A Monoidal View on Fixpoint Checks*

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Abstract. Fixpoints are ubiquitous in computer science as they play a central role in providing a meaning to recursive and cyclic definitions. Bisimilarity, behavioural metrics, termination probabilities for Markov chains and stochastic games are defined in terms of least or greatest fixpoints. Here we show that our recent work which proposes a technique for checking whether the fixpoint of a function is the least (or the largest) admits a natural categorical interpretation in terms of gs-monoidal categories. The technique is based on a construction that maps a function to a suitable approximation and the compositionality properties of this mapping are naturally interpreted as a gs-monoidal functor. This guides the realisation of a tool, called UDEfix that allows to build functions (and their approximations) like a circuit out of basic building blocks and subsequently perform the fixpoints checks. We also show that a slight generalisation of the theory allows one to treat a new relevant case study: coalgebraic behavioural metrics based on Wasserstein liftings.

1 Introduction

For the compositional modelling of graphs and graph-like structures it has proven useful to use the notion of monoidal categories [17], i.e., categories equipped with a tensor product. There are several extensions of such categories, such as gs-monoidal categories that have been shown to be suitable for specifying term rewriting (see e.g. [15,16]). In essence gs-monoidal categories describe graph-like structures with dedicated input and output interfaces, operators for disjoint union (tensor), duplication and termination of wires, quotiented by the axioms satisfied by these operators. Particularly useful are gs-monoidal functors that preserve such operators and hence naturally describe compositional operations.

We show that gs-monoidal categories and the composition concepts that come with them can be fruitfully used in a scenario that – at first sight – might seem

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quite unrelated: methods for fixpoints checks. In particular, we build upon [8] where a theory is proposed for checking whether a fixpoint of a given function is the least (greatest) fixpoint. The theory applies to a variety of fairly diverse application scenarios, such as bisimilarity [21], behavioural metrics [4,10,13,24], termination probabilities for Markov chains [3] and simple stochastic games [11].

More precisely, the theory above deals with non-expansive functions $f\colon \mathbb{M}^Y \to \mathbb{M}^Y$, where $\mathbb{M}$ is a set of values (more precisely, an MV-chain) and Y is a finite set. The rough idea consists in mapping such functions to corresponding approximations, whose fixpoints can be computed effectively and give insights on the fixpoints of the original function.

We show that the approximation framework and its compositionality properties can be naturally interpreted in categorical terms. This is done by introducing two gs-monoidal categories in which the concrete functions respectively their approximations live as arrows, together with a gs-monoidal functor, called $\#$, mapping one to the other. Besides shedding further light on the theoretical approximation framework of [8], this view guided the realisation of a tool, called UDEfix that allows to build functions (and their approximations) like a circuit out of basic building blocks and subsequently perform the fixpoints checks.

We also show that the functor $\#$ can be extended to deal with functions $f : \mathbb{M}^Y \to \mathbb{M}^Y$ where Y is not necessarily finite, becoming a lax functor. We prove some properties of this functor that enable us to give a recipe for finding approximations for a special type of functions: predicate liftings that have been introduced for coalgebraic modal logic [19,22]. This recipe allows us to include a new case study for the machinery for fixpoint checking: coalgebraic behavioural metrics, based on Wasserstein liftings.

The paper is organized as follows: In Sect. 2 we give some high-level motivation, while in Sect. 3 we review the theory from [8]. Subsequently in Sect. 4 we introduce two (gs-monoidal) categories $\mathbb{C}$, $\mathbb{A}$ (of concrete and abstract functions), show that the approximation $\#$ is a (lax) functor between these categories and prove some of its properties, which are used to handle predicate liftings (Sect. 5) and behavioural metrics (Sect. 6). Next, we show that the categories $\mathbb{C}$, $\mathbb{A}$ and the functor $\#$ are indeed gs-monoidal (Sect. 7) and lastly discuss the tool UDEfix in Sect. 8. We end by giving a conclusion (Sect. 9). Proofs and further material can be found in the full version of the paper [5].

2 Motivation

We start with some motivations for our theory and the tool UDEfix, which is based on it, via a case study on behavioural metrics. We consider probabilistic transition systems (Markov chains) with labelled states, given by a finite set of states X, a function $\delta\colon X \to \mathcal{D}(X)$ mapping each state $x \in X$ to a probability distribution on X and a labelling function $\ell\colon X \to \Lambda$, where Λ is a fixed set of labels (for examples see Fig. 1). Our aim is to determine the behavioural distance of two states, whose definition is based on the so-called Kantorovich or Wasserstein lifting [25] that measures the distance of two probability distributions on

X, based on a distance $d\colon X \times X \to [0,1]$. In more detail: given d, we define $d^{\mathcal{D}}\colon \mathcal{D}(X) \times \mathcal{D}(X) \to [0,1]$ as

$$d^{\mathcal{D}}(p_1, p_2) = \inf\{ \sum_{x_1,x_2 \in X} d(x_1,x_2) \cdot t(x_1,x_2) \mid t \in \Gamma(p_1,p_2)\}$$

where $\Gamma(p_1,p_2)$ is the set of couplings of p_1, p_2 (i.e., distributions $t\colon X \times X \to [0,1]$ such that $\sum_{x_2 \in X} t(x_1,x_2) = p_1(x_1)$, $\sum_{x_1 \in X} t(x_1,x_2) = p_2(x_2)$). The Wasserstein lifting gives in fact the solution of a transport problem, where we interpret p_1, p_2 as the supply respectively demand at each point $x \in X$. Transporting a unit from x_1 to x_2 costs $d(x_1,x_2)$ and t is a transport plan (= coupling) whose marginals are p_1, p_2. In other words: $d^{\mathcal{D}}(p_1,p_2)$ is the cost of the optimal transport plan, moving the supply p_1 to the demand p_2.

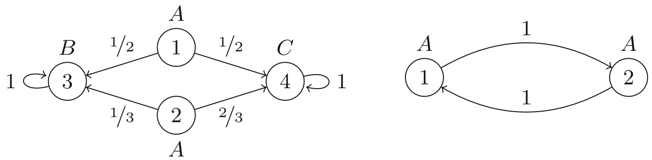

Fig. 1. Two probabilistic transition systems.

The behavioural metric is then defined as the least fixpoint of the function $f\colon [0,1]^{X\times X} \to [0,1]^{X \times X}$ where $f(d)(x_1,x_2) = 1$ if $\ell(x_1) \neq \ell(x_2)$ and $f(d)(x_1,x_2) = d^{\mathcal{D}}(\delta(x_1), \delta(x_2))$ otherwise. For instance, the best transport plan for the system on the left-hand side of Fig. 1 and the distributions $\delta(1), \delta(2)$ is t with $t(3,3) = 1/3$, $t(3,4) = 1/6$, $t(4,4) = 1/2$ and 0 otherwise.

One can observe that the function f can be decomposed as

$$f = \max_\rho \circ (c_k + (\delta \times \delta)^* \circ \min_u \circ \tilde{\mathcal{D}}),$$

where $+$ stands for disjoint union and we use the functions given in Table 1.[1] More concretely, the types of the components and the parameters k, u, ρ are given as follows, where $Y = X \times X$:

- $c_k\colon [0,1]^{\emptyset} \to [0,1]^Y$ where $k(x,x') = 1$ if $\ell(x) \neq \ell(x')$ and 0 otherwise.
- $\tilde{D}\colon [0,1]^Y \to [0,1]^{\mathcal{D}(Y)}$.
- $\min_u\colon [0,1]^{\mathcal{D}(Y)} \to [0,1]^{\mathcal{D}(X)\times\mathcal{D}(X)}$ where $u\colon \mathcal{D}(Y) \to \mathcal{D}(X) \times \mathcal{D}(X)$, $u(t) = (p,q)$ with $p(x) = \sum_{x' \in X} t(x,x')$, $q(x) = \sum_{x' \in X} t(x',x)$.
- $(\delta \times \delta)^*\colon [0,1]^{\mathcal{D}(X)\times\mathcal{D}(X)} \to [0,1]^Y$.
- $\max_\rho\colon [0,1]^{Y+Y} \to [0,1]^Y$ where $\rho\colon Y + Y \to Y$ is the obvious map from the coproduct (disjoint union) $Y+Y$ to Y.

In fact this decomposition can be depicted diagrammatically, as in Fig. 2.

[1] If the underlying sets are infinite, min, max can be replaced by inf, sup.

