Functorial Concurrent Semantics for Petri Nets with Read and Inhibitor Arcs*

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Abstract. We propose a functorial concurrent semantics for Petri nets extended with *read* and *inhibitor* arcs, that we call *inhibitor nets*. Along the lines of the seminal work of Winskel on safe nets, the truly concurrent semantics is given at a categorical level via a chain of functors leading from the category **SW-IN** of semi-weighted inhibitor nets to the category **Dom** of finitary prime algebraic domains. As an intermediate semantic model, we introduce *inhibitor event structures*, an extension of prime event structures able to faithfully capture the dependencies among events which arise in the presence of read and inhibitor arcs.

Introduction

Several generalizations of Petri nets [15] have been proposed in the literature to overcome the expressiveness limitations of the classical model. At a very basic level Petri nets have been extended with two new kinds of arcs, namely *read arcs* (also called test, read, activator or positive contextual arcs) [7, 11, 8, 16] and *inhibitor arcs* (also called negative contextual arcs) [1, 11], which allow a transition to check for the presence, resp. absence of tokens, without consuming them. Read arcs have been shown to be useful to model in a natural way several practical situations (see e.g. [3, 2] for more references). A study of the expressiveness of inhibitor arcs, along with a comparison with other extensions proposed in the literature, namely priorities, exclusive-or transitions and switches, is carried out in [13]. In particular, we recall that inhibitor arcs make the model Turing complete, essentially because they allow to simulate the zero-testing operation of RAM machines, not expressible only with flow and read arcs.

The purpose of this paper is to give a truly concurrent semantics to Petri nets extended with read and inhibitor arcs, that we will call *inhibitor nets*. We follow the seminal work on ordinary safe nets of [12, 18], where the semantics is given at a categorical level via a chain of coreflections, leading from the category **S-N** of safe (marked) P/T nets to the category **Dom** of finitary prime algebraic domains, through the categories **O-N** of occurrence nets and **PES** of prime event structures (PES's), the last step being an equivalence of categories.

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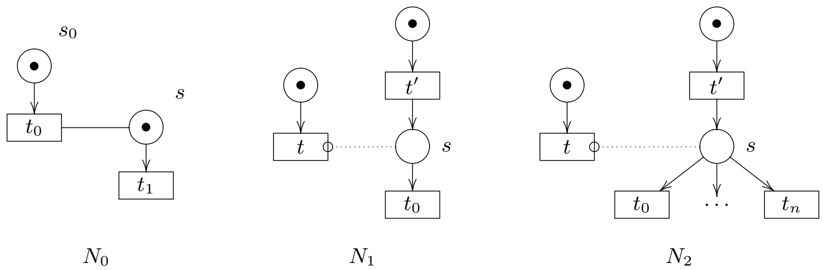

Fig. 1. Some basic contextual and inhibitor nets.

$$\begin{tikzcd}[column sep=large] \mathbf{S\text{-}N} \arrow[r, shift right, "\mathscr{U}"'] \arrow[r, phantom, "\perp"] & \mathbf{O\text{-}N} \arrow[l, hook', shift right] \arrow[r, shift right, "\mathscr{E}"'] \arrow[r, phantom, "\perp"] & \mathbf{PES} \arrow[l, shift right, "\mathscr{N}"'] \arrow[r, shift right, "\mathscr{L}"'] \arrow[r, phantom, "\sim"] & \mathbf{Dom} \arrow[l, shift right, "\mathscr{P}"'] \end{tikzcd}$$

As shown in [10] essentially the same construction applies to the wider category of *semi-weighted* nets, i.e. P/T nets in which the initial marking and the post-set of each transition is a set, rather than a proper multiset.

The mentioned approach has been extended in [4] to nets with read arcs, referred to as *contextual nets* (see also [17]). The key problem for the treatment of contextual nets is illustrated by the net N_0 of Fig. 1 where the same place s is “read” by transition t_0 and “consumed” by transition t_1 (a read arc is represented by an undirected, horizontal line). The firing of t_1 prevents t_0 to be executed, so that t_0 can never follow t_1 in a computation, while the converse is not true, since t_1 *can* fire after t_0. This situation can be interpreted naturally as an *asymmetric conflict* between the two transitions and cannot be represented faithfully in a PES. To model the behaviour of contextual nets, the paper [4] introduces *asymmetric event structures* (AES’s), an extension of prime event structures where the symmetric conflict is replaced by an asymmetric conflict relation. Such a feature is still necessary to be able to model the dependencies arising between events in a net with inhibitor arcs (also in the absence of read arcs). However the nonmonotonic features introduced by inhibitor arcs (negative conditions) make the situation far more complicated.

Consider the net N_1 in Fig. 1 where the place s, which inhibits transition t, is in the post-set of transition t' and in the pre-set of t_0 (an inhibitor arc is depicted as a dotted line from s to t, ending with an empty circle). The execution of t' inhibits the firing of t, which can be enabled again by the firing of t_0. Thus t can fire before or after the “sequence” $t'; t_0$, but not in between the two transitions. Roughly speaking there is a sort of atomicity of the sequence $t'; t_0$ w.r.t. t. The situation can be more involved since many transitions $t_0, \ldots, t_n$ may have the place s in their pre-set (see the net N_2 in Fig. 1). Therefore, after the firing of t', the transition t can be re-enabled by any of the conflicting transitions $t_0, \ldots, t_n$. This leads to a sort of *or*-causality, but only when t fires after t'. With a logical terminology we can say that t causally depends on $t' \Rightarrow t_0 \vee t_1 \vee \ldots \vee t_n$.

To face these complications in this paper we introduce *inhibitor event structures* (IES's), a generalization of AES's equipped with a ternary relation $\vdash\!\circ(\cdot,\cdot,\cdot)$, called *DE-relation (disabling-enabling relation)*, which allows one to model the dependency between transitions in N_2 as $\vdash\!\circ(\{t'\}, t, \{t_0, \ldots, t_n\})$. The configurations of an IES, endowed with a computational order, form a prime algebraic domain, and Winskel's equivalence between **PES** and **Dom** generalizes to a coreflection between the category **IES** of inhibitor event structures and **Dom**.

