let $z = u \sqcap v$ be the meet of u and v. Hence the following diagram is a pushout and a pullback.

$$\begin{tikzcd} D(z) \arrow[r, tail, "D(z\sqsubseteq u)"] \arrow[d, tail, "D(z\sqsubseteq v)"'] & D(u) \arrow[d, tail, "D(u\sqsubseteq x)"] \\ D(v) \arrow[r, tail, "D(v\sqsubseteq x)"] & D(x) \end{tikzcd}$$

Using the fact that mediating morphisms from pushouts exist and are unique, one can check that $(\varphi^A)^x$ is a colimit in a straightforward manner. The crucial proof obligation that remains is to show that $(\varphi^A)^x$ is actually VK.

Let $D'^x : (\mathbb{I} - x) \to \mathbb{C}$ be another diagram with cocone $(\varphi^B)^x : D'^x \Rightarrow B$ and let $\beta^x : D'^x \Rightarrow D^x$ be a cartesian natural transformation which satisfies the equation $(\varphi^B)^x; d = \beta^x; (\varphi^A)^x$.

We extend D'^x to D' by constructing the following pushout.

$$\begin{tikzcd}
D'(z) \arrow[r, tail, "D'(z \sqsubseteq u)"] \arrow[d, tail, "D'(z \sqsubseteq v)"'] & D'(u) \arrow[d, tail, "m'"] \\
D'(v) \arrow[r, tail, "n'"] & X'
\end{tikzcd}$$

Now let $D' : \mathbb{I} \to \mathbb{C}$ be the unique extension of D'^x which satisfies the two equations $D'(v \sqsubseteq x) = n'$ and $D'(u \sqsubseteq x) = m'$. (That such an extension exists and is unique follows again from the fact that mediating morphisms from pushouts exist and are unique.) Moreover, there are unique extensions of $(\varphi^A)^x$ and $(\varphi^B)^x$ to cocones $\varphi^A : D \Rightarrow A$ and $\varphi^B : D' \Rightarrow B$, respectively.

Finally, we extend β^x to β; there is only one choice for β_x, which must be the unique mediating morphism which makes the following diagram commute.

$$\begin{tikzcd}
D'(z) \arrow[rr, tail] \arrow[dr, tail] \arrow[dd, "\beta_z"'] & & D'(u) \arrow[dr, tail] \arrow[dd, "\beta_u"' near end] & \\
& D'(v) \arrow[rr, tail] \arrow[dd, "\beta_v"' near start] & & D'(x) \arrow[dd, dashed, "\beta_x"] \\
D(z) \arrow[rr, tail] \arrow[dr, tail] & & D(u) \arrow[dr, tail] & \\
& D(v) \arrow[rr, tail] & & D(x)
\end{tikzcd}$$

Since we work in an adhesive category, the resulting natural transformation β can be proved to be cartesian. First, the following squares below are pullbacks – this is a direct consequence of the VK-square property.

$$\begin{tikzcd}
D'(v) \arrow[r, tail, "D'(v \sqsubseteq x)"] \arrow[d, "\beta_v"'] & D'(x) \arrow[d, "\beta_x"] \\
D(v) \arrow[r, tail, "D(v \sqsubseteq x)"] & D(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
D'(u) \arrow[r, tail, "D'(u \sqsubseteq x)"] \arrow[d, "\beta_u"'] & D'(x) \arrow[d, "\beta_x"] \\
D(u) \arrow[r, tail, "D(u \sqsubseteq x)"] & D(x)
\end{tikzcd}$$

It remains to check that also the remaining new naturality squares are pullbacks. Let $x \sqsubseteq y$ be an arrow in $\mathbb{I}$ from the object x. Then we have the following situation.

$$\begin{tikzcd}
D'(z) \arrow[rr, tail] \arrow[dr, tail] \arrow[dd] & & D'(u) \arrow[dr, tail] \arrow[dd] \arrow[drr, bend left] & & \\
& D'(v) \arrow[rr, tail] \arrow[dd] \arrow[rrr, bend right] & & D'(x) \arrow[r] \arrow[dd] & D'(y) \arrow[dd] \\
D(z) \arrow[rr, tail] \arrow[dr, tail] & & D(u) \arrow[dr, tail] \arrow[drr, bend left] & & \\
& D(v) \arrow[rr, tail] \arrow[rrr, bend right] & & D(x) \arrow[r] & D(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
D'(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{PB?}"] & D'(y) \arrow[d] \\
D(x) \arrow[r] & D(y)
\end{tikzcd}$$

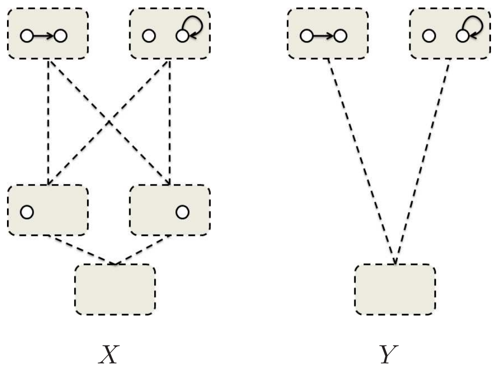


Fig. 5. X is a weakly $\sqcap$-closed subset of $Sub(A)$ (in Fig. 4), while Y is not.

All vertical squares are known to be pullbacks, except for the rightmost square in the display above. However, reusing the proof idea of Proposition 4.4 of (Heindel, 2009), also this square can be shown to be a pullback. For arrows $y \sqsubseteq x$ in $\mathbb{I}$, the relevant naturality square can similarly be shown to be a pullback, this time using the pushouts based on the $\mathbb{I}$-objects $z \sqcap y, u \sqcap y, v \sqcap y, y$.

We also need to show that $\varphi^B; d = \beta; \varphi^A$, which can be derived as follows:

$$D'(u \sqsubseteq x); \varphi^B_x; d = \varphi^B_u; d = \beta_u; \varphi^A_u = \beta_u; D(u \sqsubseteq x); \varphi^A_x = D'(u \sqsubseteq x); \beta_x; \varphi^A_x$$

and $D'(v \sqsubseteq x); \varphi^B_x; d = D'(v \sqsubseteq x); \beta_x; \varphi^A_x$ is obtained in the same way. Then the uniqueness of mediating morphisms implies the desired $\varphi^B_x; d = \beta_x; \varphi^A_x$.