Table 1. Basic functions of type $\mathbb{M}^Y \to \mathbb{M}^Z$, $a\colon Y \to \mathbb{M}$.

Function	c_k $k\colon Z \to \mathbb{M}$	g^* $g\colon Z \to Y$	$\min_u$ $u\colon Y \to Z$	$\max_u$ $u\colon Y \to Z$	$\mathrm{av}_D = \tilde{D}$ $\mathbb{M} = [0,1]$, $Z = \mathcal{D}(Y)$
Name	constant	reindexing	minimum	maximum	expectation
$a \mapsto \dots$	k	$a \circ g$	$\lambda z. \min\limits_{u(y)=z} a(y)$	$\lambda z. \max\limits_{u(y)=z} a(y)$	$\lambda z.\lambda y. \sum\limits_{y \in Y} z(y) \cdot a(y)$

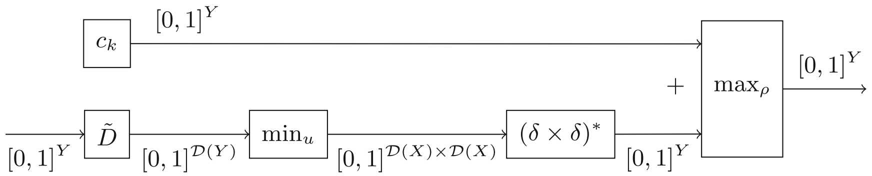

Fig. 2. Decomposition of the fixpoint function for computing behavioural metrics.

The function f is a monotone function on a complete lattice, hence it has a least fixpoint by Knaster-Tarski's fixpoint theorem [23], which is the behavioural metric. By giving a transport plan as above, it is possible to provide an upper bound for the Wasserstein lifting and hence there are strategy iteration algorithms that can approach a fixpoint from above. The problem with these algorithms is that they might get stuck at a fixpoint that is not the least. Hence, it is essential to be able to determine whether a given fixpoint is indeed the smallest one (cf. [2]).

Consider for instance the transition system in Fig. 1 on the right. It contains two states $1, 2$ on a cycle. In fact these two states should be indistinguishable and hence $d(1,2) = d(2,1) = 0$ if $d = \mu f$ is the least fixpoint of f. However, the metric a with $a(1,2) = a(2,1) = 1$ (0 otherwise) is also a fixpoint and the question is how to determine that it is not the least.

For this, we use the techniques developed in [8] that require in particular that f is non-expansive (i.e., given two metrics d_1, d_2, the sup-distance of $f(d_1), f(d_2)$ is smaller or equal than the sup-distance of d_1, d_2). In this case we can associate f with an approximation $f_\#^a$ on subsets of $X \times X$ such that, given $Y' \subseteq X \times X$, $f_\#^a(Y')$ intuitively contains all pairs (x_1, x_2) such that, decreasing function a by some value δ over Y', resulting in a function b (defined as $b(x_1,x_2) = a(x_1,x_2) - \delta$ if $(x_1,x_2) \in Y'$ and $b(x_1,x_2) = a(x_1,x_2)$ otherwise) and applying f, we obtain a function $f(b)$, where the same decrease took place at (x_1,x_2) (i.e., $f(b)(x_1,x_2) = f(a)(x_1,x_2) - \delta$). More concretely, here $f_\#^a(\{(1,2)\}) = \{(2,1)\}$, since a decrease at $(1,2)$ will cause a decrease at $(2,1)$ in the next iteration. In fact the greatest fixpoint of $f_\#^a$ (here: $\{(1,2),(2,1)\}$) gives us those elements that have a potential for decrease (intuitively there is "slack" or "wiggle room") and form a vicious cycle as above. It holds that a is the least fixpoint of f iff the the greatest fixpoint of $f_\#^a$ is the empty set, a non-trivial result [6,8].

The importance of the decomposition stems from the fact that the approximation is in fact compositional, that is $f^a_\#$ can be built out of the approximations of $\max_\rho$, c_k, $(\delta \times \delta)^*$, $\min_u$, $\tilde{D} = \mathrm{av}_D$, which can be easily determined (see [8]). For general functors, beyond the distribution functor, the characterization is however still missing and will be provided in this paper.

We anticipate that in our tool UDEfix we can draw a diagram as in Fig. 2, from which the approximation and its greatest fixpoint is automatically computed in a compositional way, allowing us to perform such fixpoint checks.

3 Preliminaries

This section reviews some background used throughout the paper. This includes the basics of lattices and MV-algebras, where the functions of interest take values. We also recap some results from [8] useful for detecting if a fixpoint of a given function is the least (or greatest).

We will also need some standard notions from *category theory*, in particular categories, functors and natural transformations. The definition of (strict) gs-monoidal categories is spelled out in detail in Definition 7.1.

For sets X, Y, we denote by $\mathcal{P}(X)$ the powerset of X and $\mathcal{P}_f(X)$ the set of finite subsets of X. The set of functions from X to Y is denoted by Y^X.

A partially ordered set $(P, \sqsubseteq)$ is often denoted simply as P, omitting the order relation. The *join* and the *meet* of a subset $X \subseteq P$ (if they exist) are denoted $\bigsqcup X$ and $\bigsqcap X$. We write $x \sqsubset y$ when $x \sqsubseteq y$ and $x \neq y$.

A *complete lattice* is a partially ordered set $(\mathbb{L}, \sqsubseteq)$ such that each subset $X \subseteq \mathbb{L}$ admits a join $\bigsqcup X$ and a meet $\bigsqcap X$. A complete lattice $(\mathbb{L}, \sqsubseteq)$ always has a least element $\bot = \bigsqcap \mathbb{L}$ and a greatest element $\top = \bigsqcup \mathbb{L}$.

A function $f : \mathbb{L} \to \mathbb{L}$ is *monotone* if for all $l, l' \in \mathbb{L}$, if $l \sqsubseteq l'$ then $f(l) \sqsubseteq f(l')$. By Knaster-Tarski's theorem [23, Theorem 1], any monotone function on a complete lattice has a least fixpoint μf and a greatest fixpoint νf.

For a set Y and a complete lattice $\mathbb{L}$, the set of functions $\mathbb{L}^Y$, with pointwise order (for $a, b \in \mathbb{L}^Y$, $a \sqsubseteq b$ if $a(y) \sqsubseteq b(y)$ for all $y \in Y$), is a complete lattice.

We are also interested in the set of finitely supported probability distributions $\mathcal{D}(Y) \subseteq [0,1]^Y$, i.e., functions $\beta : Y \to [0,1]$ with finite support such that $\sum_{y \in Y} \beta(y) = 1$.

An *MV-algebra* [18] is a tuple $\mathbb{M} = (M, \oplus, 0, \overline{(\cdot)})$ where $(M, \oplus, 0)$ is a commutative monoid and $\overline{(\cdot)} : M \to M$ maps each element to its *complement*, such that for all $x, y \in M$ (i) $\overline{\overline{x}} = x$; (ii) $x \oplus \overline{0} = \overline{0}$; (iii) $\overline{(\overline{x} \oplus y)} \oplus y = \overline{(\overline{y} \oplus x)} \oplus x$.

We define $1 = \overline{0}$ and subtraction $x \ominus y = \overline{\overline{x} \oplus y}$.

MV-algebras are endowed with a partial order, the so-called *natural order*, defined for $x, y \in M$, by $x \sqsubseteq y$ if $x \oplus z = y$ for some $z \in M$. When $\sqsubseteq$ is total, $\mathbb{M}$ is called an *MV-chain*. We will write $\mathbb{M}$ instead of M.

The natural order gives an MV-algebra a lattice structure where $\bot = 0$, $\top = 1$, $x \sqcup y = (x \ominus y) \oplus y$ and $x \sqcap y = \overline{\overline{x} \sqcup \overline{y}} = x \ominus (x \ominus y)$. We call the MV-algebra *complete* if it is a complete lattice, which is not true in general, e.g., $([0,1] \cap \mathbb{Q}, \leq)$.

Example 3.1. *A prototypical MV-algebra is* $([0,1], \oplus, 0, \overline{(\cdot)})$ *where* $x \oplus y = \min\{x+y, 1\}$, $\overline{x} = 1 - x$ *and* $x \ominus y = \max\{0, x-y\}$ *for* $x, y \in [0,1]$. *The natural order is* $\leq$ *(less or equal) on the reals. Another example is* $K = (\{0,\ldots,k\}, \oplus, 0, \overline{(\cdot)})$ *where* $n \oplus m = \min\{n+m, k\}$, $\overline{n} = k - n$ *and* $n \ominus m = \max\{n-m, 0\}$ *for* $n, m \in \{0,\ldots,k\}$. *Both MV-algebras are complete and MV-chains.*

We next briefly recap the theory from [8] which will be helpful in the paper for checking whether a fixpoint is the least or the greatest fixpoint of some underlying endo-function. For the purposes of the present paper we actually need a generalisation of the theory which provides the approximation also for functions with an infinite domain (while the theory in [8] was restricted to finite sets). Hence in the following, sets Y and Z are possibly infinite.

