Petri nets are dioids*

Paolo Baldan[1] and Fabio Gadducci[2]

[1] Dipartimento di Matematica Pura e Applicata, Università Padova
Via Trieste 63, 35121 Padova, Italia
[2] Dipartimento di Informatica, Università di Pisa
Polo "Guglielmo Marconi", via dei Colli 90, 19100 La Spezia, Italia
`baldan@math.unipd.it, gadducci@di.unipi.it`

Abstract. In a seminal paper Montanari and Meseguer showed that an algebraic interpretation of Petri nets in terms of commutative monoids can be used to provide an elegant characterisation of the deterministic computations of a net, accounting for their sequential and parallel composition. Here we show that, along the same lines, by adding an (idempotent) operation and thus taking dioids (commutative semirings) rather than monoids, one can faithfully characterise the non-deterministic computations of a Petri net.

Introduction

Petri nets [12] are one of the best studied and most widely known models for concurrent systems. Due to the conceptual simplicity of the model and its intuitive graphical presentation, since their introduction, which dates back to the 60's [11], Petri nets have attracted the interest of both theoreticians and practitioners.

The basic operational behaviour of a Petri net can be straightforwardly defined in terms of the so-called token game and of firing sequences. Concurrency in computations can be made explicit by resorting to a semantics given in terms of *(non-sequential) deterministic processes* à la Goltz-Reisig [5]. A process describes the events occurring in a computation and their mutual dependency relations. Concretely, a deterministic processes is an acyclic, deterministic net whose structure induces a partial order on transitions which can be seen as occurrences of transition firings in the original net. A deterministic process thus captures an abstract notion of concurrent computation, in the sense that it can be seen as a representative of a full class of firing sequences differing only for the order of independent events, i.e., all the firing sequences corresponding to linearisations of the underlying partial ordering.

Different (concurrent) computations can be merged into a single non-deterministic process [3], a structure which, besides concurrency, captures also the intrinsic non-deterministic nature of Petri nets. A non-deterministic process is again a special Petri net, satisfying suitable acyclicity requirement, but where, roughly speaking, transitions can compete for the use of tokens, thus leading to a branching structure.

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The concurrent nature of Petri net computations has been expressed in an elegant algebraic way in the so-called "Petri nets are monoids" approach [10]. A Petri net N is seen as a kind of signature Σ, and the computational model of the net is characterised as a symmetric monoidal category $\mathcal{P}(N)$ freely generated from N, in the same way as the cartesian category $\mathcal{L}(\Sigma)$ of terms and substitutions is freely generated from Σ. As (tuples of) terms in the free algebra $T_\Sigma(X)$ are arrows of $\mathcal{L}(\Sigma)$, processes of N are arrows of $\mathcal{P}(N)$. The functoriality of the monoidal operator $\otimes$ is shown to capture the essence of concurrency in net computations. The construction of $\mathcal{P}(N)$ provides a concise description of the concurrent operational semantics of P/T nets, and, as $\mathcal{P}(N)$ can be finitely axiomatized, one also gets an axiomatization of deterministic processes.

After the original paper, further proposals for adding suitable operators to the category of (deterministic) net computations were introduced, as summed up in [9]. However, to the best of our knowledge, no explicit connection was drawn from a categorical model to any set-theoretical notion of non-deterministic process, thus re-establishing the same connection as with $\mathcal{P}(N)$ and the deterministic processes of N. In this paper we show how the algebraic approach of [10] can be naturally generalised in order to capture the non-deterministic computations of Petri nets. The algebraic model of a net described above is extended by adding a second monoidal operator $\oplus$ which is intended to exactly model the non-deterministic composition of computations.

The presence of two symmetric monoidal operators $\otimes$ and $\oplus$, where the former distributes over the latter, naturally leads to consider the so-called *bi-monoidal* (or *rig*) categories which, roughly speaking, are the categorical counterpart of semirings (or rigs) pretty much as monoidal categories corresponds to monoids. Additionally, the branching structure of non-deterministic computations is captured by the presence of a natural transformation $\nabla_a : a \to a \oplus a$. As this recalls the idempotency axioms of $\oplus$ in tropical semirings or dioids, we denoted the corresponding categorical structure as a *diodal category*.

More in detail, we introduce a category of concatenable non-deterministic processes $\mathbf{CNP}(N)$ for a Petri net N which generalises the category of deterministic processes originally defined in [2,14]. Then we show that the category of concatenable non-deterministic processes can be characterised as the free diodal category $\mathcal{NP}(N)$ built over N. As a consequence the non-deterministic processes of a net N, as introduced in [3], turn out to be in one to one correspondence with a suitable class of arrows of $\mathcal{NP}(N)$, quotiented under natural axioms.

The rest of the paper is organised as follows. In Section 1, after recalling some basics about Petri nets, we review the notions of deterministic and non-deterministic process. In Section 2 we present the construction of the category of concatenable non-deterministic processes for a Petri net. Section 3 recalls some basic notions on symmetric monoidal categories, introduces diodal categories and presents the main theorem, concerning the correspondence between non-deterministic processes and arrows of a free diodal category. The paper is rounded up with some remarks on our construction and pointers to further works.

1 Petri nets and non-deterministic processes

Given a set X, we denote by $X^{\otimes}$ the free commutative monoid over X (finite multisets over X) with unit the empty set $\emptyset$. Furthermore, given a function $f : X \to Y^{\otimes}$ we denote by $f^{\otimes} : X^{\otimes} \to Y^{\otimes}$ its commutative monoidal extension. Given $u \in X^{\otimes}$, we denote by $[\![u]\!]$ the underlying subset of X defined in the obvious way. When set relations are used over multisets, we implicitly refer to the underlying set. E.g., for $u, v \in X^{\otimes}$ by $x \in u$ we mean $x \in [\![u]\!]$ and similarly $u \cap v$ means $[\![u]\!] \cap [\![v]\!]$.

Definition 1 (P/T net). *A* P/T Petri net *is a tuple $N = (\zeta_0, \zeta_1, S, T)$, where S is a set of* places*, T is a set of* transitions*, and $\zeta_0, \zeta_1 : T \to S^{\otimes}$ are functions assigning multisets called source and target, respectively, to each transition.*

Hereafter, for any net N we assume $N = (\zeta_0, \zeta_1, S, T)$, with subscripts and superscripts carrying over the names of the components. In order to simplify the presentation, we require that for any net N, for all $t \in T$, $\zeta_0(t) \neq \emptyset \neq \zeta_1(t)$.

Given a net N, a multiset $u \in S^{\otimes}$, representing a state of the net, is often referred to as a *marking* of N. It is called *safe* if any place occurs at most once in it, i.e., $u = [\![u]\!]$.

The notion of net morphism naturally arises from an algebraic view, where places and transitions play the role of sorts and operators.