As for ordinary and contextual nets, the connection between nets and event structures is established via an unfolding construction which maps each net into an occurrence net. Then the unfolding can be naturally abstracted to an IES, having the transitions of the net as events. The main difference with respect to the case of ordinary and contextual nets is the absence of a functor performing the backward step from IES's to occurrence inhibitor nets. Hence the problem of characterizing the passage from occurrence inhibitor nets to event structures as a coreflection, and thus of fully extending Winskel's approach to inhibitor nets, remains open. We refer the reader to [3, 2] for a detailed treatment of the material presented in this paper and for a discussion of some related issues.

1 Inhibitor event structures

This section introduces the class of event structures that we consider adequate for modelling the complex phenomena which arise in the dynamics of inhibitor nets. Let us fix some notational conventions. The powerset of a set X is denoted by $\mathbf{2}^X$, while $\mathbf{2}^X_{fin}$ denotes the set of finite subsets of X and $\mathbf{2}^X_1$ the set of subsets of X of cardinality at most one. Hereafter generic subsets of events will be denoted by upper case letters $A, B, \ldots$, and singletons or empty subsets by $a, b, \ldots$

Definition 1 (pre-inhibitor event structure). *A* pre-inhibitor event structure *(pre-*IES*) is a pair $I = \langle E, \vdash\!\circ\rangle$, where E is a set of* events *and $\vdash\!\circ \subseteq \mathbf{2}^E_1 \times E \times \mathbf{2}^E$ is a ternary relation called* disabling-enabling relation (DE-relation)*.*

Informally, if $\vdash\!\circ(\{e'\}, e, A)$ then the event e' inhibits the event e, which can be enabled again by one of the events in A. The first argument of the relation can be also the empty set $\emptyset$, $\vdash\!\circ(\emptyset, e, A)$ meaning that the event e is inhibited in the initial state of the system. Also the third argument A can be empty, $\vdash\!\circ(\{e'\}, e, \emptyset)$ meaning that no events can re-enable e' after it has been disabled by e.

The DE-relation allows to represent both causality and asymmetric conflict and thus, concretely, it is the only relation of a (pre-)IES. In fact, if $\vdash\!\circ(\emptyset, e, \{e'\})$ then the event e can be executed only after e' has been fired. This is exactly what happens in a PES when e' causes e, or in symbols when $e' < e$. More generally, if $\vdash\!\circ(\emptyset, e, A)$ then we can imagine A as a set of disjunctive causes for e, since at least one of the events in A will appear in every history of the event e. This generalization of causality, restricted to the case in which the set A is pairwise conflictive (namely all distinct events in A are in conflict), will be represented in symbols as $A < e$. Similar notions of or-causality have been studied in general event structures [18], flow event structures [5] and in bundle event structures [9].

Furthermore, if $\vdash\!\circ(\{e'\}, e, \emptyset)$ then e can never follow e' in a computation since there are no events which can re-enable e after the execution of e'. Instead the converse order of execution is admitted, namely e *can* fire before e'. This situation is naturally interpreted as an *asymmetric conflict* between the two events and it is written $e \nearrow e'$. It can be seen also as a *weak* form of *causal dependency*, in the sense that if $e \nearrow e'$ then e precedes e' in all computations containing both events. This explains why a rule below imposes asymmetric conflict to include (also generalized) causality, by asking that $A < e$ implies $e' \nearrow e$ for all $e' \in A$.

Finally, cycles of asymmetric conflict are used to define a notion of conflict on sets of events. If $e_0 \nearrow e_1 \ldots e_n \nearrow e_0$ then all such events cannot appear together in the same computation, since each one should precede the others. This fact is formalized via a conflict relation on sets of events $\#\{e_0, e_1 \ldots, e_n\}$. In particular, binary (symmetric) conflict is represented by asymmetric conflict in both directions.

Definition 2 (dependency relations). *Let $I = \langle E, \vdash\!\circ \rangle$ be a pre-*IES*. The relations of* (generalized) causality $< \subseteq \mathbf{2}^E \times E$, asymmetric conflict $\nearrow \subseteq E \times E$ *and* conflict $\# \subseteq \mathbf{2}^E_{fin}$ *are defined by the following set of rules:*

$$\frac{\vdash\!\circ(\emptyset, e, A) \quad \#_p A}{A < e}\ (<1) \qquad \frac{A < e \quad \forall e' \in A.\ A_{e'} < e' \quad \#_p(\cup\{A_{e'} \mid e' \in A\})}{(\cup\{A_{e'} \mid e' \in A\}) < e}\ (<2)$$

$$\frac{\vdash\!\circ(\{e'\}, e, \emptyset)}{e \nearrow e'}\ (\nearrow 1) \qquad \frac{e \in A < e'}{e \nearrow e'}\ (\nearrow 2) \qquad \frac{\#\{e, e'\}}{e \nearrow e'}\ (\nearrow 3)$$

$$\frac{e_0 \nearrow \ldots \nearrow e_n \nearrow e_0}{\#\{e_0, \ldots, e_n\}}\ (\#1) \qquad \frac{A' < e \quad \forall e' \in A'.\ \#(A \cup \{e'\})}{\#(A \cup \{e\})}\ (\#2)$$

where $\#_p A$ means that A is pairwise conflictive, namely $\#\{e, e'\}$ for all $e, e' \in A$ with $e \neq e'$. We will write $e\#e'$ for $\#\{e, e'\}$ and $e < e'$ for $\{e\} < e'$.

The basic rules (<1), $(\nearrow 1)$ and $(\#1)$, as well as $(\nearrow 2)$ and $(\nearrow 3)$ are justified by the discussion above. Rule (<2) generalizes the transitivity of the causality relation, while rule $(\#2)$ expresses a kind of hereditarity of the conflict with respect to causality.

An inhibitor event structure is a pre-IES where the DE-relation satisfies some further requirements suggested by its intended meaning, and causality and asymmetric conflict are induced "directly" by the DE-relation.