Given $a \in \mathbb{M}^Y$ we define its *norm* as $\|a\| = \sup\{a(y) \mid y \in Y\}$. A function $f : \mathbb{M}^Y \to \mathbb{M}^Z$ is *non-expansive* if for all $a, b \in \mathbb{M}^Y$ it holds $\|f(b) \ominus f(a)\| \sqsubseteq \|b \ominus a\|$. It can be seen that non-expansive functions are monotone. A number of standard operators are non-expansive (e.g., constants, reindexing, max and min over a relation, average in Table 1), and non-expansiveness is preserved by composition and disjoint union (see [8]). Given $Y' \subseteq Y$ and $\delta \in \mathbb{M}$, we write $\delta_{Y'}$ for the function defined by $\delta_{Y'}(y) = \delta$ if $y \in Y'$ and $\delta_{Y'}(y) = 0$, otherwise.

Let $f : \mathbb{M}^Y \to \mathbb{M}^Y$, $a \in \mathbb{M}^Y$ and $0 \sqsubset \delta \in \mathbb{M}$. Define $[Y]^a = \{y \in Y \mid a(y) \neq 0\}$ and consider the functions $\alpha^{a,\delta} : \mathcal{P}([Y]^a) \to [a \ominus \delta, a]$ and $\gamma^{a,\delta} : [a \ominus \delta, a] \to \mathcal{P}([Y]^a)$, defined, for $Y' \in \mathcal{P}([Y]^a)$ and $b \in [a \ominus \delta, a]$, by

$$\alpha^{a,\delta}(Y') = a \ominus \delta_{Y'} \qquad\qquad \gamma^{a,\delta}(b) = \{y \in [Y]^a \mid a(y) \ominus b(y) \sqsupseteq \delta\}.$$

Here $[a,b] = \{c \in \mathbb{M}^Y \mid a \sqsubseteq c \sqsubseteq b\}$. In fact, for suitable values of δ, the functions $\alpha^{a,\delta}, \gamma^{a,\delta}$ form a Galois connection.

For a non-expansive function $f : \mathbb{M}^Y \to \mathbb{M}^Z$ and $\delta \in \mathbb{M}$, define $f_\#^{a,\delta} \colon \mathcal{P}([Y]^a) \to \mathcal{P}([Z]^{f(a)})$ as $f_\#^{a,\delta} = \gamma^{f(a),\delta} \circ f \circ \alpha^{a,\delta}$. The function $f_\#^{a,\delta}$ is antitone in the parameter δ and we define the *a-approximation* of f as

$$f_\#^a = \bigcup_{\delta \sqsupset 0} f_\#^{a,\delta}.$$

For finite sets Y and Z there exists a suitable value $\iota_f^a \sqsupset 0$, such that all functions $f_\#^{a,\delta}$ for $0 \sqsubset \delta \sqsubseteq \iota_f^a$ are equal. Here, the a-approximation is given by $f_\#^a = f_\#^{a,\delta}$ for $\delta = \iota_f^a$.

Intuitively, given some Y', the set $f_\#^a(Y')$ contains the points where a decrease of the values of a on the points in Y' "propagates" through the function f. The greatest fixpoint of $f_\#^a$ gives us the subset of Y where such a decrease is propagated in a cycle (so-called "vicious cycle"). Whenever $\nu f_\#^a$ is non-empty, one can argue that a cannot be the least fixpoint of f since we can decrease the value in all elements of $\nu f_\#^a$, obtaining a smaller prefixpoint. Interestingly, for non-expansive functions, it is shown in [8] that also the converse holds, i.e., emptiness of the greatest fixpoint of $f_\#^a$ implies that a is the least fixpoint.

Theorem 3.2 (soundness and completeness for fixpoints). *Let $\mathbb{M}$ be a complete MV-chain, Y a finite set and $f : \mathbb{M}^Y \to \mathbb{M}^Y$ be a non-expansive function. Let $a \in \mathbb{M}^Y$ be a fixpoint of f. Then $\nu f_\#^a = \emptyset$ if and only if $a = \mu f$.*

Using the above theorem we can check whether some fixpoint a of f is the least fixpoint. Whenever a is a fixpoint, but not yet the least fixpoint of f, it can be decreased by a fixed value in $\mathbb{M}$ (see [8] for the details) on the points in $\nu f_\#^a$ to obtain a smaller pre-fixpoint.

Lemma 3.3. *Let $\mathbb{M}$ be a complete MV-chain, Y a finite set and $f : \mathbb{M}^Y \to \mathbb{M}^Y$ a non-expansive function, $a \in \mathbb{M}^Y$ a fixpoint of f, and let $f_\#^a$ be the corresponding a-approximation. If a is not the least fixpoint and thus $\nu f_\#^a \neq \emptyset$ then there is $0 \sqsubset \delta \in \mathbb{M}$ such that $a \ominus \delta_{\nu f_\#^a}$ is a pre-fixpoint of f.*

The above theory can easily be dualised [8].

4 A Categorical View of the Approximation Framework

The framework from [8], summarized in the previous section, is not based on category theory, but – as we shall see – can be naturally reformulated in a categorical setting. In particular, casting the compositionality results into a monoidal structure (see Sect. 7) is a valuable basis for our tool. But first, we will show how the operation $\#$ of taking the a-approximation of a function can be seen as a (lax) functor between two categories: a concrete category $\mathbb{C}$ whose arrows are the non-expansive functions for which we seek the least (or greatest) fixpoint and an abstract category $\mathbb{A}$ whose arrows are the corresponding approximations.

More precisely, recall that given a non-expansive function $f : \mathbb{M}^Y \to \mathbb{M}^Z$, the approximation of f is relative to a fixed map $a \in \mathbb{M}^Y$. Hence objects in $\mathbb{C}$ are elements $a \in \mathbb{M}^Y$ and an arrow from $a \in \mathbb{M}^Y$ to $b \in \mathbb{M}^Z$ is a non-expansive function $f : \mathbb{M}^Y \to \mathbb{M}^Z$ required to map a into b. The approximations instead live in $\mathbb{A}$. Recall that the approximation is $f_\#^a : \mathcal{P}([Y]^a) \to \mathcal{P}([Z]^b)$. Since their domains and codomains are dependent again on a map a, we still employ elements of $\mathbb{M}^Y$ as objects, but functions between powersets as arrows.

Definition 4.1 (concrete and abstract categories). *The concrete category $\mathbb{C}$ has as objects maps $a \in \mathbb{M}^Y$ where Y is a (possibly infinite) set. Given $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ an arrow $f : a \dashrightarrow b$ is a non-expansive function $f \colon \mathbb{M}^Y \to \mathbb{M}^Z$, such that $f(a) = b$. The abstract category $\mathbb{A}$ has again maps $a \in \mathbb{M}^Y$ as objects. Given $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ an arrow $f : a \dashrightarrow b$ is a monotone (wrt. inclusion) function $f \colon \mathcal{P}([Y]^a) \to \mathcal{P}([Z]^b)$. Arrow composition and identities are the obvious ones.*

The lax functor $\# \colon \mathbb{C} \to \mathbb{A}$ is defined as follows: for an object $a \in \mathbb{M}^Y$, we let $\#(a) = a$ and, given an arrow $f : a \dashrightarrow b$, we let $\#(f) = f_\#^a$.

Note that abstract arrows are dashed ($\dashrightarrow$), while the underlying functions are represented by standard arrows ($\to$).

Lemma 4.2 (well-definedness). *The categories* $\mathbb{C}$ *and* $\mathbb{A}$ *are well-defined and* $\#$ *is a lax functor, i.e., identities are preserved and* $\#(f \circ g) \subseteq \#(f) \circ \#(g)$ *for composable arrows* f, g *in* $\mathbb{C}$.

It will be convenient to restrict to the subcategory of $\mathbb{C}$ where arrows are reindexings and to subcategories of $\mathbb{C}$, $\mathbb{A}$ with maps on finite sets.