Definition 2 (net morphism). *A* Petri net morphism $f = \langle f_{\mathrm{s}}, f_{\mathrm{t}} \rangle : N \to N'$ *is a pair where $f_{\mathrm{s}} : S^{\otimes} \to S'^{\otimes}$ is a monoid homomorphism, and $f_{\mathrm{t}} : T \to T'$ is a function such that $\zeta'_i \circ f_{\mathrm{t}} = f_{\mathrm{s}} \circ \zeta_i$, for any $i \in \{0, 1\}$. The category of P/T nets (as objects) and their morphisms (as arrows) is denoted by* **Petri***.*

In the sequel, when the meaning is clear from the context, we omit the subscripts from the morphism components, thus writing f instead of f_{s} and f_{t}.

Let N be a P/T net. The *causality relation* is the least transitive relation $<_N \subseteq (S \cup T) \times (S \cup T)$ such that

i. if $s \in \zeta_0(t)$ then $s <_N t$; ii. if $s \in \zeta_1(t)$ then $t <_N s$.

Given a place or transition $x \in S \cup T$, the set of *causes* of x in T is defined as $\lfloor x \rfloor = \{t \in T \mid t <_N x\} \cup \{t\}$; and, for $X \subseteq S \cup T$, $\lfloor X \rfloor = \bigcup_{x \in X} \lfloor x \rfloor$. The *conflict relation* $\#_N \subseteq (S \cup T) \times (S \cup T)$ is the least symmetric relation such that

i. if $t \neq t'$ and $\zeta_0(t) \cap \zeta_0(t') \neq \emptyset$ then $t \#_N t'$;
ii. if $x \#_N x'$ and $x' <_N x''$ then $x \#_N x''$.

Definition 3 (occurrence net). *An* occurrence net *is a P/T net N where $\zeta_o(t)$, $\zeta_1(t)$ are safe for all $t \in T$ and (i) causality $<_N$ is a strict partial order and, for any transition t, the set of causes $\lfloor t \rfloor$ is finite; (ii) there are no backward conflicts, i.e., for any $t \neq t'$, $\zeta_1(t) \cap \zeta_1(t') = \emptyset$; (iii) conflict $\#_N$ is irreflexive. The sets of minimal and maximal places of N w.r.t. $<_N$ are denoted by* $\min(O)$ *and* $\max(O)$*. An occurrence net is* deterministic *if it has no forward conflicts, i.e., for any $t \neq t'$, $\zeta_0(t) \cap \zeta_0(t') = \emptyset$.*

A monomorphism $e : O_1 \to O_2$ such that $e^{\otimes}(\min(O_1)) = \min(O_2)$ is referred to as an embedding *of O_1 in O_2.*

An occurrence O net can be seen as the representation of a possibly non-deterministic computation starting from $\min(O)$. Reachable states in O can be characterised statically by using the dependency relation.

Definition 4 (cuts). *Let O be an occurrence Petri net. A* cut *in O is a maximal subset X of places such that neither $s <_O s'$ nor $s \#_O s'$ for all $s, s' \in X$. The set of cuts of O is denoted by $cuts(O)$. A subset of cuts $W \subseteq cuts(O)$ is called a* covering *of O if $T = \bigcup\{\lfloor X \rfloor : X \in W\}$.*

It can be shown that any cut $X \in cuts(O)$ is reachable by executing all the transitions in $\lfloor X \rfloor$ in any order compatible with $<_O$. The only nonstandard notion is that of covering: if a subset W of cuts is intended to represent the final states of a set of possible computations of O, then W is a covering for O if any possible transition in O is used in one of those computations.

We next review the notion of deterministic and non-deterministic process for P/T nets. A process is represented as a morphism $\pi : O \to N$ from an occurrence Petri net O to the original net N [5]. Since morphisms are simulations, the morphism maps computations of O into computations of N in such a way that the process can be seen as a representative of a set of possible computations of N. The occurrence net makes explicit the causal structure of such computations since each transition is fired at most once and each place is filled with at most one token during each computation. In this way transitions and places of O can be thought of, respectively, as firing of transitions and tokens in places of the original net. Actually, to allow for such an interpretation, some further restrictions have to be imposed on the morphism π, namely it must map places into places (rather than into multisets of places).

Let us call a net morphism $f : N \to N'$ *elementary* if for any $s \in S$, $f_s(s) \in S'$ (places are sent to single places rather than to proper multisets).

Definition 5 (process). *Let N be a P/T net. A* non-deterministic process *of N is an elementary net morphism $\pi : O \to N$ where O is a finite occurrence net and if $\pi(t) = \pi(t')$ and $\zeta_0(t) = \zeta_0(t')$ then $t = t'$ for any $t, t' \in T_O$ (irredundancy).*

The process π is deterministic *if the underlying occurrence net O is so. For a finite process π we write* $\min(\pi)$, $\max(\pi)$ *and $cuts(\pi)$ to refer to the sets* $\min(O)$, $\max(O)$ *and $cuts(O)$ in the underlying occurrence net. We also write $\zeta_0(\pi)$ for $\pi^{\otimes}(\min(\pi))$ and $\zeta_1(\pi)$ for $\pi^{\otimes}(\max(\pi))$.*

Intuitively, a process π represents a set of possible computations starting at the marking $\zeta_0(\pi)$. Not every elementary morphism is a process, as it might fail to satisfy the irredundancy condition, which essentially imposes that the non-deterministic composition of a computation with itself gives back the original computation [3]. However we can easily show that the following result holds.

Proposition 1 (collapsing). *Let N be a P/T net, O an occurrence net and $\xi : O \to N$ an elementary morphism. Then there exists a unique (up to isomorphism) factorisation $\xi = \beta; \pi(\xi)$, where β is epi and $\pi(\xi)$ is a process such that $\zeta_0(\pi(\xi)) = \xi^{\otimes}(\min(O))$. The process $\pi(\xi)$ is called the* collapsing *of ξ.*

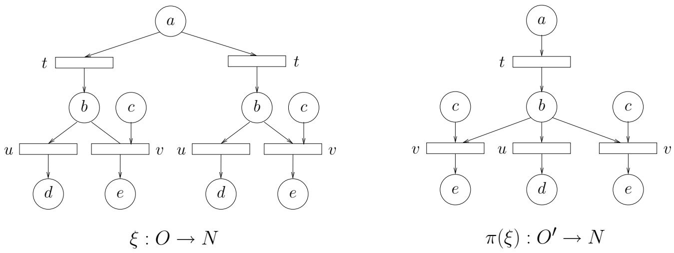

Fig. 1. An elementary morphism and its collapsing.

Intuitively, the collapsing of ξ is obtained from ξ by merging pairs of transitions which violate the irredundancy requirement. As an example, Fig. 1 presents on the left an elementary morphism from an occurrence net to a Petri net, and on the right its collapsing. The morphism is represented by labelling places and transitions of the occurrence net by their images over N (and the net N which they are mapped to is not relevant here and thus omitted).