Finally, we can show both implications of the VK-property: first, let the squares consisting of morphisms $(\varphi^B)^x_y, \beta^x_y, (\varphi^A)^x_y, d$ be pullbacks. This means that all squares of the form $\varphi^B_y, \beta_y, \varphi^A_y, d$ for $y \neq x$ are pullbacks by assumption and by the above argument, all other squares must also be pullbacks. Hence, since φ^A is a VK-colimit, φ^B must be a colimit and thus $(\varphi^B)^x$ is a colimit for D'^x. For the other direction let $(\varphi^B)^x$ be a colimit. This means that φ^B is also a colimit and hence the squares $\varphi^B_y, \beta_y, \varphi^A_y, d$ are pullbacks and – a fortiori – the same is true for the squares $(\varphi^B)^x_y, \beta^x_y, (\varphi^A)^x_y, d$ for all $y \in (\mathbb{I} - x)$. $\square$

The result stating that any finite object of an adhesive category can be obtained as the colimit of its weak irreducibles will be obtained as a corollary of a more general result. For this, we need the notion of weak $\sqcap$-closure.

Definition 10 (Weakly $\sqcap$-closed)**.** Let $(D, \sqsubseteq)$ be a lattice. We say that $X \subseteq D$ is *weakly* $\sqcap$-*closed* if for all $x, y \in X$ there exists $Y \subseteq X$ with $Y \neq \emptyset$ such that $x \sqcap y = \bigsqcup Y$.

Intuitively, X is weakly $\sqcap$-closed if for any two elements, their meet is possibly not in X, but it can be generated as the join of elements in X. Fig. 5 shows two subsets of $Sub(A)$ (in Fig. 4): one is weakly $\sqcap$-closed and the other is not.

Additionally, given a subset of the subobject lattice of an object, we will need to view it as a diagram. This is formalised below.

Definition 11 (Diagram of subobjects)**.** Given a subset of the subobject lattice $X \subseteq Sub(A)$, we will write $D[X]$ to refer to the diagram $|\cdot|_{|X}\colon (X, \sqsubseteq) \to \mathbb{C}$.

That is, $D[X]$ is a diagram whose scheme is given by the underlying partial order and subobject inclusions as arrows, i.e., $D[X](a') = |a'|$ for any $a' \in X$ and, if $a' \sqsubseteq a''$ then $D[X](a' \sqsubseteq a'')$ is the corresponding inclusion arrow.

We can now prove the desired result.

Proposition 3. Let A be an object of $\mathbb{C}$ and let $X \subseteq \mathit{Sub}(A)$ be a finite subset with $X \neq \emptyset$, which is weakly $\sqcap$-closed. Then $D[X]$ has $\varphi : D[X] \Rightarrow \bigsqcup X$, where each φ_x is the inclusion arrow, as a VK-colimit.

Proof. Set

$$\overline{X} = \{\bigsqcup X' \mid X' \subseteq X, X' \neq \emptyset\}.$$

Clearly $\overline{X}$ is a finite set. Since X is weakly $\sqcap$-closed we can easily show that $\overline{X}$ is a sublattice of $\mathit{Sub}(A)$. In fact, let $a', a'' \in \overline{X}$, hence $a = \bigsqcup X'$ and $a'' = \bigsqcup X''$, with $X', X'' \subseteq X$. Then their meet is

$$\begin{aligned}
&a' \sqcap a'' = (\bigsqcup X') \sqcap (\bigsqcup X'') = \\
&\quad = \bigsqcup\{b' \sqcap b'' \mid b' \in X' \ \wedge \ b'' \in X''\} \qquad \text{[by distributivity]} \\
&\quad = \bigsqcup\{\bigsqcup\{c \in X \mid c \sqsubseteq b' \sqcap b''\} \mid b' \in X' \ \wedge \ b'' \in X''\} \qquad \text{[by weak } \sqcap\text{-closure]} \\
&\quad = \bigsqcup\{c \in X \mid c \sqsubseteq b' \sqcap b'' \text{ for some } b' \in X' \ \wedge \ b'' \in X''\} \\
&\quad \in \overline{X} \qquad \text{[by construction]}
\end{aligned}$$

Similarly, their join $a' \sqcup a'' = (\bigsqcup X') \sqcup (\bigsqcup X'') = \bigsqcup (X' \cup X'') \in \overline{X}$. Thus by Lemma 2 the diagram $D[\overline{X}]$ is VK and has colimit $\bigsqcup X$, which, by construction, is the top of $\overline{X}$.

Now take any linearisation of $\overline{X} \backslash X$, listing elements in descending order with respect to $\sqsubseteq$, i.e, $z_1, \ldots, z_n$ with $z_i \not\sqsubseteq z_k$ if $i < k$. We will remove these objects from $D[\overline{X}]$ one after the other: assume that $z_1, \ldots, z_{j-1}$ have already been removed, the resulting diagram D has a colimit object $\bigsqcup X$ and it is VK.

Now, in the next step, we remove z_j which has the form $z_j = \bigsqcup X'$ for some $X' \subseteq X$, $|X'| > 1$ (in fact, note that if $|X'| = 1$ then z_j would be in X, contradicting the fact that by construction $z_j \in \overline{X} \backslash X$). Therefore z_j is the join of two elements of $\overline{X} \backslash \{z_1, \ldots, z_{j-1}\}$ and can be removed by using Lemma 3. □

For an example take the graph A in Fig. 4 and the subset X in Fig. 5. The colimit of $D[X]$ coincides with $\bigsqcup X$ in $\mathit{Sub}(A)$ (which, in this particular instance, is A itself). Now, consider the subset Y (in Fig. 5) that is not weakly $\sqcap$-closed. The colimit of $D[Y]$ consists of a graph with four nodes, while $\bigsqcup Y$ is the original graph A. This example shows that the above proposition does not hold if the set is not weakly $\sqcap$-closed.

We are now ready to prove the main result of this section.

Corollary 1. Let A be a finite object in $\mathbb{C}$. Then A is the colimit of $D[\mathcal{J}_\perp(A)]$ and the colimit is VK.

Proof. Immediate from Proposition 3, as $\mathcal{J}_\perp(A)$ is weakly $\sqcap$-closed. In fact any element of the lattice can be expressed as the join of weak irreducibles, and thus this holds in particular for the meet of two weak irreducibles. □

For an example, look at Fig. 4. The colimit of $D[\mathcal{J}_\perp(A)]$ is A itself.