Definition 3 (inhibitor event structure). *An* inhibitor event structure (IES) *is a pre-*IES *$I = \langle E, \vdash\!\circ \rangle$ satisfying, for all $e \in E$, $a \in \mathbf{2}^E_1$ and $A \subseteq E$,*

1. *if $\vdash\!\circ(a, e, A)$ then $\#_p A$ and $\forall e' \in a.\ \forall e'' \in A.\ e' < e''$;*
2. *if $A < e$ then $\vdash\!\circ(\emptyset, e, A)$;*
3. *if $e \nearrow e'$ then $\vdash\!\circ(\{e'\}, e, \emptyset)$.*

Given a pre-IES I satisfying only (1) it is always possible to "saturate" the relation $\vdash\!\circ$ in order to obtain an IES, where the relations of causality and (asymmetric) conflict are exactly the same as in I. The *"saturated"* IES is defined as $\overline{I} = \langle E, \vdash\!\circ' \rangle$, where $\vdash\!\circ' = \vdash\!\circ \cup \{(\emptyset, e, A) \mid A < e\} \cup \{(\{e\}, e', \emptyset) \mid e \nearrow e'\}$.

Definition 4 (category IES). *Let $I_0 = \langle E_0, \mapstochar\!\!\circ_0\rangle$ and $I_1 = \langle E_1, \mapstochar\!\!\circ_1\rangle$ be two* IES*'s. An* IES*-morphism $f : I_0 \to I_1$ is a partial function $f : E_0 \to E_1$ such that for all $e_0, e'_0 \in E_0$, $A_1 \subseteq E_1$, if $f(e_0)$ and $f(e'_0)$ are defined then*

1. $(f(e_0) = f(e'_0)) \wedge (e_0 \neq e'_0) \quad \Rightarrow \quad e_0 \#_0 e'_0$;
2. $A_1 < f(e_0) \quad \Rightarrow \quad \exists A_0 \subseteq f^{-1}(A_1).\ A_0 < e_0$;
3. $\mapstochar\!\!\circ_1(\{f(e'_0)\}, f(e_0), A_1) \quad \Rightarrow \quad \exists A_0 \subseteq f^{-1}(A_1).\ \exists a_0 \subseteq \{e'_0\}.\ \mapstochar\!\!\circ_0(a_0, e_0, A_0)$.

We denote by **IES** *the category of inhibitor event structures and* IES*-morphisms.*

Condition (1) is standard. Condition (2) generalizes the requirement of preservation of causes $\lfloor f(e) \rfloor \subseteq f(\lfloor e \rfloor)$ of PES (and AES) morphisms. Condition (3), as it commonly happens for event structures morphisms, just imposes the preservation of computations by asking, whenever some events in the image are constrained in some way, that stronger constraints are present in the pre-image. More precisely suppose that $\mapstochar\!\!\circ(\{f(e'_0)\}, f(e_0), A_1)$. Thus we can have a computation where $f(e'_0)$ is executed first and $f(e_0)$ can occur only after one of the events in A_1. Otherwise the computation can start with $f(e_0)$. According to condition (3), e_0 and e'_0 are subject in I_0 to the same constraint of their images or, when $a_0 = \emptyset$ or $A_0 = \emptyset$, to stronger constraints selecting one of the possible orders of execution.

The category **PES** of prime event structures can be viewed as a full subcategory of **IES**. The full embedding functor $\mathscr{J}_i : \mathbf{PES} \hookrightarrow \mathbf{IES}$ maps each PES $P = \langle E, \leq, \# \rangle$ to the IES $\langle E, \mapstochar\!\!\circ \rangle$ where the DE-relation is defined by $\mapstochar\!\!\circ(\emptyset, e, \{e''\})$ if $e'' < e$ and $\mapstochar\!\!\circ(\{e'\}, e, \emptyset)$ if $e \# e'$. For any PES morphism $f : P_1 \to P_2$ its image is $\mathscr{J}_i(f) = f$. More generally, the category of asymmetric event structures [4] fully embeds into **IES** (see [2]), and also (extended) bundle event structures [9] and prime event structures with possible events [14] can be seen as special IES's.

2 Inhibitor event structures and domains

The paper [18] shows that the categories **PES** of prime event structures and **Dom** of finitary prime algebraic domains are equivalent, via the functors $\mathscr{P} : \mathbf{Dom} \to \mathbf{PES}$ and $\mathscr{L} : \mathbf{PES} \to \mathbf{Dom}$. This section establishes a connection between IES's and finitary prime algebraic domains, by showing that the mentioned result generalizes to the existence of a categorical coreflection between **IES** and **Dom**. Then we study the problem of removing the non-executable events from an IES, by characterizing the full subcategory $\mathbf{IES^e}$ consisting of the IES's where all events are executable, as a coreflective subcategory of **IES**.

The domain of configurations

The domain associated to an IES is obtained by considering the family of its configurations with a suitable order. To understand the notion of IES configuration, consider a set of events C of an inhibitor event structure I, and suppose $e', e, e'' \in C$ and $\mapstochar\!\!\circ(\{e'\}, e, A)$ for some A, with $e'' \in A$. Note that two distinct orders of execution of the three events are possible (either $e; e'; e''$ or $e'; e''; e)$, which should not be confused from the point of view of causality. Hence, a configuration is not simply a set of events C, but some additional information must be added, in the form of a *choice relation*, to choose among the possible different orders of execution of events in C constrained by the DE-relation (e.g., in the above example a choice relation specifies wether e precedes e' or e'' precedes e).

We first introduce, for a given set of events C, the set $choices(C)$, a relation on C which "collects" *all* the possible precedences between events induced by the DE-relation. A choice relation for C is then a suitable subset of $choices(C)$.