2 Concatenable processes

In this section, after reviewing the theory of concatenable deterministic process [2,14], we propose a notion of concatenable non-deterministic process. This leads to a category $\mathbf{CNP}(N)$ of non-deterministic processes for a net N, where objects are states and arrows model non-deterministic computations of N.

2.1 Concatenable deterministic processes

Concatenable processes for Petri nets have been introduced in [2,14] as a refinement of Goltz-Reisig deterministic processes, endowed with operations of sequential and parallel composition that are consistent with the causal structure of computations. In order to properly define such operations we need to impose a suitable ordering over the places in $\min(\pi)$ and $\max(\pi)$ for each process π. Such an ordering allows to distinguish among "interface" places of O_π which are mapped to the same place of the original net, a capability which is essential to track causal dependencies.

Definition 6. *Let A, B be sets and $f : A \to B$ a function. An* f-indexed ordering *is a family $\alpha = \{\alpha_b \mid b \in B\}$ of bijections $\alpha_b : f^{-1}(b) \to [|f^{-1}(b)|]$, where $[i]$ denotes the subset $\{1, \ldots, i\}$ of $\mathbb{N}$, and $f^{-1}(b) = \{a \in A \mid f(a) = b\}$.*

The f-indexed ordering α is often identified with the function from A to $\mathbb{N}$ that it naturally induces (formally defined as $\bigcup_{b \in B} \alpha_b$). Let $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$, with $A_1 \cap A_2 = \emptyset$, so that $f = f_1 \cup f_2 : A_1 \cup A_2 \to B$ is a function. Consider two f_i-indexed orderings α_i, $i \in \{1, 2\}$. Then we denote by $\alpha_1 \otimes \alpha_2$ the f-indexed ordering defined by $\alpha_1 \otimes \alpha_2(a) = \alpha_1(a)$ if $a \in A_1$ and $\alpha_1 \otimes \alpha_2(a) = \alpha_2(a) + |f_1^{-1}(f_2(a))|$, otherwise.

Definition 7 (concatenable process). *A* concatenable process *of a net N is a triple $\delta = \langle \mu, \pi, \nu \rangle$, where π is a deterministic process of N, μ is a π-indexed ordering of* $\min(\pi)$ *and ν is a π-indexed ordering of* $\max(\pi)$.

Isomorphism of concatenable processes is defined in the usual way (see e.g. [2]) and an isomorphism class of processes is called *(abstract) concatenable process* and denoted by $[\delta]$, for δ is a member of the class. Often the word "abstract" is omitted and δ denotes the corresponding isomorphism class.

Definition 8 (sequential and parallel composition). *Let $\delta_1 = \langle \mu_1, \pi_1, \nu_1 \rangle$ and $\delta_2 = \langle \mu_2, \pi_2, \nu_2 \rangle$ be two concatenable processes of a net N.*

- *Let $\zeta_1(\pi_1) = \zeta_0(\pi_2)$. Suppose $T_1 \cap T_2 = \emptyset$ and $S_1 \cap S_2 = \max(\pi_1) = \min(\pi_2)$, with $\pi_1(s) = \pi_2(s)$ and $\nu_1(s) = \mu_2(s)$ for each $s \in S_1 \cap S_2$. Then $\delta_1; \delta_2$ is the concatenable process $\delta = \langle \mu_1, \pi, \nu_2 \rangle$, where the process π is the (component-wise) union of π_1 and π_2.*
- *Suppose $T_1 \cap T_2 = S_1 \cap S_2 = \emptyset$. Then $\delta_1 \otimes \delta_2$ is the concatenable process $\delta = \langle \mu, \pi, \nu \rangle$, where the process π is the (component-wise) union of π_1 and π_2, $\mu = \mu_1 \otimes \mu_2$ and $\nu = \nu_1 \otimes \nu_2$.*

The premise of the first item means that δ_1 and δ_2 overlap only on $\max(\pi_1) = \min(\pi_2)$, and on such places the labelling on the original net and the ordering coincide. Then, their concatenation is the process obtained by gluing the maximal places of π_1 and the minimal places of π_2 according to their ordering. Parallel composition is instead obtained simply by juxtaposing the two processes.

Concatenation and parallel composition clearly induce well-defined operations on abstract processes, independent of the choice of representatives.

Definition 9 (category of concatenable processes). *Let N be a net. The* category of (abstract) concatenable processes *of N, denoted by $\mathbf{CP}(N)$, is defined as follows. Objects are multisets of places of N, namely elements of $S^{\otimes}$. Each (abstract) concatenable process $[\langle \mu, \pi, \nu \rangle]$ of N is an arrow from $\zeta_0(\pi)$ to $\zeta_1(\pi)$. Parallel composition $\otimes$ makes $\mathbf{CP}(N)$ a symmetric monoidal category.*

2.2 Concatenable non-deterministic processes

Intuitively, a concatenable non-deterministic process is a set of non-deterministic processes, which, starting from a set of possible initial states, produces a set of possible final states. For technical reasons, it is preferable to consider sequences of processes rather than sets. Additionally, as in the deterministic case, in order to allow for a sequential composition of computations keeping track of the causal dependencies, initial and final states of computations are decorated.

Definition 10 (concatenable non-deterministic process). *Let N be a net. A* concatenable non-deterministic process *for N is a triple of finite non-empty lists $\eta = \langle \alpha, \boldsymbol{\pi}, \omega \rangle$ with*

- *$\boldsymbol{\pi} = \pi_1 \ldots \pi_n$, where each π_i is a non-deterministic process;*

- $\alpha = \alpha_1 \dots \alpha_n$, *where each* α_i *is a* π_i*-indexed ordering of* $\min(\pi_i)$*;*
- $\omega = \omega_1 \dots \omega_\ell$, *where*
  - *for each* $j \in \{1, \dots, \ell\}$, $\omega_j : X \to \mathbb{N}$ *with* $X \in \mathit{cuts}(\pi_i)$ *for some* i *and* ω_j *a* π_i*-indexed ordering of* X*;*
  - *for any* i *the cuts of* O_i *which occur in* ω*, i.e.,* $\{X \in \mathit{cuts}(\pi_i) \mid \exists j.\, w_j : X \to \mathbb{N}\}$*, are a covering of* O_i*.*

The source of η *is the list* $\zeta_0(\eta) = \zeta_0(\pi_1) \dots \zeta_0(\pi_n)$*, i.e., the list of the sources of the component processes, while the target of* η *is* $\zeta_1(\eta) = u_1 \dots u_\ell$ *where* $u_j = \pi_i^{\otimes}(X)$ *if* $\omega_j : X \to \mathbb{N}$ *and* $X \in \mathit{cuts}(O_i)$*.*

In order to ease notation we fix a naming scheme. We assume concatenable non-deterministic processes to be of the kind $\eta = \langle \alpha, \boldsymbol{\pi}, \omega \rangle$, with $\boldsymbol{\pi} = \pi_1 \dots \pi_n$ and $n = |\boldsymbol{\pi}|$. In turn, for each process π_i in $\boldsymbol{\pi}$ we assume $\pi_i : O_i \to N$, where O_i has S_i and T_i as place and transition sets. Processes π_i are supposed to be pairwise disjoint. Superscripts carry over the name of the components.