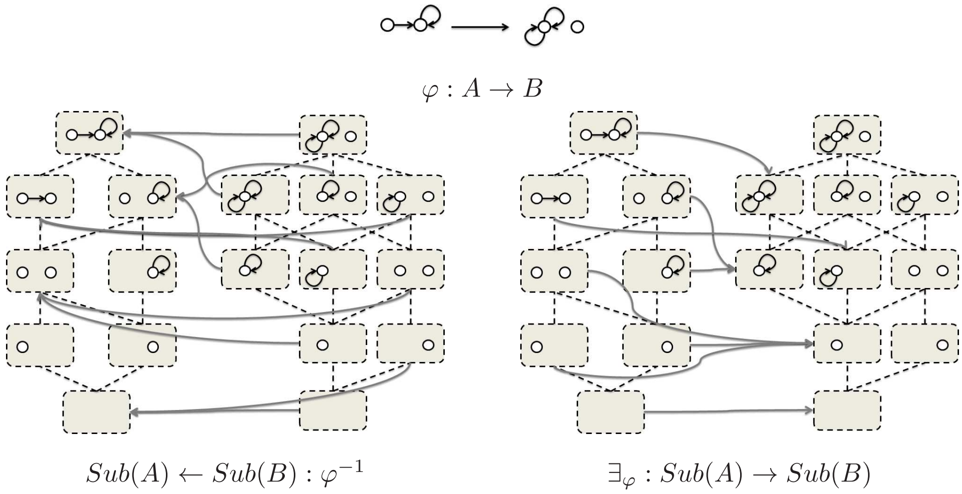


Fig. 6. An arrow $\varphi : A \to B$, the lattice homomorphism φ^{-1} and its left adjoint $\exists_\varphi$. The arrow φ maps both the nodes of A into the leftmost node of B, the self-loop into the upper self-loop of B and the edge into the lower self-loop B.

4. From Arrows to Lattice Homomorphisms

4.1. On the homomorphism induced by an arrow

In this and the next section, we establish a close relationship between arrows in an adhesive category and lattice homomorphisms between the corresponding subobject lattices. We start by proving that for any arrow $\varphi : A \to B$ in an adhesive category, we can define a map $\varphi^{-1} : Sub(B) \to Sub(A)$ that is a $\top$-lattice homomorphism.

Definition 12 (φ^{-1})**.** Let $\varphi : A \to B$ be an arrow in $\mathbb{C}$. The mapping $\varphi^{-1} : Sub(B) \to Sub(A)$ is defined as follows: every subobject $[b' : B' \rightarrowtail B]$ is mapped to a subobject $[a' : A' \rightarrowtail A]$ obtained by taking the following pullback.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, tail, "a'"'] & B' \arrow[d, tail, "b'"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

As an example, Fig. 6 depicts an arrow $\varphi : A \to B$ in the category of graphs and the corresponding map $\varphi^{-1} : Sub(B) \to Sub(A)$.

We will now show that φ^{-1} is always a $\top$-lattice homomorphism. Note that while the preservation of meets is obvious, the preservation of joins is specific to adhesive categories as it depends on the VK-square property discussed after Definition 3.

Proposition 4 (Arrows induce $\top$-homomorphisms)**.** For any arrow $\varphi : A \to B$ in $\mathbb{C}$, the mapping $\varphi^{-1} : Sub(B) \to Sub(A)$ is a $\top$-homomorphism.

Proof. The mapping preserves the $\top$ element. In fact the following square is a pullback

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, tail, "id_A"'] & B \arrow[d, tail, "id_B"] \\
A \arrow[r, "\varphi"] & B
\end{tikzcd}$$

which means that $\varphi^{-1}(\top_{Sub(B)}) = \varphi^{-1}([id_B]) = [id_A] = \top_{Sub(A)}$.

We now show that the mapping preserves meets. Let $[b_3] = [b_1] \sqcap [b_2]$, hence the square below on the right (B, B_1, B_2, B_3) is a pullback. Furthermore by using φ^{-1} and standard pullback splitting we obtain the four squares in the middle which are also pullbacks. Now it can be shown that the square B, B_2, A_1, A_3 is a pullback (as it arises as the combination of two pullbacks B_1, B_3, A_1, A_3 and B, B_1, B_2, B_3). Then since B, B_2, A, A_2 is also a pullback, it follows that the square (A, B, A_2, A_3) on the left is also a pullback.

$$\begin{tikzcd}
& A_3 \arrow[rrr] \arrow[dl, tail] \arrow[dr, tail] & & & B_3 \arrow[dl, tail] \arrow[dr, tail] & \\
A_1 \arrow[rrr] \arrow[dr, tail, "a_1"'] & & A_2 \arrow[rrr] \arrow[dl, "a_2"] & B_1 \arrow[dr, tail, "b_1"'] & & B_2 \arrow[dl, "b_2"] \\
& A \arrow[rrr, "\varphi"'] & & & B &
\end{tikzcd}$$

Thus $[a_1] = \varphi^{-1}([b_1])$, $[a_2] = \varphi^{-1}([b_2])$, $[a_3] = \varphi^{-1}([b_3])$ and $[a_3] = [a_1] \sqcap [a_2]$.

In a second step we show that φ^{-1} also preserves joins, so let $[b'] = [b_1] \sqcup [b_2]$ with $[b_3] = [b_1] \sqcap [b_2]$. Hence we can construct a diagram as above with all squares pullbacks. Now take the pushout of $B_3 \rightarrowtail B_1$, $B_3 \rightarrowtail B_2$, obtaining B' with mediating arrow $b' : B' \rightarrowtail B$. Then take the pullback of b' and φ, obtaining A' with arrow $a' : A' \rightarrowtail A$ and mediating morphisms $a_1' : A_1 \rightarrowtail A'$, $a_2' : A_2 \rightarrowtail A'$. Due to pullback splitting we have that the squares B', B_1, A', A_1 and B', B_2, A', A_2 are pullbacks. Hence, since the right-hand pushout over B' is a VK-square we can infer that A', A_1, A_2, A_3 is also a pushout.

$$\begin{tikzcd}
& A_3 \arrow[rrr] \arrow[dl, tail] \arrow[dr, tail] & & & B_3 \arrow[dl, tail] \arrow[dr, tail] & \\
A_1 \arrow[rrr] \arrow[dr, tail, "a_1'"'] \arrow[ddr, tail, bend right, "a_1"'] & & A_2 \arrow[rrr] \arrow[dl, "a_2'"] \arrow[ddl, tail, bend left, "a_2"] & B_1 \arrow[dr, tail] \arrow[ddr, tail, bend right, "b_1"'] & & B_2 \arrow[dl] \arrow[ddl, tail, bend left, "b_2"] \\
& A' \arrow[rrr] \arrow[d, tail, "a'"'] & & & B' \arrow[d, tail, "b'"'] & \\
& A \arrow[rrr, "\varphi"'] & & & B &
\end{tikzcd}$$

This gives us $[a_1] = \varphi^{-1}([b_1])$, $[a_2] = \varphi^{-1}([b_2])$, $[a'] = \varphi^{-1}([b'])$ and $[a'] = [a_1] \sqcup [a_2]$, as desired. $\square$

As hinted at earlier, arrows in adhesive categories are not necessarily $\bot$-homomorphisms. Consider for instance the category of pointed sets, i.e., the category of sets having a distinguished element $\bullet$ and functions which preserve $\bullet$. Now consider the unique morphism $\varphi\colon A \to B$ with $A = \{a, \bullet\}$, $B = \{\bullet\}$ (note that B is the final object). Subobject lattices here consist of pointed subsets ordered by subset inclusion. The bottom element of the subobject lattice $Sub(B)$ is $[id_B]$, but the pullback of φ and id_B gives us $[id_A]$, which is *not* the bottom element of $Sub(A)$. In Section 6, we will show that by requiring the existence of a strict initial object, we can prove that φ^{-1} also preserves bottom.