Definition 5 (choice). *Let $I = \langle E, \mapsto \rangle$ be an* IES *and let $C \subseteq E$. We denote by $choices(C)$ the set*

$$\{(e, e') \mid \exists A.\ \mapsto_C(\{e'\}, e, A)\} \cup \{(e'', e) \mid \exists A.\ \mapsto_C(a, e, A) \wedge\ e'' \in A\} \subseteq C \times C,$$

where the restriction of $\mapsto(,,)$ to C is defined by $\mapsto_C(a, e, A)$ iff $\mapsto(a, e, A')$ for some A', with $e \in C$, $a \subseteq C$ and $A = A' \cap C$.
A choice *for C is an irreflexive relation $\hookrightarrow_C \subseteq choices(C)$ such that*

1. *if $\mapsto_C(a, e, A)$ then $\exists e' \in a.\ e \hookrightarrow_C e'$ or $\exists e'' \in A.\ e'' \hookrightarrow_C e$;*
2. *$(\hookrightarrow_C)^*$ is a finitary partial order.*

Condition (1) intuitively requires that whenever the DE-relation permits two possible orders of execution, the relation $\hookrightarrow_C$ chooses one of them. The fact that $\hookrightarrow_C \subseteq choices(C)$ ensures that $\hookrightarrow_C$ does not impose more precedences than necessary. Condition (2) guarantees that the precedences specified by $\hookrightarrow_C$ are not cyclic and that each event must be preceded only by finitely many others.

Definition 6 (configuration). *Let $I = \langle E, \mapsto \rangle$ be an* IES*. A* configuration *of I is a pair $\langle C, \hookrightarrow_C \rangle$, where $C \subseteq E$ and $\hookrightarrow_C \subseteq C \times C$ is a* choice *for C.*

It can be shown that the above definition generalizes the notion of PES and AES configuration since the property of admitting a choice implies causal closedness and conflict freeness. In the sequel, with abuse of notation, we will often denote a configuration and the underlying set of events with the same symbol C, referring to the corresponding choice relation as $\hookrightarrow_C$.

The computational order on configurations is a generalization of that introduced in [4] for AES's.

Definition 7 (extension). *Let $I = \langle E, \mapsto \rangle$ be an* IES *and let C and C' be configurations of I. We say that C'* extends *C, written $C \sqsubseteq C'$, if $C \subseteq C'$ and*

1. *$\forall e \in C.\ \forall e' \in C'.\ e' \hookrightarrow_{C'} e \quad \Rightarrow \quad e' \in C$;*
2. *$\hookrightarrow_C \subseteq \hookrightarrow_{C'}$.*

The poset of all configurations of I, ordered by extension, is denoted by $Conf(I)$.

As expressed by condition (1), a configuration C can be extended only by adding events which are not supposed to happen before other events already in C. Moreover, condition (2) ensures, together with (1), that the past history of events in C remains the same in C'. Indeed if $C \sqsubseteq C'$ then $\hookrightarrow_C = \hookrightarrow_{C'} \cap (C \times C)$, and thus, roughly speaking, C coincides with a "truncation" of C'.

The history of an event in a configuration C is formally defined as a subconfiguration of C. More precisely, for a configuration C and an event $e \in C$ the *history* of e in C is the configuration $\langle C[\![e]\!], \hookrightarrow_{C[\![e]\!]}\rangle$, where $C[\![e]\!] = \{e' \in C \mid e' \hookrightarrow^*_C e\}$ and $\hookrightarrow_{C[\![e]\!]} = \hookrightarrow_C \cap (C[\![e]\!] \times C[\![e]\!])$. Then it is possible to show that the poset of configurations of an IES has the desired algebraic structure.

Theorem 1 (domain of configurations). *For any* IES *I the poset $Conf(I)$ is a (finitary prime algebraic) domain. Its complete primes are the possible histories of events in I, i.e. $Pr(Conf(I)) = \{C[\![e]\!] \mid C \in Conf(I), e \in C\}$.*

The construction which associates the domain of configurations to an IES lifts to a functor from **IES** to **Dom**. Observe that since configurations are not simply sets of events it is not completely obvious, a priori, what should be the image of a configuration through a morphism. Let $f : I_0 \to I_1$ be an IES-morphism and let $\langle C_0, \hookrightarrow_0\rangle$ be a configuration of I_0. It is possible to show that $\hookrightarrow_1 = f(\hookrightarrow_0) \cap$ $choices(f(C_0))$ is the the unique choice relation on $f(C_0)$ included in $f(\hookrightarrow_{C_0})$. Furthermore the function $f^* : Conf(I_0) \to Conf(I_1)$ which associates to each configuration $\langle C_0, \hookrightarrow_0\rangle$ the configuration $\langle f(C_0), \hookrightarrow_1\rangle$ is a domain morphism.

This means that the construction taking an IES into its domain of configurations can be viewed as a functor $\mathscr{L}_i : \mathbf{IES} \to \mathbf{Dom}$ defined as $\mathscr{L}_i(I) = Conf(I)$ for each IES I and $\mathscr{L}_i(f) = f^*$ for each IES-morphism $f : I_0 \to I_1$.

A functor $\mathscr{P}_i : \mathbf{Dom} \to \mathbf{IES}$ going back from domains to IES's can be obtained as the composition of Winskel's functor $\mathscr{P} : \mathbf{Dom} \to \mathbf{PES}$ with the full embedding $\mathscr{J}_i : \mathbf{PES} \to \mathbf{IES}$ defined at the end of Section 1.

Theorem 2 (coreflection IES $\hookrightarrow$ Dom). *The functor $\mathscr{P}_i : \mathbf{Dom} \to \mathbf{IES}$ is left adjoint to $\mathscr{L}_i : \mathbf{IES} \to \mathbf{Dom}$. The counit of the adjunction at an* IES *I is the function $\epsilon_I : \mathscr{P}_i \circ \mathscr{L}_i(I) \to I$, mapping each history of an event e into the event e itself, i.e., $\epsilon_I(C[\![e]\!]) = e$, for all $C \in Conf(I)$ and $e \in C$.*

The above result, together with Winskel's equivalence between the categories **Dom** of domains and **PES** of prime event structures, allows to translate an IES I into a PES $\mathscr{P}(\mathscr{L}_i(I))$. The PES is obtained from the IES essentially by introducing an event for each possible different history of events in the IES.