Two concatenable non-deterministic processes $\eta = \langle \alpha, \boldsymbol{\pi}, \omega \rangle$, $\eta' = \langle \alpha', \boldsymbol{\pi}', \omega' \rangle$ are *isomorphic* if $|\boldsymbol{\pi}| = |\boldsymbol{\pi'}|$ and there exist non-deterministic process isomorphisms between π_i and π'_i, with $i \in \{1, \dots, |\boldsymbol{\pi}|\}$, consistent with the decorations and the ordering of sources and targets. Abstract concatenable non-deterministic processes, i.e., isomorphism classes of processes, are often identified with one of the representatives, i.e., we write η to refer to the corresponding abstract process.

Graphically, a concatenable non-deterministic process $\eta = \langle \alpha, \boldsymbol{\pi}, \omega \rangle$, with $\boldsymbol{\pi} = \langle \pi_1 \dots \pi_n \rangle$, is represented by enclosing in a box the list of the nets $O_1, \dots, O_n$, underlying the component subprocesses, separated by vertical bars. Places and transitions of O_i are labelled by their images through π_i (the net N which they are mapped to is not relevant here and thus omitted). The decoration of the source of each process π_i is represented by listing on the top of the process itself the places in $\min(\pi_i)$ in an order compatible with α_i, i.e., if $s, s' \in \min(O_i)$ and $\pi_i(s) = \pi_i(s')$ and $\alpha_i(s) < \alpha_i(s')$ then s is listed first. Similarly, in the bottom part of the box, we represent $w = w_1 \dots \omega_\ell$ as a list of elements, one for each ω_j. If $\omega_j : X \to \mathbb{N}$, with $X \in \mathit{cuts}(\pi_i)$, then the corresponding element is itself a sequence which lists the places in X in an order compatible with ω_j. A process $\eta = \langle \alpha_1\alpha_2, \pi_1\pi_2, \omega_1\omega_2\omega_3 \rangle$ consisting of two component processes π_1 and π_2, with three targets can be found in Fig. 2. In this case, for instance, α_1 is the function $\alpha_1(s_1) = 0$, $\alpha_1(s_2) = 0$ and $\alpha_1(s_3) = 1$. Concerning the targets, $\{s_5, s_3\}, \{s_6, s_7\} \in \mathit{cuts}(\pi_1)$ and $\{s_{10}\} \in \mathit{cuts}(\pi_2)$. It is easy to see that the cuts $\{s_5, s_3\}, \{s_6, s_7\}$ form a covering of O_1, and similarly $\{s_{10}\}$ is a covering for O_2.

Sequential and parallel composition for concatenable non-deterministic processes can be defined as follows.

Definition 11 (sequential composition). *Let* $\eta = \langle \alpha, \boldsymbol{\pi}, \omega \rangle$ *and* $\eta' = \langle \alpha', \boldsymbol{\pi}', \omega' \rangle$ *be two concatenable non-deterministic processes of a net* N *such that* $\zeta_1(\eta) = \zeta_0(\eta')$*, and thus* $|\omega| = |\alpha'|$*. Suppose that for any* i, j *it holds* $T_i \cap T'_j = \emptyset$ *and, for all* j*, if* $\omega_j : X \to \mathbb{N}$*, with* $X \in \mathit{cuts}(\pi_i)$ *then* $S_i \cap S'_j = X = \min(\pi'_j)$*, with* $\pi_i(s) = \pi'_j(s)$ *and* $\omega_i(s) = \alpha'_j(s)$ *for each* $s \in X$*. Then* $\eta_1; \eta_2$ *is the concatenable process* $\eta = \langle \alpha, \boldsymbol{\pi}'', \omega' \rangle$*, where* $\boldsymbol{\pi}'' = \pi''_1 \dots \pi''_{|\boldsymbol{\pi}|}$ *and each process* π''_i *is*

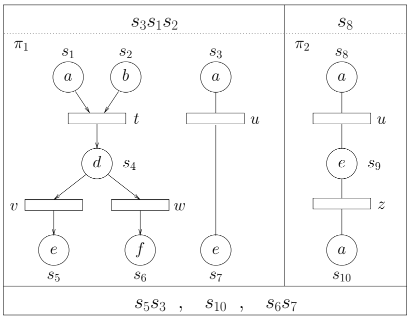

Fig. 2. A concatenable non-deterministic process.

obtained as follows: take the (component-wise) union of π_i *with all processes* π'_j *such that* $\omega_j : X \to \mathbb{N}$ *with* $X \in cuts(\pi_i)$ *thus getting an elementary morphism* $\xi_i : O''_i \to N$ *and then consider the collapsing* $\pi(\xi_i)$ *of such morphism.*

Roughly, for any $j \in \{1, \ldots, |\omega|\}$, if $\omega_j : X \to \mathbb{N}$ where X is a cut in π_i, then the process π'_j in η' must be attached to the set of places X in π_i. Assuming that π_i and π'_j overlap only on $X = \min(\pi'_j)$, and on such places the labelling on the original net and the ordering imposed by the two processes coincide, then attaching π_j to π_i reduces to taking their component-wise union. Therefore the composition has $|\boldsymbol{\pi}|$ components, each one obtained as the component-wise union of each π_i with all π'_j which must be connected to π_i.

Definition 12 (parallel composition). *Let* $\eta = \langle \alpha, \boldsymbol{\pi}, \omega \rangle$ *and* $\eta' = \langle \alpha', \boldsymbol{\pi}', \omega' \rangle$ *be two concatenable non-deterministic processes. Suppose* $|\boldsymbol{\pi}| = n$, $|\boldsymbol{\pi}'| = n'$, *and* $T_i \cap T'_j = S_i \cap S'_j = \emptyset$ *for any* i, j. *Then* $\eta \otimes \eta'$ *is the concatenable process* $\eta'' = \langle \alpha'', \boldsymbol{\pi}'', \omega'' \rangle$, *with*

$$\boldsymbol{\pi}'' = \pi_{1,1} \ldots \pi_{n,1} \pi_{1,2} \ldots \pi_{n,2} \ldots \pi_{n',1} \ldots \pi_{n',n}$$

where each $\pi_{i,j}$ *is the (component-wise) union of* π_i *and* π'_j. *Similarly* $\alpha'' = \alpha_{1,1} \ldots \alpha_{n',n}$ *with* $\alpha_{i,j} = \alpha_i \otimes \alpha'_j$ *and* $\omega'' = \omega_{1,1} \ldots \omega_{\ell',\ell}$ *with* $\omega_{i,j} = \omega_i \otimes \omega'_j$.