4.2. On the left adjoint of the homomorphism induced by an arrow

In this subsection we present a characterization of $\exists_\varphi$, the left adjoint of φ^{-1}, which exists by Lemma 1 because we just proved that φ^{-1} is a lattice $\top$-homomorphism, and thus it preserves arbitrary meets. At the same time, we show that for finite objects adhesive categories enjoy unique cover-mono factorization that we exploit to characterize $\exists_\varphi$. This result will be used later in Section 5.2.

Let us start by introducing the notion of cover (Freyd and Scedrov, 1990).

Definition 13 (Cover)**.** An arrow $c : A \to B$ is called a cover if whenever $c = a; b$ with b mono, then b is an iso.

Covers will in the following be denoted by arrows of type $\twoheadrightarrow$. It can be easily seen that every iso is a cover. Additionally, one can prove the following facts about covers.

Lemma 4. All epis are covers in adhesive categories. In a category with equalizers all covers are epis.

Proof. We first show that in an adhesive category all epis are covers. Assume that c is an epi and $c = a; b$ with b mono. Then b is also an epi (as second arrow in the decomposition of an epi). Since it is mono and epi it must be iso (in fact, as shown in (Lack and Sobociński, 2005), in adhesive categories arrows that are both mono and epi are isos).

Next, we prove that in a category with equalizers all covers are epis. Take a cover $c : A \twoheadrightarrow B$ and two arrows $a, b : B \to C$ such that $c; a = c; b$. Now take the equalizer $e : E \rightarrowtail B$ of a, b, which must be a mono (since all equalizers are mono). Due to the universal property there exists a unique mediating arrow $m : A \to E$. Then e must be an iso and from $e; a = e; b$ it follows that $a = b$.

$$\begin{tikzcd}
A \arrow[r, two heads, "c"] \arrow[d, dotted, "m"'] & B \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & C \\
E \arrow[ur, tail, "e"'] & &
\end{tikzcd}$$

□

The exact relationship between epis and covers in adhesive categories is still not completely understood. In the following we use the notion of cover, instead of epi, as it integrates much better with the rest of the theory.

Proposition 5 ($\exists_\varphi$ induces covers)**.** Let A, B be finite objects in $\mathbb{C}$, $\varphi : A \to B$ be an arrow, consider the $\top$-homomorphism $\varphi^{-1} : Sub(B) \to Sub(A)$ and let $\exists_\varphi$ be the corresponding left adjoint. Then, for any pair of monos $a : A' \rightarrowtail A$, $b : B' \rightarrowtail B$ with $[b] = \exists_\varphi([a])$, there exists an arrow $\varphi' : A' \to B'$ such that $\varphi'; b = a; \varphi$ (obviously unique, as b is mono) and φ' is a cover.

$$\begin{tikzcd}
A' \arrow[r, dotted, "\varphi'"] \arrow[d, tail, "a"'] & B' \arrow[d, tail, "b"] \\
A \arrow[r, "\varphi"] & B
\end{tikzcd}$$

Proof. Take the pullback of φ and b, resulting in the pullback object $\overline{A}$. Since $\overline{a} = \varphi^{-1}(\exists_\varphi([a])) \sqsupseteq [a]$ there exists a mono $m : A' \rightarrowtail \overline{A}$ making everything commute. Define $\varphi' = m; f$ where f is the upper leg of the pullback.

The arrow φ' is a cover. In fact, let $\varphi' = f'; m'$ where m' is mono.

$$\begin{tikzcd}
 & A'' \arrow[r] \arrow[d, tail, "a'"] & B'' \arrow[d, tail, "m'"] \\
A' \arrow[ur, dotted] \arrow[urr, bend left, "f'"] \arrow[r, tail, "m"'] \arrow[dr, tail, "a"'] & \overline{A} \arrow[r, "f"'] \arrow[d, tail, "\overline{a}"] & B' \arrow[d, tail, "b"] \\
 & A \arrow[r, "\varphi"] & B
\end{tikzcd}$$

Take the pullback of f and m', resulting in the pullback object A''. Consider the subobject $a'' = a'; \bar{a}$ and observe that $a'' = \varphi^{-1}([m'; b])$ since the square A'', B', B, A is a pullback (by pullback chasing). Note that the mediating arrow from A' to A'' must be a mono, since m is mono. Hence $a \sqsubseteq a'' = \varphi^{-1}([m'; b])$. Thus, by definition of $\exists_\varphi$

$$b = \exists_\varphi([a]) \sqsubseteq \exists_\varphi(\varphi^{-1}([m'; b])) \sqsubseteq m'; b \sqsubseteq b$$

which implies that m' is an iso. $\square$

The above proposition allows us to prove that each arrow between finite objects in an adhesive category can be uniquely factorized into a cover and a mono.

Proposition 6 (Cover-mono factorizations in adhesive categories)**.** Let A, B be finite objects in $\mathbb{C}$ and let $\varphi : A \to B$ be an arrow. Then there exists a unique cover-mono factorization of φ, i.e., there is a unique cover c and a unique mono m such that $\varphi = c; m$.

Proof. *Existence* of a cover-mono factorization can be shown as follows. Let $\varphi : A \to B$ be an arrow and let $\exists_\varphi : \mathit{Sub}(A) \to \mathit{Sub}(B)$ be the left-adjoint to φ^{-1}. Take $\exists_\varphi([id_A : A \to A]) = [b : B' \rightarrowtail B]$. Then, by Proposition 5, there is a cover $c : A \twoheadrightarrow B'$ such that the diagram below commutes, thus providing the desired cover-mono factorization.

$$\begin{tikzcd}
A \arrow[r, tail, two heads, "c"] \arrow[d, tail, "id_A"'] & B' \arrow[d, tail, "b"] \\
A \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

As far as *uniqueness* is concerned, we derive it by showing that the diagonalization property holds: assume the square on the left below commutes where c is a cover and b is mono. Now take the pullback of b and d and obtain the square in the middle. Since c is a cover, b' must be an iso and $b'^{-1}; d'$ is the required diagonal arrow.