Removing non-executable events

The non-executability of events in an IES is not completely captured by the proof system of Definition 2, in the sense that we cannot derive $\#\{e\}$ for every non-executable event. Here we propose a semantic approach to rule out unused events from an IES, namely we simply remove from a given IES all events which do not appear in any configuration.

Definition 8. *We denote by* $\mathbf{IES^e}$ *the full subcategory of* **IES** *consisting of the* IES*'s $I = \langle E, \vdash\!\circ\rangle$ such that for any $e \in E$ there exists $C \in Conf(I)$ with $e \in C$.*

Any IES is turned into an $\mathbf{IES^e}$ object by forgetting the events which do not appear in any configuration.

Definition 9. *We denote by* $\Psi : \mathbf{IES} \to \mathbf{IES^e}$ *the functor mapping each* IES I *into the* $\mathbf{IES^e}$ *object* $\Psi(I) = \overline{\langle \psi(E), \vdash\!\circ_{\psi(E)} \rangle}$*, where* $\psi(E)$ *is the set of executable events in* I*, namely* $\psi(E) = \{e \in E \mid \exists C \in \mathit{Conf}(I).\ e \in C\}$*. Moreover if* $f : I_0 \to I_1$ *is an* IES*-morphism then* $\Psi(f) = f_{|\psi(E_0)}$*. With* $\mathscr{J}_{ies} : \mathbf{IES^e} \to \mathbf{IES}$ *we denote the inclusion.*

The inclusion of $\mathbf{IES^e}$ into $\mathbf{IES}$ is left adjoint to Ψ, i.e., $\Psi \vdash \mathscr{J}_{ies}$, and thus $\mathbf{IES^e}$ is a coreflective subcategory of $\mathbf{IES}$. Furthermore the coreflection between $\mathbf{IES}$ and $\mathbf{Dom}$ restricts to a coreflection between $\mathbf{IES^e}$ and $\mathbf{Dom}$, i.e., if $\mathscr{P}_i^{\mathbf{e}} : \mathbf{Dom} \to \mathbf{IES^e}$ and $\mathscr{L}_i^{\mathbf{e}} : \mathbf{IES^e} \to \mathbf{Dom}$ denote the restrictions of the functors $\mathscr{P}_i$ and $\mathscr{L}_i$ then $\mathscr{P}_i^{\mathbf{e}} \dashv \mathscr{L}_i^{\mathbf{e}}$.

3 A category of inhibitor nets

Inhibitor nets are an extension of ordinary Petri nets where, by means of read and inhibitor arcs, transitions can check both for the presence and for the absence of tokens in places of the net. To give the formal definition we need some notation for multisets. Let A be a set; a *multiset* of A is a function $M : A \to \mathbb{N}$. The set of multisets of A is denoted by μA. The usual operations and relations on multisets, like multiset union $+$ or multiset difference $-$, are used. We write $M \leq M'$ if $M(a) \leq M'(a)$ for all $a \in A$. If $M \in \mu A$, we denote by $[\![M]\!]$ the multiset defined as $[\![M]\!](a) = 1$ if $M(a) > 0$ and $[\![M]\!](a) = 0$ otherwise; sometimes $[\![M]\!]$ will be confused with the corresponding subset $\{a \in A \mid [\![M]\!](a) = 1\}$ of A. A *multirelation* $f : A \to B$ is a multiset of $A \times B$. We will limit our attention to finitary multirelations, namely multirelations f such that the set $\{b \in B \mid f(a,b) > 0\}$ is finite. Multirelation f induces in an obvious way a function $\mu f : \mu A \to \mu B$, defined as $\mu f(\sum_{a \in A} n_a \cdot a) = \sum_{b \in B} \sum_{a \in A} (n_a \cdot f(a,b)) \cdot b$ (possibly partial, since infinite coefficients are disallowed). If f satisfies $f(a,b) \leq 1$ for all $a \in A$ and $b \in B$, i.e. $f = [\![f]\!]$, then we sometimes confuse it with the corresponding set-relation and write $f(a,b)$ for $f(a,b) = 1$.

Definition 10 (inhibitor net). *A* (marked) inhibitor Petri net (i-net) *is a tuple* $N = \langle S, T, F, C, I, m \rangle$*, where* S *is a set of* places*,* T *is a set of* transitions *(with* $S \cap T = \emptyset$*),* $F = \langle F_{pre}, F_{post} \rangle$ *is a pair of multirelations from* T *to* S*,* C *and* I *are relations between* T *and* S*, called the* context *and* inhibitor relation*, respectively, and* m *is a multiset of* S*, called the* initial marking*. If the inhibitor relation* I *is empty then* N *is called a* contextual net (c-net)*.*

We require that for each $t \in T$, $F_{pre}(t,s) > 0$ for some place $s \in S$. Hereafter, when considering an i-net N, we will assume that $N = \langle S, T, F, C, I, m \rangle$. Subscripts on the net name carry over the names of the net components.

As usual, given a finite multiset of transitions $A \in \mu T$ we write ${}^{\bullet}A$ for its *pre-set* $\mu F_{pre}(A)$ and $A^{\bullet}$ for its *post-set* $\mu F_{post}(A)$. Moreover, by $\underline{A}$ we denote the *context* of A, defined as $\underline{A} = C([\![A]\!])$, and by ${}^{\circledcirc}A = I([\![A]\!])$ the *inhibitor set* of A. The same notation is used to denote the functions from S to $\mathbf{2}^T$ defined as, for $s \in S$, ${}^\bullet s = \{t \in T \mid F_{post}(t,s) > 0\}$, $s^\bullet = \{t \in T \mid F_{pre}(t,s) > 0\}$, $\underline{s} = \{t \in T \mid C(t,s)\}$ and ${}^{\circledcirc}s = \{t \in T \mid I(t,s)\}$.

Let N be an i-net. A finite multiset of transitions A is enabled at a marking M, if M contains the pre-set of A and an additional multiset of tokens which covers the context of A. Furthermore no token must be present nor produced by the transitions in the places of the inhibitor set of A. Formally, a finite multiset $A \in \mu T$ is *enabled* at M if ${}^\bullet A + \underline{A} \le M$ and $[\![M + A^\bullet]\!] \cap {}^{\circledcirc}A = \emptyset$. In this case, to indicate that the execution of A in M produces the new marking $M' = M - {}^\bullet A + A^\bullet$ we write $M\,[A\rangle\,M'$. Step and firing sequences, as well as reachable markings are defined in the usual way.