Note that when composing in parallel two non-deterministic processes η and η', we compose each component of η with each component of η'. Intuitively, this means that parallel composition distributes over non-deterministic composition.

Finally, we can easily define a notion of non-deterministic composition, which is obtained by juxtaposing the two processes.

Definition 13 (non-deterministic composition). *Let* $\eta = \langle \alpha, \boldsymbol{\pi}, \omega \rangle$ *and* $\eta' = \langle \alpha', \boldsymbol{\pi}', \omega' \rangle$ *be concatenable non-deterministic processes. Then* $\eta \oplus \eta' = \langle \alpha\alpha', \boldsymbol{\pi}\boldsymbol{\pi}', \omega\omega' \rangle$, *where the juxtaposition of two lists denotes their concatenation.*

It can be shown that, as in the deterministic case, concatenation and parallel composition induce well-defined operations on abstract processes, independent of the particular choice of the representatives.

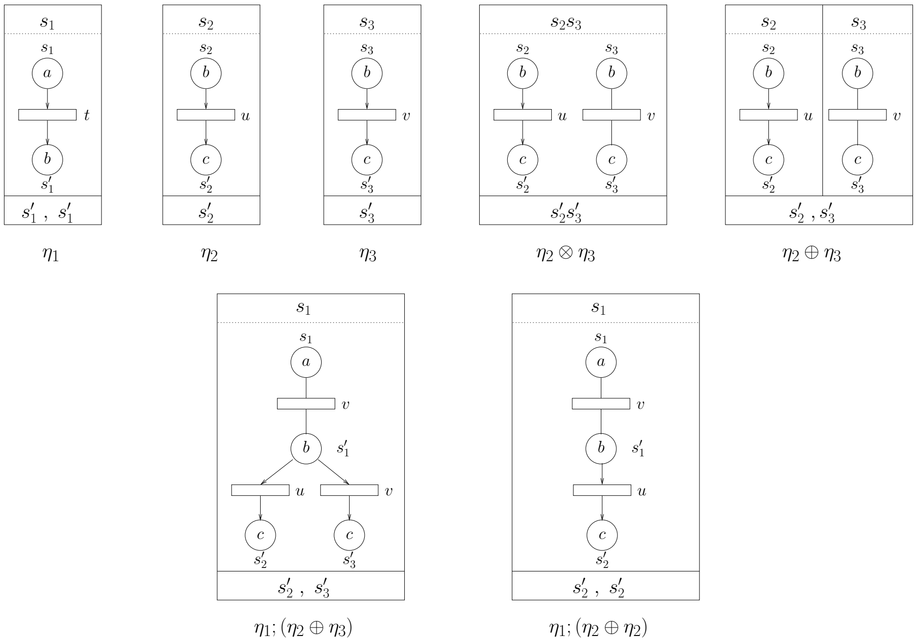

Fig. 3. Three simple concatenable non-deterministic processes η_1, η_2, η_3 and some processes arising from their composition.

As an example consider the three simple processes η_1, η_2, η_3 in Fig. 3, consisting of one transition only. Note that η_1 nondeterministically offers two copies of s'_1 as target. The same figure reports the parallel composition $\eta_2 \otimes \eta_3$, the non-deterministic composition $\eta_2 \oplus \eta_3$, the processes $\eta_1; (\eta_2 \oplus \eta_3)$ and $\eta_1; (\eta_2 \oplus \eta_2)$. For the last process observe that the two non-deterministic copies of u are joined as an effect of the composition (yet the composite process still non-deterministically offers two copies of s'_2 as target).

Definition 14 (category of concatenable non-deterministic processes). *Let N be a net. The* category of (abstract) concatenable non-deterministic processes *of N, denoted by $\mathbf{CNP}(N)$, is defined as follows. Objects are finite non-empty lists of elements of $S^{\otimes}$. Each (abstract) concatenable non-deterministic process of N is an arrow. Both parallel $\otimes$ and non-deterministic $\oplus$ composition make $\mathbf{CNP}(N)$ a symmetric monoidal category.*

Obviously, the non-deterministic processes of a net N, as given in Definition 5, correspond to the arrows of $\mathbf{CNP}(N)$ consisting of a single process $\eta = \langle \alpha_1, \pi_1, \omega \rangle$, once we forget the decoration.

3 Embedding processes into terms

This section presents the main result of the paper, namely, the description of the abstract concatenable non-deterministic processes of a net N, as defined in Section 2, as terms of a suitable algebra. Along the *Petri nets are monoids* paradigm, this is a sort of monoidal category, freely generated from the net itself.

3.1 Categorical notions

Here we introduce the relevant categorical notions that are needed for the algebraic description of processes. Most definitions are standard: for the presentation of monoidal categories we closely follow [1].

Definition 15 (monoidal categories). *A (strict)* monoidal category *is a triple $\langle \mathcal{C}, \_ \oplus \_, e \rangle$, where $\mathcal{C}$ is the underlying category, the* tensor product *$\_ \oplus \_ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a functor satisfying the law $(t_1 \oplus t_2) \oplus t_3 = t_1 \oplus (t_2 \oplus t_3)$, and e is an object of $\mathcal{C}$ satisfying the law $t \oplus e = t = e \oplus t$, for all arrows $t, t_1, t_2, t_3 \in \mathcal{C}$.*

A symmetric monoidal category *is a 4-tuple $\langle \mathcal{C}, \_ \oplus \_, e, \gamma \rangle$, where $\langle \mathcal{C}, \_ \oplus \_, e \rangle$ is a monoidal category, and $\gamma : \_{1} \oplus \_{2} \Rightarrow \_{2} \oplus \_{1} : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a natural isomorphism*[3] *satisfying the coherence axioms $\gamma_{a,e} = a$ and*

$$
\begin{tikzcd}
a \oplus b \oplus c \arrow[r, "a \oplus \gamma_{b,c}"] \arrow[dr, "\gamma_{a \oplus b, c}"'] & a \oplus c \oplus b \arrow[d, "\gamma_{a,c} \oplus b"] \\
 & c \oplus a \oplus b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \oplus b \arrow[r, "\gamma_{a,b}"] \arrow[dr, "a \oplus b"'] & b \oplus a \arrow[d, "\gamma_{b,a}"] \\
 & a \oplus b
\end{tikzcd}
$$