Definition 4.3 (reindexing subcategory). *We denote by* $\mathbb{C}^*$ *the lluf*[2] *subcategory of* $\mathbb{C}$ *where arrows are reindexing, i.e., given objects* $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ *we consider only arrows* $f : a \dashrightarrow b$ *such that* $f = g^*$ *for some* $g \colon Z \to Y$ *(hence, in particular,* $b = g^*(a) = a \circ g$*). We denote* $E \colon \mathbb{C}^* \hookrightarrow \mathbb{C}$ *the embedding functor.*

Definition 4.4 (finite subcategories). *We denote by* $\mathbb{C}_f$, $\mathbb{A}_f$ *the full subcategories of* $\mathbb{C}, \mathbb{A}$ *where objects are of the kind* $a \in \mathbb{M}^Y$ *for a* finite *set* Y.

Lemma 4.5. *The lax functor* $\# \colon \mathbb{C} \to \mathbb{A}$ *restricts to* $\# \colon \mathbb{C}_f \to \mathbb{A}_f$, *which is a (proper) functor.*

5 Predicate Liftings

In this section we discuss how predicate liftings [19,22] can be integrated into our theory. In this context the idea is to view a map in $\mathbb{M}^Y$ as a predicate over Y with values in $\mathbb{M}$ (e.g., if $\mathbb{M} = \{0, 1\}$ we obtain Boolean predicates). Then, given a functor F, a predicate lifting transforms a predicate over Y (a map in $\mathbb{M}^Y$), to a predicate over FY (a map in $\mathbb{M}^{FY}$). It must be remarked that every complete MV-algebra is a quantale[3] with respect to $\oplus$ and the inverse of the natural order (see [14]) and predicate liftings for arbitrary quantales have been studied, for instance, in [9].

First, we characterise which predicate liftings are non-expansive and second, derive their approximations. We will address both these issues in this section and then use predicate liftings to define behavioural metrics in Sect. 6.

The fact that there are some functors F, for which FY is infinite, even if Y is finite, is the reason why the categories $\mathbb{C}$ and $\mathbb{A}$ also include infinite sets. However note, that the resulting fixpoint function will be always defined for finite sets, although intermediate functions might not conform to this.

Definition 5.1 (predicate lifting). *Given a functor* $F \colon \mathbf{Set} \to \mathbf{Set}$, *a predicate lifting is a family of functions* $\tilde{F}_Y \colon \mathbb{M}^Y \to \mathbb{M}^{FY}$ *(where* Y *is a set), such that for* $g \colon Z \to Y$, $a \colon Y \to \mathbb{M}$ *it holds that* $(Fg)^*(\tilde{F}_Y(a)) = \tilde{F}_Z(g^*(a))$.

That is, predicate liftings must commute with reindexings. The index Y will be omitted if clear from the context. Such predicate liftings are in one-to-one

[2] A *lluf sub-category* is a sub-category that contains all objects.

[3] A quantale is a complete lattice with an associative operator that distributes over arbitrary joins.

correspondence to so called evaluation maps $ev\colon F\mathbb{M} \to \mathbb{M}$.[4] Given ev, we define the corresponding lifting to be $\tilde{F}(a) = ev \circ Fa\colon FY \to \mathbb{M}$, where $a\colon Y \to \mathbb{M}$.

In the sequel we will only consider well-behaved liftings [4,9], i.e., we require that (i) $\tilde{F}$ is monotone; (ii) $\tilde{F}(0_Y) = 0_{FY}$ where 0 is the constant 0-function; (iii) $\tilde{F}(a \oplus b) \sqsubseteq \tilde{F}(a) \oplus \tilde{F}(b)$ for $a, b\colon Y \to \mathbb{M}$; (iv) F preserves weak pullbacks.

We aim to have not only monotone, but non-expansive liftings.

Lemma 5.2. *Let $ev\colon F\mathbb{M} \to \mathbb{M}$ be an evaluation map and assume that its corresponding lifting $\tilde{F}\colon \mathbb{M}^Y \to \mathbb{M}^{FY}$ is well-behaved. Then $\tilde{F}$ is non-expansive iff for all $\delta \in \mathbb{M}$ it holds that $\tilde{F}\delta_Y \sqsubseteq \delta_{FY}$.*

Example 5.3. *We consider the (finitely supported) distribution functor $\mathcal{D}$ that maps a set X to all maps $p\colon X \to [0,1]$ that have finite support and satisfy $\sum_{x\in X} p(x) = 1$. (Here $\mathbb{M} = [0,1]$.) One evaluation map is $ev\colon \mathcal{D}[0,1] \to [0,1]$ with $ev(p) = \sum_{r\in[0,1]} r \cdot p(r)$, where p is a distribution on $[0,1]$ (expectation). It is easy to see that $\tilde{D}$ is well-behaved and non-expansive. The latter follows from $\tilde{D}(\delta_Y) = \delta_{\mathcal{D}Y}$.*

Example 5.4. *Another example is given by the finite powerset functor $\mathcal{P}_f$. We are given the evaluation map $ev\colon \mathcal{P}_f\mathbb{M} \to \mathbb{M}$, defined for finite $S \subseteq \mathbb{M}$ as $ev(S) = \max S$, where $\max\emptyset = 0$. The lifting $\tilde{\mathcal{P}}_f$ is well-behaved (see [4]) and non-expansive. To show the latter, observe that $\tilde{\mathcal{P}}_f(\delta_Y) = \delta_{\mathcal{P}_f(Y)\setminus\{\emptyset\}} \sqsubseteq \delta_{\mathcal{P}_f(Y)}$.*

Non-expansive predicate liftings can be seen as functors $\tilde{F}\colon \mathbb{C}^* \to \mathbb{C}^*$. To be more precise, $\tilde{F}$ maps an object $a \in \mathbb{M}^Y$ to $\tilde{F}(a) \in \mathbb{M}^{FY}$ and an arrow $g^* : a \dashrightarrow a \circ g$, , where $g\colon Z \to Y$, to $(Fg)^* : \tilde{F}a \dashrightarrow \tilde{F}(a \circ g)$.

Proposition 5.5. *Let $\tilde{F}$ be a (non-expansive) predicate lifting. There is a natural transformation $\beta\colon \#E \Rightarrow \#E\tilde{F}$ between (lax) functors $\#E, \#E\tilde{F}\colon \mathbb{C}^* \to \mathbb{A}$, whose components, for $a \in \mathbb{M}^Y$, are $\beta_a\colon a \dashrightarrow \tilde{F}(a)$ in $\mathbb{A}$, defined by $\beta_a(U) = \tilde{F}^a_\#(U)$ for $U \subseteq [Y]^a$.*

That is, the following diagrams commute for every $g\colon Z \to Y$ (on the left the diagram with formal arrows, omitting the embedding functor E, and on the right the functions with corresponding domains). Note that $\#(g) = g^{-1}$.

$$\begin{tikzcd}
\#(a) \arrow[r, dashed, "\#(g^*)"] \arrow[d, dashed, "\beta_a"'] & \#(a \circ g) \arrow[d, dashed, "\beta_{a\circ g}"] \\
\#(\tilde{F}a) \arrow[r, dashed, "\#(\tilde{F}(g^*))"] & \#(\tilde{F}(a \circ g))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}([Y]^a) \arrow[r, "g^{-1}"] \arrow[d, "\tilde{F}^a_\#"'] & \mathcal{P}([Z]^{a\circ g}) \arrow[d, "\tilde{F}^{a\circ g}_\#"] \\
\mathcal{P}([FY]^{\tilde{F}(a)}) \arrow[r, "(Fg)^{-1}"] & \mathcal{P}([FZ]^{\tilde{F}(a\circ g)})
\end{tikzcd}$$

[4] This follows from the Yoneda lemma, see e.g. [17].

6 Wasserstein Lifting and Behavioural Metrics

In this section we show how the framework for fixpoint checking described before can be used to deal with coalgebraic behavioural metrics.

We build on [4], where an approach is proposed for canonically defining a behavioural pseudometric for coalgebras of a functor $F\colon \mathbf{Set} \to \mathbf{Set}$, that is, for functions of the form $\xi\colon X \to FX$ where X is a set. Intuitively ξ specifies a transition system whose branching type is given by F. Given such a coalgebra ξ, the idea is to endow X with a pseudo-metric $d_\xi\colon X \times X \to \mathbb{M}$ defined as the least fixpoint of the map $d \mapsto d^F \circ (\xi \times \xi)$ where $\_^F$ lifts a metric $d\colon X \times X \to \mathbb{M}$ to a metric $d^F\colon FX \times FX \to \mathbb{M}$. Here we focus on the so-called Wasserstein lifting and show how approximations of the functions involved in the definition of the pseudometric can be determined.