$$
\begin{tikzcd}
\bullet \arrow[r, two heads, "c"] \arrow[d, "a"'] & \bullet \arrow[d, "d"] \\
\bullet \arrow[r, tail, "b"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[rr, two heads, "c"] \arrow[dd, "a"'] \arrow[dr, dotted, hook, "c'"] & & \bullet \arrow[dd, "d"] \\
& \bullet \arrow[ur, tail, "b'"'] \arrow[dl, "d'"] & \\
\bullet \arrow[rr, tail, "b"'] & & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[rr, two heads, "c"] \arrow[dd, "a"'] \arrow[dr, dotted, hook, "c'"] & & \bullet \arrow[dd, "d"] \arrow[dl, "\sim"', "b'^{-1}"] \\
& \bullet \arrow[dl, "d'"] & \\
\bullet \arrow[rr, tail, "b"'] & & \bullet
\end{tikzcd}
$$

We now show the uniqueness of the diagonal arrow. Assume that there were two diagonal arrows d'_1, d'_2. Since $d'_1; b = d = d'_2; b$ and b is mono we can conclude that $d'_1 = d'_2$. $\square$

An immediate consequence of the uniqueness of cover-mono factorization is that the left adjoint to the $\top$-homomorphism induced by an arrow is uniquely determined by the cover-mono factorization in $\mathbb{C}$.

Corollary 2 (Characterization of $\exists_\varphi$)**.** Let $\varphi : A \to B$ be an arrow in $\mathbb{C}$ where A, B are finite objects, let $\varphi^{-1} : Sub(B) \to Sub(A)$ be the induced $\top$-homomorphism, and let $\exists_\varphi : Sub(A) \to Sub(B)$ be its left adjoint. Then for every $[a : A' \rightarrowtail A] \in Sub(A)$, it holds $\exists_\varphi([a]) = [b : B' \rightarrowtail B]$, where $c; b$ is the cover-mono factorization of $a; \varphi$:

$$
\begin{tikzcd}
A' \arrow[r, tail, two heads, "c"] \arrow[d, tail, "a"'] & B' \arrow[d, tail, "b"] \\
A \arrow[r, "\varphi"'] & B
\end{tikzcd}
$$

Proof. Immediate, by Propositions 5 and 6. $\square$

An example for a mapping $\exists_\varphi$ arising from a graph morphism φ is shown in the right part of Fig. 6. Note that, while φ^{-1} is a $\top$-homomorphism, $\exists_\varphi$ is not, since it does not preserve meets and top.

5. From Lattice Homomorphisms to Arrows

The converse of Proposition 4 does not hold, i.e., not every lattice homomorphisms from $Sub(B)$ to $Sub(A)$ is induced as the inverse image functor of an arrow of $\mathbb{C}$. This should not be surprising: in fact, a morphism $\varphi : A \to B$ does not induce simply a lattice $\top$-homomorphism $\varphi^{-1} : Sub(B) \to Sub(A)$, but a much richer structure. In fact, for any $[b_1 : B_1 \rightarrowtail B]$, we also get a morphism $\Phi_{b_1} : \varphi^{-1}(B_1) \to B_1$, and for any pair of subobjects $[b_1 : B_1 \rightarrowtail B]$ and $[b_2 : B_2 \rightarrowtail B]$, such that $[b_1] \sqsubseteq [b_2]$, the following square is a pullback

$$
\begin{tikzcd}
\varphi^{-1}(B_1) \arrow[r, tail] \arrow[d, "\Phi_{b_1}"'] & \varphi^{-1}(B_2) \arrow[d, "\Phi_{b_2}"] \\
B_1 \arrow[r, tail] & B_2
\end{tikzcd}
$$

In other words, we have a cartesian natural transformation $\Phi : \varphi^{-1}; |\cdot|_A \Rightarrow |\cdot|_B$.

As an immediate consequence, a $\top$-homomorphism $Sub(B) \to Sub(A)$ cannot be induced by a morphism $\varphi : A \to B$ if such additional structure does not exist. For instance, in a category of labeled graphs with label preserving homomorphisms, consider two graphs each consisting of a single node, which is labeled a in the first graph and b in the second one. The two subobject lattices are clearly isomorphic, but of course there is no homomorphism between these graphs.

We next identify suitable conditions under which a $\top$-homomorphism between subobject lattices is induced by an arrow between the corresponding objects. Note that we concentrate on finite objects in order to exploit Birkhoff's representation theorem. Throughout the section $\mathbb{C}$ is implicitly assumed to be an adhesive category.

5.1. Arrows and Cartesian Natural Transformations

We look for sufficient conditions to show that a given lattice homomorphism $\gamma : \mathit{Sub}(B) \to \mathit{Sub}(A)$ is induced by an arrow $A \to B$. Inspired by the above considerations, the first solution consists of requiring the existence of a cartesian natural transformation between the image under γ of the diagram of irreducibles of B and the diagram itself, i.e., for each irreducible B' of B we ask for the existence of an arrow from $\gamma(B')$ to B', in a way that the resulting squares are all pullbacks.

We first need to introduce some notation. Let A, B be objects in $\mathbb{C}$, let $X \subseteq \mathit{Sub}(B)$ and let $f : \mathit{Sub}(B) \to \mathit{Sub}(A)$ be a monotone mapping. Let $D[X]$ be the diagram as in Definition 11. We denote by $f * D[X] : X \to \mathbb{C}$ the diagram with the same scheme of $D[X]$ and which maps any $b \in X$ to $|f(b)|_A$. An example is provided in the left part of Fig. 7, which represents the diagram $\gamma * D[\mathcal{J}_\perp(B)]$ where γ is the homomorphism φ^{-1} in Fig. 6. Note that the scheme category of this diagram is a poset, while the target is the category of graphs. In order to stress this difference we have used dashed lines in the graphical representation of the former category and straight lines for the latter.

Proposition 7. Let A, B be finite objects in $\mathbb{C}$. Let $\gamma : \mathit{Sub}(B) \to \mathit{Sub}(A)$ be a $\top$-homomorphism such that there exists a cartesian natural transformation $\Phi : \gamma * D[\mathcal{J}_\perp(B)] \Rightarrow D[\mathcal{J}_\perp(B)]$. Then there exists a unique arrow $\varphi : A \to B$ such that $\varphi^{-1} = \gamma$.

Proof. The colimit of $D[\mathcal{J}_\perp(B)]$ is B and it is a VK-colimit by Corollary 1.