A i-monoidal category *is a 5-tuple $\langle \mathcal{C}, \_ \oplus \_, e, \gamma, \nabla \rangle$, where $\langle \mathcal{C}, \_ \oplus \_, e, \gamma \rangle$ is a symmetric monoidal category and $\nabla : \_{1} \Rightarrow \_{1} \oplus \_{1} : \mathcal{C} \longrightarrow \mathcal{C}$ is a natural transformation satisfying the coherence axioms $\nabla_e = e$ and*

$$
\begin{tikzcd}
a \arrow[r, "\nabla_a"] \arrow[d, "\nabla_a"'] & a \oplus a \arrow[d, "a \oplus \nabla_a"] \\
a \oplus a \arrow[r, "\nabla_a \oplus a"'] & a \oplus a \oplus a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\nabla_a"] \arrow[dr, "\nabla_a"'] & a \oplus a \arrow[d, "\gamma_{a,a}"] \\
 & a \oplus a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \oplus b \arrow[r, "\nabla_a \oplus \nabla_b"] \arrow[dr, "\nabla_{a \oplus b}"'] & a \oplus a \oplus b \oplus b \arrow[d, "a \oplus \gamma_{a,b} \oplus b"] \\
 & a \oplus b \oplus a \oplus b
\end{tikzcd}
$$

While symmetric monoidal categories are a staple of theoretical computer science, at least since the seminal work by Meseguer and Montanari [10], we introduced i-monoidality in order to capture the idempotency of the additive operator. Making each object s a cosemigroup object (not yet a comonoid object, since the arrow $s \to e$ is missing [8]) and requiring the naturality of ∇ are suggested by the need of equating somehow the addition of terms, yet banning the identity $t = t \oplus t$: we offer further remarks in the concluding section.

[3] Given functors $F, G : \mathcal{A} \to \mathcal{B}$, a *natural transformation* $\tau : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$ is a family of arrows of $\mathcal{B}$ indexed by objects of $\mathcal{A}$, $\tau = \{\tau_a : F(a) \to G(a) \mid a \in O_{\mathcal{A}}\}$, such that for every arrow $f : a \to a'$ in $\mathcal{A}$, $\tau_a; G(f) = F(f); \tau_{a'}$ in $\mathcal{B}$. We say that τ is an *isomorphism* if all its components τ_a's are so.

Definition 16 (diodal categories). *A* bimonoidal category *is a 8-tuple* $\langle \mathcal{C}, \_\oplus\_, e, \gamma, \_\otimes\_, o, \rho, \delta\rangle$*, where* $\langle \mathcal{C}, \_\oplus\_, e, \gamma\rangle$ *and* $\langle \mathcal{C}, \_\otimes\_, o, \rho\rangle$ *are symmetric monoidal categories satisfying the law* $t \otimes e = e$ *for all arrows* $t \in \mathcal{C}$ *and the coherence axiom* $\rho_{a,e} = e$*, and* $\delta : \_{1} \otimes (\_{2} \oplus \_{3}) \Rightarrow (\_{1} \otimes \_{2}) \oplus (\_{1} \otimes \_{3}) : \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \longrightarrow \mathcal{C}$ *is a natural isomorphism satisfying the axioms* $\delta_{o,b,c} = b \oplus c$*,* $\delta_{a,b,e} = a \otimes b$ *and*

$$\begin{tikzcd} a\otimes(b\oplus c) \arrow[r, "\delta_{a,b,c}"] \arrow[d, "a\otimes\gamma_{b,c}"'] & (a\otimes b)\oplus(a\otimes c) \\ a\otimes(c\oplus b) \arrow[r, "\delta_{a,c,b}"'] & (a\otimes c)\oplus(a\otimes b) \arrow[u, "\gamma_{a\otimes c, a\otimes b}"'] \end{tikzcd}$$

Finally, a diodal category *is a 9-tuple* $\langle \mathcal{C}, \_\oplus\_, o, \rho, \nabla, \_\otimes\_, e, \gamma, \delta\rangle$*, where* $\langle \mathcal{C}, \_\oplus\_, o, \rho, \nabla\rangle$ *is a i-monoidal category and* $\langle \mathcal{C}, \_\oplus\_, o, \rho, \_\otimes\_, e, \gamma, \delta\rangle$ *is a bimonoidal category, satisfying the coherence axiom*

$$\begin{tikzcd} a\otimes b \arrow[r, "a\otimes\nabla_b"] \arrow[dr, "\nabla_{a\otimes b}"'] & a\otimes(b\oplus b) \arrow[d, "\delta_{a,b,b}"] \\ & (a\otimes b)\oplus(a\otimes b) \end{tikzcd}$$

Bimonoidal categories, and their coherence laws, have been considered quite early on in the literature [7]. Recently they surfaced, sometimes with the name *rig* or *semiring* categories, in the definition of models for quantum programming [6].

We introduced diodal categories in order to obtain a categorical counterpart of dioids, i.e., semirings where the additive operator is idempotent. In the following, we consider diodal categories satisfying an additional requirement.

Definition 17 (bipermutative and dipermutative categories). *A* bipermutative *category is a bimonoidal category such that* δ *is an identity, so that the objects* $a\otimes(b\oplus c)$ *and* $(a\otimes b)\oplus(a\otimes c)$ *coincide; and moreover* $\rho_{a,b\oplus c} = \rho_{a,b} \oplus \rho_{a,c}$*.* Dipermutative categories *are diodal categories based on bipermutative categories.*

3.2 Categories of processes

In this part we introduce a concrete category, out of the transitions of the net, and we prove that it forms a diodal category. More importantly, those arrows exactly correspond to non-deterministic processes, along the lines of the characterisation of deterministic processes via the category $\mathcal{P}(N)$ in [2].

Notation. Given a monoid $\langle M, \otimes, 1\rangle$, we denote by $M^{\oplus}$ the free monoid over M (finite non-empty lists over M): the unit of $\oplus$ is the list $\langle 1\rangle$ containing only the unit of the monoid. Note that the $\otimes$ operator can be extended set-wise to the monoid $M^{\oplus}$, resulting in a semiring (not yet a dioid, since $\oplus$ is not idempotent). So, assuming that M is $X^{\otimes}$, the resulting structure is denoted simply as $X^{\otimes,\oplus}$, and it coincides with the free $\otimes$-commutative semiring on X.

$$\frac{s \in S_N^{\otimes,\oplus}}{id_s : s \to s \in \mathcal{NP}(N)} \qquad \frac{t \in T_N}{t : \zeta_0(t) \to \zeta_1(t) \in \mathcal{NP}(N)} \qquad \frac{s, s' \in S_N^{\otimes,\oplus}}{\rho_{s,s'} : s \otimes s' \to s' \otimes s \in \mathcal{NP}(N)}$$

$$\frac{t : s \to s', t_1 : s' \to s_1 \in \mathcal{NP}(N)}{t; t_1 : s \to s_1 \in \mathcal{NP}(N)} \qquad \frac{t : s \to s', t_1 : s_1 \to s'_1 \in \mathcal{NP}(N)}{t \otimes t_1 : s \otimes s_1 \to s' \otimes s'_1 \in \mathcal{NP}(N)}$$

Fig. 4. The deterministic fragment of the set of inference rules generating $\mathcal{NP}(N)$.

$$t; id_{s'} = t = id_s; t \qquad t = t \otimes id_\emptyset \qquad (t_1 \otimes t_2); (t_3 \otimes t_4) = (t_1; t_3) \otimes (t_2; t_4)$$

$$\rho_{s,s'}; \rho_{s',s} = id_{s \otimes s'} = id_s \otimes id_{s'} \qquad \rho_{s,s' \otimes s''} = (id_s \otimes \rho_{s',s''}); (\rho_{s,s''} \otimes id_{s'}) \qquad \rho_{s,\emptyset} = id_s$$

$$\rho_{s_1,s_2}; (t_1 \otimes t_2) = (t_2 \otimes t_1); \rho_{s'_2,s'_1} \qquad \rho_{a,b} = id_{a \otimes b} \text{ for } a \neq b \in S_N \qquad \rho; t = t; \rho' \text{ for } t \in T_N$$

Fig. 5. The set of axioms for deterministic processes quotienting $\mathcal{DP}(N)$.