6.1 Wasserstein Lifting

Hereafter, F denotes a fixed endofunctor on **Set** and $\xi\colon X \to FX$ is a coalgebra over a finite set X. We also fix a well-behaved non-expansive predicate lifting $\tilde{F}$.

In order to define a Wasserstein lifting, a first ingredient is that of a coupling. Given $t_1, t_2 \in FX$ a *coupling* of t_1 and t_2 is an element $t \in F(X \times X)$, such that $F\pi_i(t) = t_i$ for $i = 1, 2$, where $\pi_i\colon X \times X \to X$ are the projections. We write $\Gamma(t_1, t_2)$ for the set of all such couplings.

Definition 6.1 (Wasserstein lifting). *The Wasserstein lifting* $\_^F\colon \mathbb{M}^{X\times X} \to \mathbb{M}^{FX\times FX}$ *is defined for* $d\colon X \times X \to \mathbb{M}$ *and* $t_1, t_2 \in FX$ *as*

$$d^F(t_1, t_2) = \inf_{t\in\Gamma(t_1,t_2)} \tilde{F}d(t)$$

For more intuition on the Wasserstein lifting see Sect. 2. Note that a coupling correspond to a transport plan. It can be shown that for well-behaved $\tilde{F}$, the lifting preserves pseudometrics (see [4,9]).

In order to make the theory for fixpoint checks effective we will need to restrict to a subclass of liftings.

Definition 6.2 (finitely coupled lifting). *We call a lifting* $\tilde{F}$ finitely coupled *if for all* X *and* $t_1, t_2 \in FX$ *there exists a finite* $\Gamma'(t_1, t_2) \subseteq \Gamma(t_1, t_2)$*, which can be computed given* t_1, t_2*, such that* $\inf_{t\in\Gamma(t_1,t_2)} \tilde{F}d(t) = \min_{t\in\Gamma'(t_1,t_2)} \tilde{F}d(t)$*.*

Observe that whenever the infimum above is a minimum, there is trivially a finite $\Gamma'(t_1, t_2)$. We however ask that there is an effective way to determine it.

The lifting in Example 5.4 (for the finite powerset functor) is obviously finitely coupled. For the lifting $\tilde{\mathcal{D}}$ from Example 5.3 we note that the set of couplings $t \in \Gamma(t_1, t_2)$ forms a polytope with a finite number of vertices, which can be effectively computed and $\Gamma'(t_1, t_2)$ consists of these vertices. The infimum (minimum) is obtained at one of these vertices [1, Remark 4.5].

6.2 A Compositional Representation

As mentioned above, for a coalgebra $\xi\colon X \to FX$ the behavioural pseudometric $d : X \times X \to \mathbb{M}$ arises as the least fixpoint of $\mathcal{W} = (\xi \times \xi)^* \circ (\_^F)$ where $(\_^F)$ is the Wasserstein lifting.

Example 6.3. *We can recover the motivating example from Sect. 2 by setting $\mathbb{M} = [0,1]$ and using the functor $FX = \Lambda \times \mathcal{D}(X)$, where Λ is a fixed set of labels. We observe that couplings of $(a_1, p_1), (a_2, p_2) \in FX$ only exist if $a_1 = a_2$ and – if they do not exist – the Wasserstein distance is the empty infimum, hence 1. If $a_1 = a_2$, couplings correspond to the usual Wasserstein couplings of p_1, p_2 and the least fixpoint of $\mathcal{W}$ equals the behavioural metrics, as explained in Sect. 2.*

Note that we do not use a discount factor to ensure contractivity and hence the fixpoint might not be unique. Thus, given some fixpoint d, the d-approximation $\mathcal{W}^d_\#$ can be used for checking whether $d = \mu\mathcal{W}$.

In the rest of the section we show how $\mathcal{W}$ can be decomposed into basic components and study the corresponding approximation.

The Wasserstein lifting can be decomposed as $\_^F = \min_u \circ \tilde{F}$ where $\tilde{F} : \mathbb{M}^{X\times X} \to \mathbb{M}^{F(X\times X)}$ is the predicate lifting – which we require to be non-expansive (cf. Lemma 5.2) – and $\min_u$ is the minimum over the coupling function $u\colon F(X\times X) \to FX \times FX$ defined as $u(t) = (F\pi_1(t), F\pi_2(t))$, which means that $\min_u\colon \mathbb{M}^{F(X\times X)} \to \mathbb{M}^{FX\times FX}$ (see Table 1).

We can now derive the corresponding d-approximation.

Proposition 6.4. *Assume that $\tilde{F}$ is finitely coupled. Let $Y = X \times X$, where X is finite. For $d \in \mathbb{M}^Y$ and $Y' \subseteq [Y]^d$ we have*

$$\begin{aligned}\mathcal{W}^d_\#(Y') = \{(x,y) \in [Y]^d \mid \exists t \in \tilde{F}^d_\#(Y'), u(t) = (\xi(x), \xi(y)),\\ \tilde{F}d(t) = \min\nolimits_{t' \in \Gamma(\xi(x),\xi(y))} \tilde{F}d(t')\}.\end{aligned}$$

Intuitively the statement of Proposition 6.4 means that the minimum must be reached in a coupling based on Y'.

For using the above result we next characterize $\tilde{F}^d_\#(Y')$. We rely on the fact that d can be decomposed into $d = \pi_1 \circ \bar{d}$, where the projection π_1 is independent of d and $\bar{d}$ is dependent on Y', and exploit the natural transformation in Proposition 5.5.

Proposition 6.5. *We fix $Y' \subseteq Y$. Let $\pi_1\colon \mathbb{M} \times \{0,1\} \to \mathbb{M}$ be the projection to the first component and $\bar{d}\colon Y \to \mathbb{M} \times \{0,1\}$ with $\bar{d}(y) = (d(y), \chi_{Y'}(y))$ where $\chi_{Y'}\colon Y \to \{0,1\}$ is the characteristic function of Y'. Then $\tilde{F}^d_\#(Y') = (F\bar{d})^{-1}(\tilde{F}^{\pi_1}_\#((\mathbb{M}\backslash\{0\}) \times \{1\}))$.*

Here $\tilde{F}^{\pi_1}_\#((\mathbb{M}\backslash\{0\}) \times \{1\}) \subseteq F(\mathbb{M} \times \{0,1\})$ is independent of d and has to be determined only once for every predicate lifting $\tilde{F}$. We will show how this set looks like for our example functors.

Lemma 6.6. *Consider the lifting of the distribution functor presented in Example 5.3 and let $Z = [0,1] \times \{0,1\}$. Then we have*

$$\tilde{D}_{\#}^{\pi_1}((0,1] \times \{1\}) = \{p \in \mathcal{D}Z \mid supp(p) \in (0,1] \times \{1\}\}.$$

This means intuitively that a decrease or "slack" can exactly be propagated for elements whose probabilities are strictly larger than 0.

Lemma 6.7. *Consider the lifting of the finite powerset functor from Example 5.4 and let $Z = \mathbb{M} \times \{0,1\}$. Then we have*

$$(\tilde{\mathcal{P}}_f)_{\#}^{\pi_1}((\mathbb{M}\backslash\{0\}) \times \{1\}) = \{S \in [\mathcal{P}_f Z]^{\tilde{\mathcal{P}}_f \pi_1} \mid \exists (s,1) \in S\ \forall (s',0) \in S:\ \ s \sqsupset s'\}.$$

The idea is that the maximum of a set S decreases if we decrease at least one its values and all values which are not decreased are strictly smaller.