Similarly, we can prove that the colimit of $\gamma * D[\mathcal{J}_\perp(B)]$ is A and it is VK. In fact, this diagram has obviously the same colimit as $D[\gamma(\mathcal{J}_\perp(B))]$ (they are the same apart for the fact that in the first diagram some objects, which arise as the non-injective image of different irreducibles through γ, are repeated). Now, we know that $\gamma(\mathcal{J}_\perp(B))$ is weakly $\sqcap$-closed, since γ is a lattice homomorphism (and thus it preserves meets and joins). Thus, by Proposition 3, $D[\gamma(\mathcal{J}_\perp(B))]$ has colimit $\bigsqcup \gamma(\mathcal{J}_\perp(B))$ and

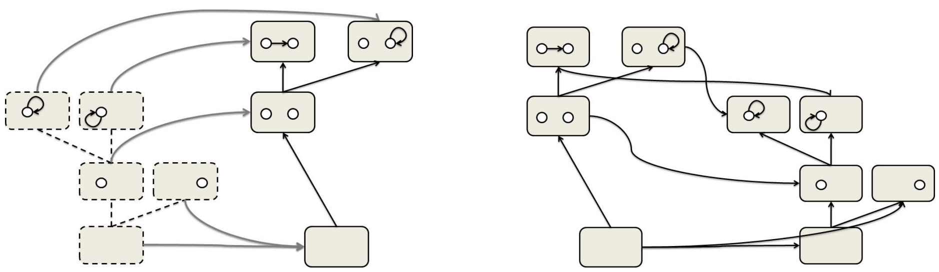
$$\bigsqcup \gamma(\mathcal{J}_\perp(B)) = \bigsqcup_{I \in \mathcal{J}_\perp(B)} \gamma(I) = \gamma(\bigsqcup_{I \in \mathcal{J}_\perp(B)} I) = \gamma([id_B]) = [id_A]$$

as γ preserves joins and the top element. Again by Proposition 3, the colimit A is VK.

Then we can get φ as a mediating arrow

$$\begin{tikzcd} \gamma * D[\mathcal{J}_\perp(B)] \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\Phi"'] & A \arrow[d, dotted, "\varphi"] \\ D[\mathcal{J}_\perp(B)] \arrow[r, Rightarrow] & B \end{tikzcd}$$

Since Φ is cartesian, the top row $\gamma * D[\mathcal{J}_\perp(B)] \Rightarrow A$ is a VK-colimit and the bottom row $D[\mathcal{J}_\perp(B)] \Rightarrow B$ is a VK-colimit, we deduce that for any weak irreducible $I \in \mathcal{J}_\perp(B)$ the following square is a pullback



$\gamma * D[\mathcal{J}_\perp(B)] : \mathcal{J}_\perp(B) \to \mathbb{C}$ $\qquad\qquad$ $\Phi : \gamma * D[\mathcal{J}_\perp(B)] \Rightarrow D[\mathcal{J}_\perp(B)]$

Fig. 7. The diagram $\gamma * D[\mathcal{J}_\perp(B)]$ (left) and the cartesian natural transformation Φ of Proposition 7 (right).

$$\begin{tikzcd}
{|\gamma(I)|} \arrow[r, tail] \arrow[d, "\Phi_{\gamma(I)}"'] & A \arrow[d, "\varphi"] \\
{|I|} \arrow[r, tail] & B
\end{tikzcd}$$

This means that φ^{-1} coincides with γ on the weak irreducibles. But since they both are $\top$-homomorphisms, they are determined by their value on the weak irreducibles and thus $\gamma = \varphi^{-1}$, as desired. $\square$

The right part of Fig. 7 shows an example of the cartesian natural transformation Φ, where γ is the map φ^{-1} in Fig. 6. The scheme category of the two diagrams is $\mathcal{J}_\perp(B)$, depicted in the left part of the figure. For each $I \in \mathcal{J}_\perp(B)$, there is a graph morphism $\Phi_I : \gamma * D[\mathcal{J}_\perp(B)](I) \to D[\mathcal{J}_\perp(B)](I)$. Note that all the squares in the figure are commuting (i.e., Φ is a natural transformation) and, most importantly, they are all pullbacks (i.e., Φ is cartesian).

5.2. *Arrows, Covers and Natural Transformations*

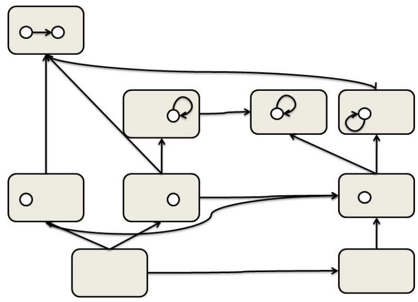
An alternative characterization is based on the notion of cover and on the existence of cover-mono factorizations in adhesive categories, which, as shown in Corollary 2, leads to a characterization of the left adjoint to the inverse image functor.

Proposition 8. Let A, B be finite objects in $\mathbb{C}$ and assume that there exists a $\top$-lattice homomorphism $\gamma : \mathit{Sub}(B) \to \mathit{Sub}(A)$. Let $\alpha : \mathit{Sub}(A) \to \mathit{Sub}(B)$ be the corresponding left adjoint (which maps weak irreducibles to weak irreducibles, by Proposition 1).

If there exists a natural transformation $\Phi : D[\mathcal{J}_\perp(A)] \Rightarrow \alpha * D[\mathcal{J}_\perp(A)]$, made of covers, then there exists a unique arrow $\varphi : A \to B$ with $\varphi^{-1} = \gamma$.

Proof. We proceed by showing the existence of an arrow $\varphi : A \to B$ such that for any weak irreducible $[a' : A' \rightarrowtail A]$, if $[b'] = \alpha([a'])$ then $\Phi_{[a']}; b' = a'; \varphi$.

$$\begin{tikzcd}
A' \arrow[r, two heads, "\Phi_{[a']}"] \arrow[d, tail, "a'"'] & B \arrow[d, tail, "b'"] \\
A \arrow[r, "\varphi"] & B
\end{tikzcd}$$



$$\Phi : D[\mathcal{J}(A)] \Rightarrow \alpha * D[\mathcal{J}(A)]$$

Fig. 8. The cartesian natural transformation Φ of Proposition 8.

Hence, by Corollary 2, since $\Phi_{[a']}$ is a cover, $\exists_\varphi([a']) = [b'] = \alpha([a'])$. Since α and $\exists_\varphi$ coincide on weak irreducibles, they preserve joins and every element is the join of weak irreducibles, we conclude that they coincide on any element, i.e., $\exists_\varphi = \alpha$. Finally, since the left adjoint determines the right adjoint, as a further consequence, also the corresponding right adjoints coincide, i.e., $\gamma = \varphi^{-1}$, as desired.

In order to prove the existence of an arrow $\varphi : A \to B$ with the desired properties, first note that by Corollary 1, A is the colimit of $D[\mathcal{J}_\perp(A)]$.