Definition 18 (a category for deterministic processes). *Let N be a P/T net. Then, $\mathcal{DP}(N)$ is the category whose objects are markings of N (i.e., elements of $S_N^{\otimes}$), while the arrows are freely generated according to the rules in Fig. 4, subject to the axioms in Fig. 5.*[4]

Since the composition operator ; is partial, axioms in Fig. 5 hold when both sides are defined; additionally, note that a, b denote places in S_N, instead of elements of $S_N^{\otimes}$. The objects of $\mathcal{DP}(N)$ are thus markings of N, representing sources and targets of deterministic processes. Its arrows are equivalence classes of concrete elements generated by the set of inference rules in Fig. 4, modulo the equations making it a symmetric monoidal category.

The further equations $\rho_{a,b} = id_a \otimes id_b$ and $\rho; t = t; \rho'$ (for permutations ρ, ρ', i.e., arrows built out of identities and $\rho_{a,b}$'s) represent a well-known idiosyncrasy of the concrete representation of deterministic processes [13], so that e.g. for transitions t_1 and t_2 with distinct sources and targets, the processes $t_1 \otimes t_2$ and $t_2 \otimes t_1$ have to be identified, since, as discussed for concrete processes, the order of distinct places is irrelevant. Analogous issues appear in the category below, extending the former in order to include non-determinism.

Definition 19 (a category for non-deterministic processes). *Let N be a P/T net. Then, $\mathcal{NP}(N)$ is the category whose objects are finite non-empty lists of markings of N (i.e., elements of $S_N^{\otimes,\oplus}$), while the arrows are freely generated according to the rules in Fig. 4 and 6, subject to the axioms in Fig. 5 and 7.*

Given a net N, the objects of $\mathcal{NP}(N)$ are lists of markings of N, each one representing the source of one non-deterministic component of the non-deterministic

[4] For the sake of space saving, we overloaded some symbols, so that for the current definition $S_N^{\otimes,\oplus}$ and $\mathcal{NP}(N)$ in Fig. 4 should read as $S_N^{\otimes}$ and $\mathcal{DP}(N)$.

$$\frac{s, s' \in S_N^{\otimes\oplus}}{\gamma_{s,s'} : s \oplus s' \to s' \oplus s \in \mathcal{NP}(N)} \qquad \frac{s \in S_N^{\otimes,\oplus}}{\nabla_s : s \to s \oplus s \in \mathcal{NP}(N)}$$

$$\frac{t : s \to s', t_1 : s_1 \to s_1' \in \mathcal{NP}(N)}{t \oplus t_1 : s \oplus s_1 \to s' \oplus s_1' \in \mathcal{NP}(N)}$$

Fig. 6. The inference rules for non-determinism generating $\mathcal{NP}(N)$.

$$t = t \oplus id_{\langle\emptyset\rangle} \qquad (t_1 \oplus t_2);(t_3 \oplus t_4) = (t_1;t_3) \oplus (t_2;t_4)$$

$$\gamma_{s,s'};\gamma_{s',s} = id_{s\oplus s'} = id_s \oplus id_{s'} \qquad \gamma_{s,s'\oplus s''} = (id_s \oplus \gamma_{s',s''});(\gamma_{s,s''} \oplus id_{s'}) \qquad \gamma_{s,\langle\emptyset\rangle} = id_s$$

$$\nabla_s;\gamma_{s,s} = \nabla_s \quad \nabla_s;(id_s \otimes \nabla_s) = \nabla_s;(\nabla_s \otimes id_s) \quad \nabla_{s\oplus s} = (\nabla_s \oplus \nabla_s);(id_s \oplus \gamma_{s,s} \oplus id_s)$$

$$\gamma_{s_1,s_2};(t_1 \oplus t_2) = (t_2 \oplus t_1);\gamma_{s_2',s_1'} \quad \nabla_{\langle\emptyset\rangle} = id_{\langle\emptyset\rangle} = \rho_{s,\langle\emptyset\rangle} \quad id_s \otimes \nabla_{s'} = \nabla_{s\otimes s'}$$

$$t \otimes (t_1 \oplus t_2) = (t \otimes t_1) \oplus (t \otimes t_2) \qquad \rho_{s,s'\oplus s''} = \rho_{s,s'} \oplus \rho_{s,s''}$$

Fig. 7. The set of axioms quotienting $\mathcal{NP}(N)$.

process. Instead, arrows are equivalence classes of elements generated by the inference rules in Fig. 4 and 6, modulo a set of equations making it a dipermutative category. Note the lack of the equation $\gamma_{a,b} = id_{a\oplus b}$.

The objects of $\mathcal{N}P(N)$ are obtained by constructing the free $\otimes$-commutative semiring, out of the initial set of places of the net N. As for arrows, the analogy with the semiring construction out of a monoid is confirmed by the characterization result stated below. For a marking s, we let s^k denote the k-times composition $s \oplus \ldots \oplus s$; while we let ∇_s^k denote the unique arrow with source s and target s^k, inductively built as $\nabla_s^1 = id_s$ and $\nabla_s^{k+1} = \nabla_s^k;(id_s \oplus \nabla_s)$.

Proposition 2. *Let $s_1, \ldots, s_l \in S_N^{\otimes}$ and $t \in \mathcal{NP}(N)$ an arrow with source $s_1 \oplus \ldots \oplus s_l$. Then, t can be decomposed as $(\nabla_{s_1}^{k_1} \oplus \ldots \oplus \nabla_{s_l}^{k_l});\gamma;(t_1 \oplus \ldots \oplus t_n)$, for $n = k_1 + \ldots + k_l$, γ a permutation and t_i's in $\mathcal{DP}(N)$.*

The permutation γ is just an arrow built out of identities and $\gamma_{a,b}$'s. The normal form can be proved unique, up-to a syntactic ordering on arrows. The corollary below exploits the axiom equating $\nabla_s;\gamma_{s,s}$ to ∇_s for $s \in S_N^{\otimes}$.