Remark 8. Note that $\#$ is a functor on the subcategory $\mathbb{C}_f$, while some liftings (e.g., the one for the distribution functor) work with infinite sets. In this case, given a finite set Y, we actually focus on a finite $D \subseteq FY$. (This is possible since we consider coalgebras with finite state space and assume that all liftings are finitely coupled.) Then we consider $\tilde{F}_Y : \mathbb{M}^Y \to \mathbb{M}^{FY}$ and $e\colon D \hookrightarrow FY$ (the embedding of D into FY). We set $f = e^* \circ \tilde{F}_Y$. Given $a\colon Y \to \mathbb{M}$, we view f as an arrow $a \dashrightarrow \tilde{F}(a) \circ e$ in $\mathbb{C}$. The approximation in this subsection adapts to the "reduced" lifting, which can be seen as follows (cf. [5]: $\#$ preserves composition if one of the arrows is a reindexing):

$$f_{\#}^{a} = \#(f) = \#(e^* \circ \tilde{F}_Y) = \#(e^*) \circ \#(\tilde{F}_Y) = e^{-1} \circ \#(\tilde{F}_Y) = \#(\tilde{F}_Y) \cap D.$$

7 GS-Monoidality

We will now show that the categories $\mathbb{C}_f$ and $\mathbb{A}_f$ can be turned into gs-monoidal categories. This will give us a way to assemble functions and their approximations compositionally and this method will form the basis for the tool. We first define gs-monoidal categories in detail:

Definition 7.1. *A strict gs-monoidal category is a strict symmetric monoidal category, where $\otimes$ denotes the tensor and e its unit and symmetries are given by $\rho_{a,b}\colon a \otimes b \to b \otimes a$. For every object a there exist morphisms $\nabla_a\colon a \to a \times a$ (duplicator) and $!_a\colon a \to e$ (discharger) satisfying the axioms given below. (See also the visualizations as string diagrams in Fig. 3.)*

1. *functoriality of tensor:*
   - $(g \otimes g') \circ (f \otimes f') = (g \circ f) \otimes (g' \circ f')$
   - $id_{a \otimes b} = id_a \otimes id_b$
2. *monoidality:*
   - $(f \otimes g) \otimes h = f \otimes (g \otimes h)$
   - $f \otimes id_e = f = id_e \otimes f$

3. *naturality:*
   - $(f' \otimes f) \circ \rho_{a,a'} = \rho_{b,b'} \circ (f \otimes f')$
4. *symmetry:*
   - $\rho_{e,e} = id_e$
   - $\rho_{b,a} \circ \rho_{a,b} = id_{a\otimes b}$
   - $\rho_{a\otimes b,c} = (\rho_{a,c} \otimes id_b) \circ (id_a \otimes \rho_{b,c})$
5. *gs-monoidality:*
   - $!_e = \nabla_e = id_e$
   - *coherence axioms:*
     - $(id_a \otimes \nabla_a) \circ \nabla_a = (\nabla_a \otimes id_a) \circ \nabla_a$
     - $id_a = (id_a \otimes !_a) \circ \nabla_a$
     - $\rho_{a,a} \circ \nabla_a = \nabla_a$
   - *monoidality axioms:*
     - $!_{a\otimes b} = !_a \otimes !_b$
     - $(id_a \otimes \rho_{a,b} \otimes id_b) \circ (\nabla_a \otimes \nabla_b) = \nabla_{a\otimes b}$
       (or, equivalently, $\nabla_a \otimes \nabla_b = (id_a \otimes \rho_{b,a} \otimes id_b) \circ \nabla_{a\otimes b}$*)*

A functor $\#\colon \mathbb{C} \to \mathbb{D}$ *is gs-monoidal if the following holds:*

1. $\mathbb{C}$ *and* $\mathbb{D}$ *are gs-monoidal categories*
2. *monoidality:*
   - $\#(e) = e'$
   - $\#(a \otimes b) = \#(a) \otimes' \#(b)$
3. *symmetry:*
   - $\#(\rho_{a,b}) = \rho'_{\#(a),\#(b)}$
4. *gs-monoidality:*
   - $\#(!_a) = !'_{\#(a)}$
   - $\#(\nabla_a) = \nabla'_{\#(a)}$

where the primed operators are from the category $\mathbb{D}$*, the others from* $\mathbb{C}$*.*

In fact, in order to obtain strict gs-monoidal categories with disjoint union, we will work with the skeleton categories where every finite set Y is represented by an isomorphic copy $\{1, \dots, |Y|\}$. This enables us to make disjoint union strict, i.e., associativity holds on the nose and not just up to isomorphism. In particular for finite sets Y, Z, we define disjoint union as $Y + Z = \{1, \dots, |Y|, |Y| + 1, \dots, |Y| + |Z|\}$.

Theorem 7.2. *The category* $\mathbb{C}_f$ *with the following operators is gs-monoidal:*

1. *The tensor* $\otimes$ *on objects* $a \in \mathbb{M}^Y$ *and* $b \in \mathbb{M}^Z$ *is defined as*

$$a \otimes b = a + b \in \mathbb{M}^{Y+Z}$$

   where for $k \in Y + Z$ *we have* $(a + b)(k) = a(k)$ *if* $k \le |Y|$ *and* $(a + b)(k) = b(k - |Y|)$ *if* $|Y| < k \le |Y| + |Z|$*.*
   On arrows $f\colon a \dashrightarrow b$ *and* $g\colon a' \dashrightarrow b'$ *(with* $a' \in \mathbb{M}^{Y'}$*,* $b' \in \mathbb{M}^{Z'}$*) tensor is given by*

$$f \otimes g\colon \mathbb{M}^{Y+Y'} \to \mathbb{M}^{Z+Z'}, \quad (f \otimes g)(u) = f(\overleftarrow{u}_Y) + g(\vec{u}_Y)$$

   for $u \in \mathbb{M}^{Y+Y'}$ *where* $\overleftarrow{u}_Y \in \mathbb{M}^Y$ *and* $\vec{u}_Y \in \mathbb{M}^{Y'}$*, defined as* $\overleftarrow{u}_Y(k) = u(k)$ $(1 \le k \le |Y|)$ *and* $\vec{u}_Y(k) = u(|Y| + k)$ $(1 \le k \le |Y'|)$*.*

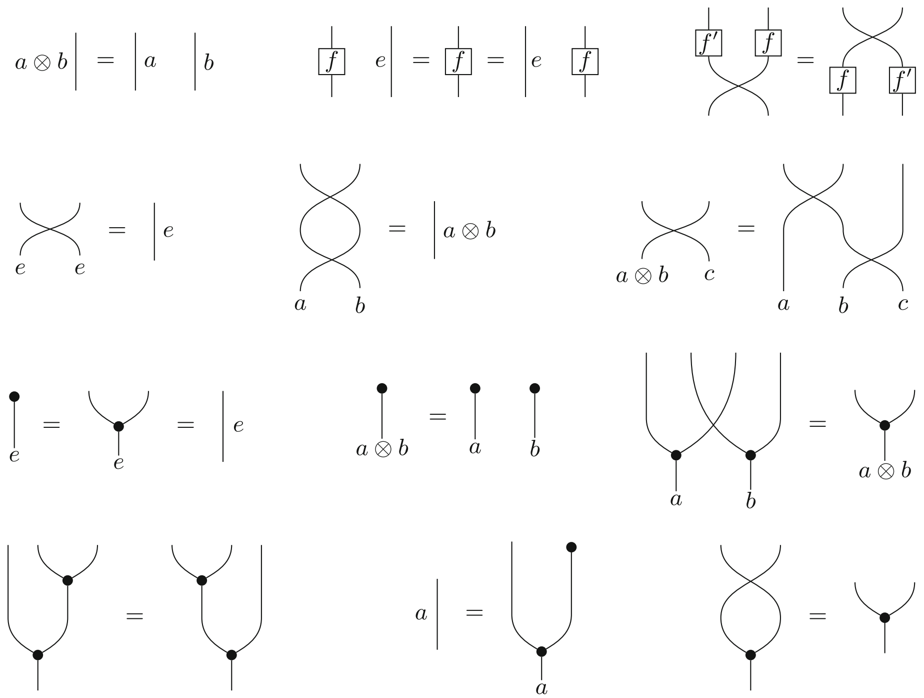

Fig. 3. String diagrams of the axioms satisfied by gs-monoidal categories.

2. *The symmetry* $\rho_{a,b} \colon a \otimes b \dashrightarrow b \otimes a$ *for* $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ *is defined for* $u \in \mathbb{M}^{Y+Z}$ *as*
$$\rho_{a,b}(u) = \vec{u}_Y + \overleftarrow{u}_Y.$$
3. *The unit* e *is the unique mapping* $e \colon \emptyset \to \mathbb{M}$.
4. *The duplicator* $\nabla_a \colon a \dashrightarrow a \otimes a$ *for* $a \in \mathbb{M}^Y$ *is defined for* $u \in \mathbb{M}^Y$ *as*
$$\nabla_a(u) = u + u.$$
5. *The discharger* $!_a \colon a \dashrightarrow e$ *for* $a \in \mathbb{M}^Y$ *is defined for* $u \in \mathbb{M}^Y$ *as* $!_a(u) = e$.