Moreover, consider $\alpha * D[\mathcal{J}_\perp(A)]$ which clearly has a cone $\alpha * D[\mathcal{J}_\perp(A)] \Rightarrow B$, given by the inclusions into B. Note that this is not necessarily a colimit.

The diagram below summarises the situation, where Φ is the natural transformation of covers that we have by hypothesis.

$$\begin{tikzcd}
D[\mathcal{J}_\perp(A)] \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\Phi"'] & A \arrow[d, dotted, "\varphi"] \\
\alpha * D[\mathcal{J}_\perp(A)] \arrow[r, Rightarrow] & B
\end{tikzcd}$$

Since A is a colimit, we deduce the existence of φ as mediating arrow and note that φ satisfies exactly the desired commutativity property. $\square$

Fig. 8 shows an example of the cartesian natural transformation Φ, where α is the map $\exists_\varphi$ in Fig. 6. The scheme category of the two diagrams is $\mathcal{J}_\perp(A)$ (depicted in Fig. 4). For each $A' \in \mathcal{J}_\perp(A)$, there is a graph morphism $\Phi_{A'} : D[\mathcal{J}(A)](A') \to \alpha * D[\mathcal{J}(A)](A')$. Note that each of these morphisms is a cover. Moreover, Φ is a natural transformation since all the squares in the figure are commuting.

6. Adhesive Categories with a Strict Initial Object

As discussed in Section 4, the lattice morphism φ^{-1} induced by an arrow φ of an adhesive category does not preserve $\perp$, in general. This is the reason why, in Section 2, we needed to consider a variation of the standard theory of duality. In the ordinary presentation, the category of finite distributive lattice has $\{\perp, \top\}$-homomorphisms as arrows, each lattice L corresponds to $\mathcal{J}(L)$ and each (possibly non-pointed) poset P corresponds to $\mathcal{O}(P)$ (the lattice consisting of the possibly empty downward-closed subsets of P).

In this section we specialise our theory by focussing on adhesive categories with a *strict initial object*. According to a result in (Lack and Sobociński, 2005) these are exactly those adhesive categories which are *extensive*. In this case, for any arrow φ the lattice morphism φ^{-1} also preserves $\bot$, and thus we can base our results on the standard Birkhoff duality.

Definition 14 (Strict initial object)**.** Let $\mathbb{C}$ be a category. An *initial object* is an object 0 such that for any other object A there exists a unique arrow $?_A : 0 \to A$. A *strict initial object* is an initial object 0 such that every arrow into 0 is an iso.

It is worth noting that not all adhesive categories have a strict initial object (hence not all adhesive categories are extensive). For instance, the category sets and partial maps, isomorphic to the category of pointed sets discussed in Section 4, is adhesive but it doesn't have a strict initial object. In fact, the empty set (or the set containing only the distinguished point, for pointed sets) is initial, but not strict initial.

We observe that the following properties hold.

Lemma 5 (Properties of strict initial objects)**.** For a strict initial object 0 the following hold:

(1) From each object there is at most one iso to 0.
(2) Every arrow $?_A : 0 \to A$ is a mono.
(3) The diagram below is always a pullback.

$$\begin{tikzcd} 0 \arrow[r, tail, "id_0"] \arrow[d, "?_A"'] & 0 \arrow[d, "?_B"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

Proof. (1) Assume that there is an object N and two isos $\psi_1, \psi_2 : N \xrightarrow{\sim} 0$. Then we have $\psi_2^{-1}; \psi_1 : 0 \to 0$ and since there is a unique arrow from 0 into itself (by initiality) we have $\psi_2^{-1}; \psi_1 = id_0$. Hence we have $\psi_1 = \psi_2$.

(2) Trivial, using (1).

(3) Consider the commuting diagram below:

$$\begin{tikzcd} N \arrow[drr, bend left, "\psi"] \arrow[dr, dotted] \arrow[ddr, bend right, "\alpha"'] & & \\ & 0 \arrow[r, tail, "id_0"] \arrow[d, "?_A"'] & 0 \arrow[d, "?_B"] \\ & A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

We require the existence of a unique mediating morphism $N \to 0$. The only possibility to make the upper right triangle commute is to choose ψ as mediating morphism. Since ψ is an iso and arrows from 0 are unique we have that $\psi^{-1}; \alpha = ?_A$, which implies $\alpha = \psi; ?_A$, which implies that the lower right triangle commutes as well.

□

From item (2) above, we have that for any object A (of an adhesive category with strict initial object), the isomorphism class $[?_A : 0 \rightarrowtail A]$ is in $Sub(A)$. Moreover, $[?_A]$ is the bottom element of $Sub(A)$, since for any other $[b : B \rightarrowtail A]$, $?_A = ?_B; b$, i.e., $[?_A] \sqsubseteq [b]$.

From item (3), we have that for any arrow $\varphi : A \to B$, the homomorphism φ^{-1} maps $[?_B]$ into $[?_A]$, i.e., it preserves $\perp$. For this reason, we have that Proposition 4 can be rephrased as follows (hereafter $\mathbb{C}$ denotes an adhesive category with a strict initial object).

Proposition 9 (Arrows are $\{\perp, \top\}$-homomorphisms)**.** For any arrow $\varphi : A \to B$ in $\mathbb{C}$, the mapping $\varphi^{-1} : \mathit{Sub}(B) \to \mathit{Sub}(A)$ is a $\{\perp, \top\}$-homomorphisms.

Now, we can rephrase our theory by employing $\mathcal{J}(A)$ in place of $\mathcal{J}_\perp(A)$. First of all, we need to change Corollary 1 as follows.

Corollary 3. Let A be a finite object in $\mathbb{C}$. Then A is the colimit of $D[\mathcal{J}(A)]$ and the colimit is VK.

Proof sketch. This follows from Corollary 1. According to this corollary A is the colimit of $D[\mathcal{J}_\perp(A)]$ and the colimit is VK. As a consequence of strict initiality (in particular of Lemma 5(2)) the bottom element in the diagram $D[\mathcal{J}_\perp(A)]$ corresponds to the subobject $[?_A\colon 0 \to A]$. Hence removing or adding the initial element does not modify the colimit, since the diagram will still commute with the cocone. Note also that the colimit of an empty diagram is 0.

Furthermore the colimit is still VK. Given a cartesian natural transformation $\beta\colon D \Rightarrow D[\mathcal{J}(A)]$ it can be converted into a cartesian natural transformation $\beta'\colon D' \Rightarrow D[\mathcal{J}_\perp(A)]$ by adding strict initial objects. According to Lemma 5(3) the new squares are all pullbacks. Hence we can apply Corollary 1 and we know that a cocone φ for D' is a colimit whenever the relevant squares are pullbacks. Finally, the fact that φ is a colimit of D' is equivalent to φ being a colimit of D. $\square$

This is all we need to translate Propositions 7 as follows.