Definition 11 (i-net morphism). *Let N_0 and N_1 be i-nets. An* i-net morphism *$h : N_0 \to N_1$ is a pair $h = \langle h_T, h_S\rangle$, where $h_T : T_0 \to T_1$ is a partial function and $h_S : S_0 \to S_1$ is a multirelation such that (1) $\mu h_S(m_0) = m_1$ and (2) for each $t \in T$,*

(a) $\mu h_S({}^\bullet t) = {}^\bullet h_T(t)$ *(c)* $\mu h_S(\underline{t}) = \underline{h_T(t)}$

(b) $\mu h_S(t^\bullet) = h_T(t)^\bullet$ *(d)* $[\![h_S]\!]^{-1}({}^{\circledcirc}h_T(t)) \subseteq {}^{\circledcirc}t$.

where $[\![h_S]\!]$ is the set relation underlying the multirelation h_S. We denote by **IN** *the category having i-nets as objects and i-net morphisms as arrows, and by* **CN** *its full subcategory of c-nets.*

Conditions (1), (2.a) - (2.b) are the defining conditions of Winskel's morphisms on ordinary nets, while (2.c) is the obvious condition which takes into account contexts. Condition (2.d) regarding the inhibitor arcs can be understood if we think of morphisms as simulations. Like preconditions and contexts must be preserved to ensure that the morphism maps computations of N_0 into computations of N_1, similarly, inhibitor conditions, which are negative conditions, must be reflected. In fact, condition (2.d) on inhibiting places can be rewritten as

$$s_1 \in [\![\mu h_S(s_0)]\!] \;\wedge\; I_1(h_T(t_0), s_1) \quad \Rightarrow \quad I_0(t_0, s_0),$$

which shows more explicitly that inhibitor arcs are reflected.

Proposition 1 (morphisms preserve the token game). *Let N_0 and N_1 be i-nets, and let $h = \langle h_T, h_S\rangle : N_0 \to N_1$ be an i-net morphism. For each $M, M' \in \mu S$ and $A \in \mu T$, if $M\,[A\rangle\,M'$ then $\mu h_S(M)\,[\mu h_T(A)\rangle\,\mu h_S(M')$. Therefore i-net morphisms preserve reachable markings, i.e. if M_0 is a reachable marking in N_0 then $\mu h_S(M_0)$ is reachable in N_1.*

As in [18, 10, 4] we will restrict our attention to a subclass of nets where each token produced in a computation has a uniquely determined history.

Definition 12 (semi-weighted and safe i-nets). *An i-net N is called* semi-weighted *if the initial marking m is a set and F_{post} is a relation (i.e., $t^\bullet$ is a set for all $t \in T$). We denote by* **SW-IN** *the full subcategory of* **IN** *having semi-weighted i-nets as objects; the corresponding subcategory of c-nets is denoted by* **SW-CN**. *A semi-weighted i-net is called* safe *if also F_{pre} is a relation and each reachable marking is a set.*

4 Occurrence i-nets and the unfolding constructions

Generally speaking, an occurrence net provides a static representation of some computations of a net, in which the events (firing of transitions) and the relationships between events are made explicit. In [4] the notion of (nondeterministic) occurrence net has been generalized to the case of nets with read arcs. Here, the presence of the inhibitor arcs and the complex kind of dependencies they induce on transitions make it hard to find an appropriate notion of occurrence i-net.

We present two different, in our opinion both reasonable, notions of occurrence i-net and, correspondingly, we develop two unfolding constructions.

In the first construction, given an i-net N, we consider the underlying contextual net N_c obtained by forgetting the inhibitor arcs, and we apply to N_c the unfolding construction for contextual nets of [4], which produces an occurrence contextual net $\mathcal{U}_a(N_c)$. Then, if a place s and a transition t were originally connected by an inhibitor arc in the net N, then we insert an inhibitor arc between each copy of s and each copy of t in $\mathcal{U}_a(N_c)$, thus obtaining the unfolding $\mathcal{U}_i(N)$ of the net N. Then the characterization of the unfolding as a universal construction can be lifted from contextual to inhibitor nets. Furthermore, in this way the unfolding of an inhibitor net is decidable, in the sense that the problem of establishing if a possible transition occurrence actually appears in the unfolding is decidable. The price to pay is that some transitions in the unfolding may not be firable, since they are generated without taking care of inhibitor places.

In the second approach, the dependency relations (of causality and asymmetric conflict) for a net are defined only with respect to a fixed assignment for the net (playing a role similar to choices) which specifies for any inhibitor arc (t, s) if the inhibited transition t is executed before or after the place s is filled and in the second case which one of the transitions in the post-set $s^\bullet$ of the inhibitor place consumes the token. Then the firability of a transition t amounts to the existence of an assignment which is acyclic on the transitions which must be executed before t. Relying on this idea we can define a notion of occurrence net where each transition is really executable. The corresponding unfolding construction produces a net where the mentioned problem of the existence of non-firable transitions disappears, but, as a consequence of the Turing completeness of inhibitor nets, the produced unfolding is not decidable.

Lifting the unfolding from contextual to inhibitor nets

In the first approach, the unfolding construction disregards the inhibitor arcs. Consequently the notion of occurrence i-net is defined without taking into account the dependencies between transitions induced by such kind of arcs.