We now turn to the abstract category $\mathbb{A}_f$. Note that here functions have as parameters sets of the form $U \subseteq [Y]^a \subseteq Y$. Hence, (the cardinality of) Y can not be determined directly from U and we need extra care with the tensor.

Theorem 7.3. *The category* $\mathbb{A}_f$ *with the following operators is gs-monoidal:*

1. *The tensor* $\otimes$ *on objects* $a \in \mathbb{M}^Y$ *and* $b \in \mathbb{M}^Z$ *is again defined as* $a \otimes b = a + b$. *On arrows* $f \colon a \dashrightarrow b$ *and* $g \colon a' \dashrightarrow b'$ *(where* $a' \in \mathbb{M}^{Y'}$, $b' \in \mathbb{M}^{Z'}$ *and* $f \colon \mathcal{P}([Y]^a) \to \mathcal{P}([Z]^{b'})$, $g \colon \mathcal{P}([Y']^{a'}) \to \mathcal{P}([Z']^{b'})$ *are the underlying functions), the tensor is given by*
$$f \otimes g \colon \mathcal{P}([Y + Y']^{a+a'}) \to \mathcal{P}([Z + Z']^{b+b'}), \quad (f \otimes g)(U) = f(\overleftarrow{U}_Y) \cup_Z g(\vec{U}_Y)$$

where $\overleftarrow{U}_Y = U \cap \{1, \ldots, |Y|\}$ *and* $\vec{U}_Y = \{k \mid |Y| + k \in U\}$. *Furthermore:*

$$U \cup_Y V = U \cup \{|Y| + k \mid k \in V\} \quad (\textit{where } U \subseteq Y)$$

2. *The symmetry* $\rho_{a,b} \colon a \otimes b \dashrightarrow b \otimes a$ *for* $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ *is defined for* $U \subseteq [Y + Z]^{a+b}$ *as*

$$\rho_{a,b}(U) = \vec{U}_Y \cup_Z \overleftarrow{U}_Y \subseteq [Z + Y]^{b+a}$$

3. *The unit* e *is again the unique mapping* $e \colon \emptyset \to \mathbb{M}$.
4. *The duplicator* $\nabla_a \colon a \dashrightarrow a \otimes a$ *for* $a \in \mathbb{M}^Y$ *is defined for* $U \subseteq [Y]^a$ *as*

$$\nabla_a(U) = U \cup_Y U \subseteq [Y + Y]^{a+a}.$$

5. *The discharger* $!_a \colon a \dashrightarrow e$ *for* $a \in \mathbb{M}^Y$ *is defined for* $U \subseteq [Y]^a$ *as* $!_a(U) = \emptyset$.

Finally, the approximation $\#$ is indeed gs-monoidal, i.e., it preserves all the additional structure (tensor, symmetry, unit, duplicator and discharger).

Theorem 7.4. $\# \colon \mathbb{C}_f \to \mathbb{A}_f$ *is a gs-monoidal functor.*

8 UDEfix: A Tool for Fixpoints Checks

We exploit gs-monoidality as discussed before and present a tool, called UDEfix, where the user can compose his or her very own function $f \colon \mathbb{M}^Y \to \mathbb{M}^Y$ as a sort of circuit. Exploiting the fact that the functor $\#$ is gs-monoidal, this circuit is then transformed automatically and in a compositional way into the corresponding abstraction $f_\#^a$, for some given $a \in \mathbb{M}^Y$. By computing the greatest fixpoint of $f_\#^a$ and checking for emptiness, UDEfix can check whether $a = \mu f$.

In fact, UDEfix can handle all functions presented in Sect. 2, where for $\min_u$, $\max_u$ we also allow u to be a relation, instead of a function. Moreover, addition and subtraction by a fixed constant (both non-expansive functions) can be handled (see [7] for details). In addition to fixpoint checks, it is possible to perform (non-complete) checks whether a given post-fixpoint a is below the least fixpoint μf. The dual checks (for greatest fixpoint and pre-fixpoints) are implemented as well.

Building the desired function $f \colon \mathbb{M}^Y \to \mathbb{M}^Y$ requires three steps:

- Choosing the MV-algebra $\mathbb{M}$. Currently the MV-chains $[0, 1]$ and $\{0, \ldots, k\}$ (for arbitrary k) are supported.
- Creating the required basic functions by specifying their parameters.
- Assembling f from these basic functions.

UDEfix is a Windows-Tool created in Python, which can be obtained from https://github.com/TimoMatt/UDEfix. The GUI of UDEfix is separated into three areas: Content area, Building area and Basic-Functions area. Under File the user can save/load contents and set the MV-algebra in Settings. Functions built in the Building area can be saved and loaded.

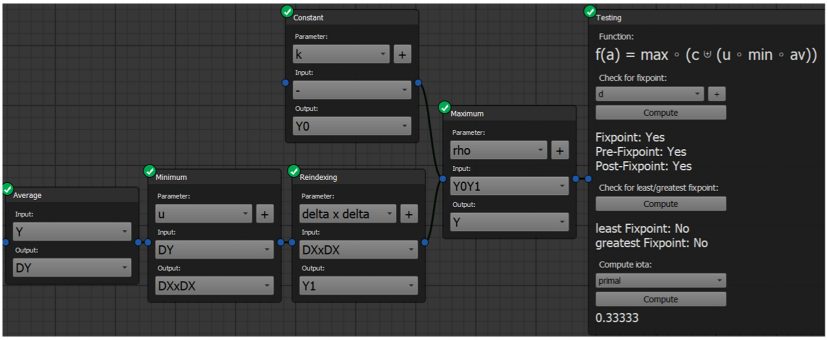

Fig. 4. Assembling the function f from Sect. 2.

Basic-Functions area: The Basic-Functions area contains the basic functions, encompassing those listed in Table 1 and additional ones. Via drag-and-drop (or right-click) these basic functions can be added to the Building area to create a Function box. Each such box requires three (in the case of $\tilde{D}$ two) Contents: The Input set, the Output set and the required parameters. These Contents are to be created in the Content area. Additionally the Basic-Functions area contains the auxiliary function Testing which we will discuss in the next paragraph.

Building area: The user can connect the created Function boxes to obtain the function of interest. Composing functions is as simple as connecting two Function boxes in the correct order and disjoint union is achieved by connecting two boxes to the same box. We note that Input and Output sets of connected Function boxes need to match, otherwise the function is not built correctly. Revisiting the example in Fig. 1, we display in Fig. 4 how this function can be assembled.

The special box Testing is always required at the end. Here, the user can enter some mapping $a\colon Y \to \mathbb{M}$, test if a is a fixpoint/pre-fixpoint/post-fixpoint of the built function f and afterwards compute the greatest fixpoint of the approximation ($\nu f^a_\#$ if we want to check whether $\mu f = a$). If the result is not the empty set ($\nu f^\#_a \neq \emptyset$) one can compute a suitable value for decreasing a, needed for iterating to the least fixpoint from above (respectively increasing a for iterating to the greatest fixpoint from below). There is additional support for comparison with pre- and post-fixpoints.

In the left-hand system in Fig. 1, the function $d\colon Y \to [0,1]$ with $d(3,3) = 0$, $d(1,1) = {}^1\!/_2$, $d(1,2) = d(2,1) = d(2,2) = {}^2\!/_3$ and 1 for all other pairs is a fixpoint of f (d is not a pseudometric). By clicking Compute in the Testing-box, UDEfix displays that d is a fixpoint and tells us that d is in fact not the least and not the greatest fixpoint. It also computes the greatest fixpoints of the approximations step by step and displays the results to the user.

Content area: Here the user can create sets, mappings and relations which are used to specify the basic functions. Creating a set is done by entering a name for the new set and clicking on the plus ("+"). The user can create a variety

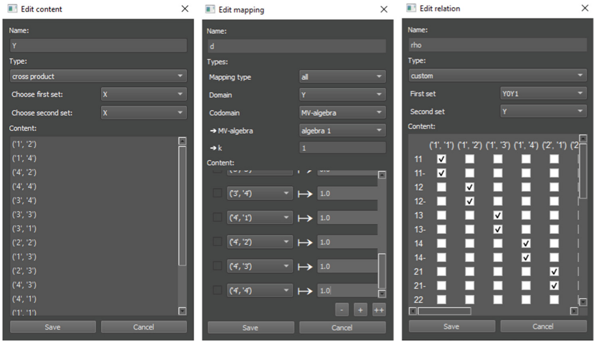

Fig. 5. Contents: Set Y, Mapping d, Relation ρ.

of different types of sets, for example the basic set $X = \{1, 2, 3, 4\}$ or the set $D = \{p_1, p_2, p_3, p_4\}$ which is a set of mappings resp. probability distributions.