Proposition 10. Let A, B be finite objects in $\mathbb{C}$. Let $\gamma : \mathit{Sub}(B) \to \mathit{Sub}(A)$ be a $\{\perp, \top\}$-homomorphism such that there exists a cartesian transformation $\Phi : \gamma * D[\mathcal{J}(B)] \Rightarrow D[\mathcal{J}(B)]$. Then there exists a unique arrow $\varphi : A \to B$ such that $\varphi^{-1} = \gamma$.

Proof sketch. Easy consequence of Proposition 7. First, the cartesian natural transformation without the strict initial object can be extended to a cartesian natural transformation with the strict initial object. By Lemma 5 adding strict initial objects automatically gives us pullback squares. (Compare also with the proof of Corollary 3.) $\square$

In order to translate Proposition 8, we have first to translate Proposition 1 into the following statement.

Proposition 11. Let L, K be finite distributive lattices, let $\gamma : K \to L$ be a $\{\perp, \top\}$-lattice homomorphism and let $\alpha : L \to K$ be its left adjoint. Then α preserves irreducibles.

Proof. We first observe that, given $l \in L$, if $\alpha(l) = \perp$ then $l = \perp$. In fact, from $\alpha(l) = \perp$, since γ preserves $\perp$ we derive $\gamma(\alpha(l)) = \perp$. Using the fact that γ is a right adjoint, we have that $l \sqsubseteq \gamma(\alpha(l)) = \perp$, i.e., $l = \perp$.

Now, by Proposition 1, we know that α preserves weak irreducibles. Thus if $l \in \mathcal{J}(L)$ is an irreducible (i.e., weak irreducible and different from $\perp$), then $\alpha(l)$ is also a weak irreducible and, by the consideration above, $\alpha(l) \neq \bot$ since $l \neq \bot$. Therefore $\alpha(l) \in \mathcal{J}(K)$. $\square$

Proposition 12. Let A, B be finite objects in $\mathbb{C}$ and assume that there exists a $\{\bot, \top\}$-lattice homomorphism $\gamma : \mathit{Sub}(B) \to \mathit{Sub}(A)$. Let $\alpha : \mathit{Sub}(A) \to \mathit{Sub}(B)$ be the corresponding left adjoint (which maps irreducibles to irreducibles, by Proposition 11).

If there exists a natural transformation $\Phi : D[\mathcal{J}(A)] \Rightarrow \alpha * D[\mathcal{J}(A)]$, made of covers, then there exists a unique arrow $\varphi : A \to B$ with $\gamma = \varphi^{-1}$.

Proof sketch. The result follows from Proposition 8: the natural transformation can be extended by adding the strict initial object 0. All squares automatically commute and the arrow $0 \to 0$ is an iso (and hence a cover). $\square$

7. Conclusion

Duality results such as the well-known Stone duality (Johnstone, 1982) are recurrent in mathematics and theoretical computer science. In this paper we have presented a representation theorem for adhesive categories, related to the Birkhoff duality. There is a huge amount of work on duality, but to our knowledge no one has so far studied the exact relationship between adhesive categories and Birkhoff's representation theorem. We are also not aware of characterization theorems in the form of Proposition 7 and 8. For simplicity we have mainly restricted ourselves to finite lattices, but a generalization to infinite lattices—including the topological concepts that come with it—would be an interesting direction for future work.

Naturally, our work is closely related to (Lack and Sobociński, 2005)—since we take adhesive categories as our basis—but also to (Cockett and Guo, 2007), which studies join restriction categories. The latter work describes under which conditions parallel partial maps can be assembled into a single partial map, which is slightly similar in spirit to the characterization of lattice homomorphisms which are arrows. Furthermore the notion of VK-colimit that we use is introduced in this paper. Our Proposition 3 is a generalization of Proposition 4.8 of (Cockett and Guo, 2007), which requires meet-closure (instead of only weak meet-closure as it is considered here). However, no direct connection to lattice theory is made in (Cockett and Guo, 2007).

There is also a connection to (Freyd and Scedrov, 1990) which studies pre-logoi and especially logoi. A logos is a regular category where the subobjects form a lattice and the inverse-image operation has a right adjoint. In those categories the inverse-image operation is necessarily a lattice-homomorphism. In addition (Freyd and Scedrov, 1990) uses the notion of cover. In order to clarify the exact relation to logoi it would be necessary to determine whether adhesive categories are always regular (i.e., whether covers are preserved by pullbacks). So far we have not been able to prove that this is the case.

As a side remark, an interesting question for which we do not have a definite answer yet, is whether *any* finite lattice homomorphism could be proved to be induced by an arrow of an adhesive category. More formally, *given a $\top$-lattice homomorphism $\gamma : K \to L$ between finite distributive lattices, is there an adhesive category $\mathbb{C}$ with two objects A_L and A_K and an arrow $\varphi : A_L \to A_K$, such that $L \simeq \mathit{Sub}(A_L)$, $K \simeq \mathit{Sub}(A_K)$, and $\gamma = \varphi^{-1}$?* We expect the answer to this question to be negative, and the preservation of covers by pullbacks would be sufficient to provide a counterexample. Note also that an obvious candidate, the category of partially ordered sets and monotone maps, is *not* adhesive.

Concerning other related work, the set of all irreducibles in a category is a generating set in the sense of (Mac Lane, 1971). That is, for every pair $f, g: A \to B$ of parallel arrows with source A and $f \neq g$ there is an arrow $i: I \rightarrowtail A$—where I is an irreducible—with $i; f \neq i; g$. This is true since, by Corollary 1, an object A can be obtained as the colimit of its irreducibles and hence the arrows from the irreducibles into A are jointly epi. However, the two notions do not fully coincide since there could be generating sets containing non-irreducible objects and the arrows into A need not be mono (by the definition of generating sets).

From a more practical perspective we are especially interested in providing new tools that can be employed for the theory of adhesive rewriting systems and hence for graph transformation (Lack and Sobociński, 2005; Ehrig et al., 2006). We believe that (graph) rewriting theory can benefit from the establishment of an explicit connection with lattice theory. For this it is also interesting to study the right adjoint of the inverse-image functor, which basically takes greatest pullback complements (as described in (Corradini et al., 2006)).

The characterization theorems (Proposition 7 and 8) describe arrows as mappings of atomic units (i.e., irreducibles) to other atomic units. This is fairly close in spirit to the definition of graph morphisms which map nodes and edges to nodes and edges. As such, it suggests the possibility of obtaining more explicit representation theorems for objects in adhesive categories, that allow to view them as some kind of "graph-like" structure. We believe that this could provide a deeper understanding of adhesive categories and of their role in the theory of rewriting of graphical structures.

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