Given a safe i-net N let us define the *read causality relation* as the least transitive relation $<_r$ on $S \cup T$ such that $s <_r t$ if $s \in {}^\bullet t$, $t <_r s$ if $s \in t^\bullet$, and $t <_r t'$ if $t^\bullet \cap \underline{t'} \neq \emptyset$, the only novelty with respect to ordinary nets being the last clause stating that a transition causally depends on transitions generating tokens in its context. The *read asymmetric conflict* $\nearrow_r$ is defined by taking $t \nearrow_r t'$ if t' consumes a token in the context of t, namely $\underline{t} \cap {}^\bullet t' \neq \emptyset$, in such a way that the firing of t' inhibits t. Moreover $t \nearrow_r t'$ if $(t \neq t' \ \wedge \ {}^\bullet t \cap {}^\bullet t' \neq \emptyset)$ to capture the usual symmetric conflict, and finally, according to the weak causality interpretation of the asymmetric conflict, $t \nearrow_r t'$ whenever $t <_r t'$.

Definition 13 (occurrence i-nets). *An* occurrence i-net N *is a safe i-net* N *where causality* $\leq_r$ *is a finitary partial order, asymmetric conflict* $\nearrow_r$ *is acyclic on the causes of each transition, for all* $s \in S$ $|{}^\bullet s| \leq 1$ *and* $m = \{s \in S \mid {}^\bullet s = \emptyset\}$. **O-IN** *denotes the full subcategory of* **SW-IN** *having occurrence i-nets as objects.*

Let us consider a functor $\mathscr{R}_{ic} : \mathbf{SW\text{-}IN} \to \mathbf{SW\text{-}CN}$ which maps each i-net into the underlying c-net, forgetting the inhibitor relation, and the inclusion $\mathscr{I}_{ci} : \mathbf{SW\text{-}CN} \to \mathbf{SW\text{-}IN}$ (see the diagram below).

$$\begin{tikzcd}
\mathbf{O\text{-}CN} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathbf{SW\text{-}CN} \arrow[l, shift left, "\mathscr{U}_a"] \arrow[r, hook, shift left, "\mathscr{I}_{ci}"] & \mathbf{SW\text{-}IN} \arrow[l, shift left, "\mathscr{R}_{ic}"] & \mathbf{O\text{-}IN} \arrow[l, hook', "\mathscr{I}_O"']
\end{tikzcd}$$

The relations $\leq_r$ and $\nearrow_r$ for an i-net N are exactly the relations of causality and asymmetric conflict of the underlying c-net $\mathscr{R}_{ic}(N)$, as defined in [4]. Thus the notion of occurrence c-net in this paper (i.e., occurrence i-net without inhibitor arcs) coincides with that of [4]. Moreover an occurrence i-net is a safe i-net N such that $\mathscr{R}_{ic}(N)$ is an occurrence c-net. Let **O-CN** be the category of occurrence c-nets, namely the full subcategory of **O-IN** having c-nets as objects. The paper [4] defines an unfolding functor $\mathscr{U}_a : \mathbf{SW\text{-}CN} \to \mathbf{O\text{-}CN}$, mapping each semi-weighted c-net to an occurrence c-net. The functor $\mathscr{U}_a$ is shown there to be right adjoint to the inclusion functor of **O-CN** into **SW-CN**. Using the functors $\mathscr{R}_{ic}$ and $\mathscr{I}_{ci}$ we can lift both the construction and the result to inhibitor nets.

Definition 14 (unfolding). *Let* N *be a semi-weighted i-net. Consider the occurrence c-net* $\mathscr{U}_a(\mathscr{R}_{ic}(N)) = \langle S', T', F', C', \emptyset, m' \rangle$ *and the folding morphism* $f_N : \mathscr{U}_a(\mathscr{R}_{ic}(N)) \to \mathscr{R}_{ic}(N)$. *Define an inhibitor relation on the net* $\mathscr{U}_a(\mathscr{R}_{ic}(N))$ *by taking for* $s' \in S'$ *and* $t' \in T'$, $I'(s', t')$ *iff* $I(f_N(s'), f_N(t'))$. *Then the unfolding* $\mathscr{U}_i(N)$ *of the i-net* N *is the occurrence i-net* $\langle S', T', F', C', I', m' \rangle$ *and the* folding morphism *is given by* f_N *seen as a morphism from* $\mathscr{U}_i(N)$ *to* N.

The i-net $\mathscr{U}_i(N)$ can be shown to be the *least* occurrence i-net which extends $\mathscr{U}_a(\mathscr{R}_{ic}(N))$ with the addition of inhibitor arcs in a way that $f_N : \mathscr{U}_i(N) \to N$ is a well defined i-net morphism.

Theorem 3. *The unfolding extends to a functor* $\mathscr{U}_i : \mathbf{SW\text{-}IN} \to \mathbf{O\text{-}IN}$ *which is right adjoint to the obvious inclusion functor* $\mathscr{I}_O : \mathbf{O\text{-}IN} \to \mathbf{SW\text{-}IN}$, *thus establishing a coreflection between* **SW-IN** *and* **O-IN**. *The component at an object* N *in* **SW-IN** *of the counit of the adjunction,* $f : \mathscr{I}_O \circ \mathscr{U}_i \stackrel{\cdot}{\to} 1$, *is the folding morphism* $f_N : \mathscr{U}_i(N) \to N$.

Executable occurrence i-nets

The second approach is inspired by the notion of deterministic process of an i-net introduced in [6], where the inhibitor arcs of the net underlying a process are partitioned into two subsets: the *before* inhibitor arcs and the *after* inhibitor arcs. Then the dependencies induced by such a partition are required to be acyclic in order to guarantee the firability of all the transitions of the net in a single computation. Following this idea, to ensure that each transition of a nondeterministic occurrence net is firable in *some* computation, we require, for each transition t, the *existence* of a so-called assignment which partitions the inhibitor arcs into before and after arcs, without introducing cyclic dependencies on the transitions which must be executed before t.

Definition 15 (assignment). *Let N be a safe i-net. An* assignment *for N is a function $\rho : I \to T$ such that, for all $(t, s) \in I$, $\rho(t, s) \in {}^{\bullet}s \cup s^{\bullet}$.*

Intuitively, an assignment ρ specifies for each inhibitor arc (t, s), if the transition t fires *before* or *after* the place s receives a token. If $\rho(t, s) \in {}^{\bullet}s$ then (t, s) is a before arc, while if $\rho(t, s) \in s^{\bullet}$ then (t, s) is an after arc. In the last case, since we are considering possibly nondeterministic nets and thus the place s may be in the pre-set of several transitions, the assignment specifies also which of the transitions in $s^{\bullet}$ consumes the token.