Once, Input and Output sets are created we can define the required parameters (cf. Table 1). Here, the created sets can be chosen as domain and co-domain. Relations can be handled in a similar fashion: Given the two sets one wants to relate, creating a relation can be easily achieved by checking some boxes. Additionally the user has access to some useful in-built relations: "is-element-of"-relation and projections to the i-th component.

To ease the use, by clicking on the "+" in a Function box a new matching content with chosen Input and Output sets is created. The additional parameters (cf. Table 1) have domains and co-domains which need to be created or are the chosen MV-algebra. The Testing function d is a mapping as well.

See Fig. 5 for examples on how to create the contents Y (set), d (distance function) and ρ (relation).

Examples: There are pre-defined functions, implementing examples, that are shipped with the tool. These concern case studies on termination probability, bisimilarity, simple stochastic games, energy games, behavioural metrics and Rabin automata. See [7,8] for more details.

9 Conclusion, Related and Future Work

We have shown how our framework from [8] can be cast into a gs-monoidal setting, justifying the development of the tool UDEfix for a compositional view on fixpoint checks. In addition we studied properties of the gs-monoidal functor #, mapping from the concrete to the abstract universe and giving us a general procedure for approximating predicate liftings.

Related work: This paper is based on fixpoint theory, coalgebras, as well as on the theory of monoidal categories. Monoidal categories [17] are categories equipped with a tensor. It has long been realized that monoidal categories can have additional structure such as braiding or symmetries. Here we base our work on so called gs-monoidal categories [12,16], called s-monoidal in [15]. These are symmetric monoidal categories, equipped with a discharger and a duplicator. Note that "gs" originally stood for "graph substitution" and such categories were first used for modelling term graph rewriting.

We view gs-monoidal categories as a means to compositionally build monotone non-expansive functions on complete lattices, for which we are interested in the (least or greatest) fixpoint. Such fixpoints are ubiquitous in computer science, here we are in particular interested in applications in concurrency theory and games, such as bisimilarity [21], behavioural metrics [4,10,13,24] and simple stochastic games [11]. In recent work we have considered strategy iteration procedures inspired by games for solving fixpoint equations [7].

Fixpoint equations also arise in the context of coalgebra [20], a general framework for investigating behavioural equivalences for systems that are parameterized – via a functor – over their branching type (labelled, non-deterministic, probabilistic, etc.). Here in particular we are concerned with coalgebraic behavioural metrics [4], based on a generalization of the Wasserstein or Kantorovich lifting [25]. Such liftings require the notion of predicate liftings, well-known in coalgebraic modal logics [22], lifted to a quantitative setting [9].

Future Work: One important question is still open: we defined a lax functor $\#$, relating the concrete category $\mathbb{C}$ of functions of type $\mathbb{M}^Y \to \mathbb{M}^Z$ – where Y, Z might be infinite – to their approximations, living in $\mathbb{A}$. It is unclear whether $\#$ is a proper functor, i.e., preserves composition. For finite sets functoriality derives from a non-trivial result in [8] and it is unclear whether it can be extended to the infinite case. If so, this would be a valuable step to extend the theory.

In this paper we illustrated the approximation for predicate liftings via the powerset and the distribution functor. It would be interesting to study more functors and hence broaden the applicability to other types of transition systems.

Concerning UDEfix, we plan to extend the tool to compute fixpoints, either via Kleene iteration or strategy iteration (strategy iteration from above and below), as detailed in [7]. Furthermore for convenience it would be useful to have support for generating fixpoint functions directly from a given coalgebra respectively transition system.

References

1. Bacci, G., Bacci, G., Larsen, K.G., Mardare, R.: On-the-fly exact computation of bisimilarity distances. Logic. Meth. Comput. Sci. **13**(2:13), 1–25 (2017)
2. Bacci, G., Bacci, G., Larsen, K.G., Mardare, R., Tang, Q., van Breugel, F.: Computing probabilistic bisimilarity distances for probabilistic automata. Logic. Meth. Comput. Sci. **17**(1) (2021)
3. Baier, C., Katoen, J.P.: Principles of Model Checking. MIT Press (2008)
4. Baldan, P., Bonchi, F., Kerstan, H., König, B.: Coalgebraic behavioral metrics. Logic. Meth. Comput. Sci. **14**(3) (2018). Selected Papers of the 6th Conference on Algebra and Coalgebra in Computer Science (CALCO 2015)
5. Baldan, P., Eggert, R., König, B., Matt, T., Padoan, T.: A monoidal view on fixpoint checks (2023). arXiv:2305.02957

6. Baldan, P., Eggert, R., König, B., Padoan, T.: Fixpoint theory - upside down (2023). arXiv:2101.08184
7. Baldan, P., Eggert, R., König, B., Padoan, T.: A lattice-theoretical view of strategy iteration. In: Proceedings of CSL 2023, vol. 252 of LIPIcs, pp. 7:1–7:19. Schloss Dagstuhl - Leibniz Center for Informatics (2023)
8. Baldan, P., Eggert, R., König, B., Padoan, T.: Fixpoint theory - upside down. In: Proceedings of FOSSACS 2021, pp. 62–81. Springer (2021). LNCS/ARCoSS 12650
9. Bonchi, F., König, B., Petrişan, D.: Up-to techniques for behavioural metrics via fibrations. In: Proceedings of CONCUR 2018, volume 118 of LIPIcs, pp. 17:1–17:17. Schloss Dagstuhl - Leibniz Center for Informatics (2018)
10. Chen, D., van Breugel, F., Worrell, J.: On the complexity of computing probabilistic bisimilarity. In: Birkedal, L. (ed.) FoSSaCS 2012. LNCS, vol. 7213, pp. 437–451. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-28729-9_29
11. Condon, A.: On algorithms for simple stochastic games. In: Advances in Computational Complexity Theory, volume 13 of DIMACS Series in Discrete Mathematics and Theoretical Computer Science, pp. 51–71 (1990)
12. Corradini, A., Gadducci, F.: An algebraic presentation of term graphs, via GS-monoidal categories. Appl. Categor. Struct. **7**, 299–331 (1999)
13. Desharnais, J., Gupta, V., Jagadeesan, R., Panangaden, P.: Metrics for labelled Markov processes. Theor. Comput. Sci. **318**, 323–354 (2004)
14. Di Nola, A., Gerla, B.: Algebras of Lukasiewicz's logic and their semiring reducts. In: Idempotent Mathematics and Mathematical Physics, vol. 377 of Proceedings of the AMS, pp. 131–144 (2005)
15. Gadducci, F.: On the Algebraic Approach To Concurrent Term Rewriting. PhD thesis, University of Pisa (1996)
16. Gadducci, F., Heckel, R.: An inductive view of graph transformation. In: Presicce, F.P. (ed.) WADT 1997. LNCS, vol. 1376, pp. 223–237. Springer, Heidelberg (1998). https://doi.org/10.1007/3-540-64299-4_36
17. Mac Lane, S.: Categories for the Working Mathematician. Springer-Verlag (1971)
18. Mundici, D.: MV-algebras. A short tutorial. http://www.matematica.uns.edu.ar/IXCongresoMonteiro/Comunicaciones/Mundici_tutorial.pdf
19. Pattinson, D.: Coalgebraic modal logic: soundness, completeness and decidability of local consequence. Theor. Comput. Sci. **309**(1), 177–193 (2003)
20. Rutten, J.J.M.M.: Universal coalgebra: a theory of systems. Theor. Comput. Sci. **249**, 3–80 (2000)
21. Sangiorgi, D.: Introduction to Bisimulation and Coinduction. Cambridge University Press (2011)
22. Schröder, L.: Expressivity of coalgebraic modal logic: the limits and beyond. Theor. Comput. Sci. **390**, 230–247 (2008)
23. Tarski, A.: A lattice-theoretical theorem and its applications. Pacific J. Math. **5**, 285–309 (1955)
24. van Breugel, F.: Probabilistic bisimilarity distances. ACM SIGLOG News **4**(4), 33–51 (2017)
25. Villani, C.: Optimal Transport - Old and New, volume 338 of A Series of Comprehensive Studies in Mathematics. Springer (2009)