Corollary 1. *Let $s \in S_N^{\otimes}$ and $t \in \mathcal{NP}(N)$ an arrow with source s. Then, t can be decomposed as $\nabla_s^k;(t_1 \oplus \ldots \oplus t_k)$, for t_i's in $\mathcal{DP}(N)$.*

As shown above, and hinted at in the beginning of the section, the insertion of the $\oplus$ operator mimics the well-known generation of a semiring from a monoid. The arrows of the resulting category $\mathcal{NP}(N)$ can indeed be seen as suitable list of arrows of $\mathcal{DP}(N)$. Recalling that $\mathcal{DP}(N)$, the sub-category obtained by restricting to the $\otimes$-fragment of $\mathcal{NP}(N)$, coincides with the symmetric category $\mathcal{P}(N)$ of deterministic processes [2,13], we can view arrows in $\mathcal{NP}(N)$ as lists of deterministic processes. This fact is deepened and formalised in the next section.

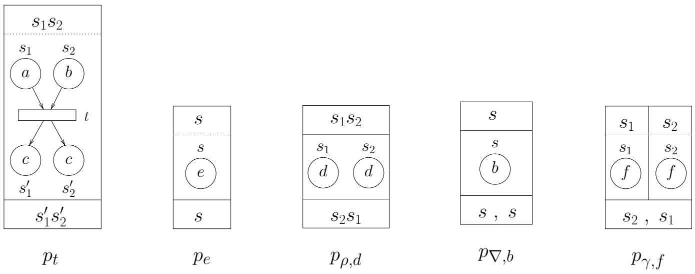

Fig. 8. Basic concatenable processes.

3.3 Processes as terms

Let us begin the section by recalling the main result concerning $\mathcal{DP}(N)$ and the category of concatenable (deterministic) processes.

For stating this result and its generalisation to the non-deterministic case we need the five basic processes represented in Fig. 8. In the discussion t represents a generic transition of a fixed net N and a, b, c, d, e, f are names for places. Any transition t can be seen as a concatenable (deterministic) process p_t. As an example, on the far left of the figure, we have a representation of the process p_t, for a transition t such that $\zeta_0(t) = a \otimes b$ and $\zeta_1(t) = c \otimes c$. Next, there is the representation of p_e, the unique (deterministic) process with no transitions from e to itself. Process $p_{\rho,d}$ is the deterministic process from $d \otimes d$ to itself, simply swapping the multiset ordering. Then $p_{\nabla,b}$ is the non-deterministic processes consisting of one place b only, with source b and as target twice the maximal cut $\{b\}$, i.e., $b \oplus b$. Finally, $p_{\gamma,f}$ represents the permutation for the two underlying identity processes: source and target are f, f.

Proposition 3 (deterministic correspondence [13]). *Let N be a net. The function $\mathcal{C}_N$ from the class of generating arrows of the category $\mathcal{DP}(N)$ to the class of basic processes of N, defined by*

$$\mathcal{C}_N(id_a) = p_a \text{ and } \mathcal{C}_N(\rho_{a,a}) = p_{\rho,a} \text{ for } a \in S_N$$

$$\mathcal{C}_N(t) = p_t \text{ for } t \in T_N$$

lifts to a full and faithful (symmetric monoidal) functor $\mathcal{P}_N : \mathcal{DP}(N) \to \mathbf{CP}(N)$.

Note that the functor induces a bijective correspondence between the arrows of the category $\mathcal{DP}(N)$ and the concatenable (deterministic) processes of the net N itself. Finally, our main result is now stated below.

Theorem 1 (non-deterministic correspondence). *Let N be a net. The function $\mathcal{CN}_N$ from the class of generating arrows of the category $\mathcal{NP}(N)$ to the class of basic processes of N, defined by extending $\mathcal{C}_N$ with*

$$\mathcal{CN}_N(\nabla_a) = p_{\nabla,a} \text{ and } \mathcal{CN}_N(\gamma_{a,a}) = p_{\gamma,a} \text{ for } a \in S_N$$

lifts to a full and faithful (diodal) functor $\mathcal{NP}_N : \mathcal{NP}(N) \to \mathbf{CNP}(N)$.

The theorem clearly exploits the decomposition result discussed in Proposition 2. For our purposes, it basically states that the arrows of $\mathcal{NP}(N)$ do capture the essence of the non-deterministic processes of a net. Note that the introduction of concatenable non-deterministic processes is indeed pivotal, since e.g. the encodings $\mathcal{CN}_N(\nabla_s; (t \oplus id_s))$ and $\mathcal{CN}_N(t)$ have the same underlying process, even if the decoration of their targets differ.

4 Conclusions and further works

Along the lines of the seminal paper [10], our work offered an algebraic presentation for non-deterministic processes of Petri nets.

A first contribution of our work is the introduction of the concatenable version of non-deterministic processes, building on the original proposal by Engelfriet [3]. To the best of our knowledge, also putting diodal categories into the limelight represents a small addition to the categorical lore. With respect to former proposals for the categorical characterization of non-determinism, our solution closely recalls *linear categories* [9]: our diodal categories lack a suitable terminal object, in order to be monoidal categories with finite products. It is precisely such a weaker structure that allows us to establish our main result: a functorial bijection between concatenable non-deterministic processes of a net N and the arrows of the free diodal category built out of N.

As for further refinements on the categorical model, as e.g. the self-dual category for modelling processes of contextual nets proposed in [4], let us just mention that we toyed with the idea of capturing the idempotency of $\oplus$ by making ∇ a natural isomorphism (hence, more in tune with the algebraic notion of dioids). The concrete description of concatenable non-deterministic processes does not allow it, since there would be no possible interpretation for the arrow $(\nabla_a)^{-1} : a \oplus a \to a$. However, this is not unfortunate, since the naturality of ∇ would actually make the diagram below commute

$$\begin{tikzcd} a\oplus a \arrow[r, "(\nabla_a)^{-1}"] \arrow[d, "\nabla_a\oplus\nabla_a"'] & a \arrow[r, "\nabla_a"] & a\oplus a \\ a\oplus a\oplus a\oplus a \arrow[rr, "a\oplus\gamma_{a,a}\oplus a"'] & & a\oplus a\oplus a\oplus a \arrow[u, "(\nabla_a)^{-1}\oplus(\nabla_a)^{-1}"'] \end{tikzcd}$$

We would e.g. infer that $(t_1 \oplus t_2); (t_3 \oplus t_4)$ is equated by functoriality to $(t_1; t_3) \oplus (t_2; t_4)$ and by naturality to $(t_1; t_3) \oplus (t_1; t_4) \oplus (t_2; t_3) \oplus (t_2; t_4)$, while those two expressions should intuitively represent different non-deterministic processes. Idempotency and functoriality do look like clashing properties for the $\oplus$ operator, and we were not ready to let the latter go.

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