Given a safe net N, once an assignment ρ for N is fixed, new dependencies arise between the transitions of the net, formalized by means of the relations $\prec_i^{\rho}$ and $\nearrow_i^{\rho}$. We define $t \prec_i^{\rho} t'$ iff $\exists s \in {}^{\circledcirc}t' \cap {}^{\bullet}t.\ \rho(t', s) = t$ and $t \nearrow_i^{\rho} t'$ iff $\exists s \in {}^{\circledcirc}t \cap t'^{\bullet}.\ \rho(t, s) = t'$. Observe that, as suggested by the adopted symbols, the additional dependencies can be seen as a kind of causality and asymmetric conflict, respectively. In fact if $t \prec_i^{\rho} t'$, then t' can happen only after t has removed the token from s, and thus t acts as a cause for t'. If $t \nearrow_i^{\rho} t'$ then if both t and t' happen in the same computation then necessarily t occurs before t', since t' generates a token in a place s which inhibits t, while according to the interpretation of ρ, t must occur before the place s is filled.

Under a fixed assignment ρ, we can introduce a kind of generalized causality and asymmetric conflict by joining the relations $\leq_r$ and $\nearrow_r$ defined before with the additional dependencies induced by the inhibitor arcs. We define $<_{\rho} = (<_r \cup \prec_i^{\rho})^+$ and $\lhd_{\rho} = <_{\rho} \cup \nearrow_r \cup \nearrow_i^{\rho}$, i.e., $\lhd_{\rho}$ records both kinds of dependency. Furthermore, for $x \in S \cup T$ we denote by $\lfloor x \rfloor_{\rho}$ the set $\{t \in T \mid t \leq_{\rho} x\}$, and similarly, for $X \subseteq S \cup T$, we define $\lfloor X \rfloor_{\rho} = \bigcup \{\lfloor x \rfloor_{\rho} \mid x \in X\}$.

Definition 16 (executable occurrence i-net). *An* executable occurrence i-net *is a safe i-net N such that (i) for all $t \in T$ there exists an assignment ρ such that $(\lhd_{\rho})_{\lfloor t \rfloor_{\rho}}$ is acyclic and $\lfloor t \rfloor_{\rho}$ is finite, (ii) for all $s \in S$, $|{}^{\bullet}s| \leq 1$, and (iii) $m = \{s \in S \mid {}^{\bullet}s = \emptyset\}$.*

Hence executable occurrence i-nets refine occurrence i-nets by considering also the dependencies induced by inhibitor arcs. We denote by $\mathbf{O\text{-}IN^e}$ the full subcategory of **O-IN** having executable occurrence i-nets as objects.

Definition 17 (concurrency). *A set of places $M \subseteq S$ is called* concurrent*, written $conc(M)$, if there exists an assignment ρ such that (i) for all $s, s' \in M$ $\neg(s <_{\rho} s')$, (ii) $\lfloor M \rfloor_{\rho}$ is finite and (iii) $\lhd_{\rho}$ acyclic on $\lfloor M \rfloor_{\rho}$.*

As for ordinary and contextual nets, a set of places M is concurrent if and only if there is a reachable marking in which all the places of M contain a token. Consequently each transition of an executable occurrence i-net can fire in some computation (and thus each place contains a token at some reachable marking), a property which justifies the name "executable".

Proposition 2. *Let N be an executable occurrence i-net. Then for each transition $t \in T$ there exists a reachable marking M such that t is enabled at M.*

We can now introduce a different unfolding construction, that, when applied to a semi-weighted i-net N, produces an executable occurrence i-net.

Definition 18 ((executable) unfolding). *Let N be a semi-weighted i-net. The* (executable) unfolding $\mathscr{U}_i^{\mathbf{e}}(N) = \langle S', T', F', C', I', m' \rangle$ *of the net N and the* folding morphism $f_N = \langle f_T, f_S \rangle : \mathscr{U}_i^{\mathbf{e}}(N) \to N$ *are the unique executable occurrence i-net and i-net morphism satisfying the following equations:*

$$
\begin{aligned}
m' &= \{\langle \emptyset, s \rangle \mid s \in m\} \\
S' &= m' \cup \{\langle t', s \rangle \mid t' \in T' \ \wedge \ s \in f_T(t')^{\bullet}\} \\
T' &= \{t' \mid t' = \langle M_p, M_c, t \rangle \ \wedge \ t \in T \ \wedge \ M_p \cup M_c \subseteq S' \ \wedge \ M_p \cap M_c = \emptyset \\
&\qquad\quad \wedge \ conc(M_p \cup M_c) \wedge \ \mu f_S(M_p) = {}^{\bullet}t \ \wedge \ \mu f_S(M_c) = \underline{t} \\
&\qquad\quad \wedge \ \exists \rho.\ (\lfloor t' \rfloor_\rho \ \mathit{finite} \ \wedge \ \lhd_\rho \ \mathit{acyclic\ on} \ \lfloor t' \rfloor_\rho)\}
\end{aligned}
$$

$$
\begin{array}{lll}
F'_{pre}(t', s') & \text{iff} & t' = \langle M_p, M_c, t \rangle \ \wedge \ s' \in M_p \quad (t \in T) \\
F'_{post}(t', s') & \text{iff} & s' = \langle t', s \rangle \quad (s \in S) \\
C'(t', s') & \text{iff} & t' = \langle M_p, M_c, t \rangle \ \wedge \ s' \in M_c \quad (t \in T) \\
I'(t', s') & \text{iff} & f_S(s', s) \ \wedge \ I(f_T(t'), s)
\end{array}
$$

$$
\begin{array}{lll}
f_T(t') = t & \text{iff} & t' = \langle M_p, M_c, t \rangle \\
f_S(s', s) & \text{iff} & s' = \langle x, s \rangle \quad (x \in T' \cup \{\emptyset\})
\end{array}
$$

As usual, places and transitions in the unfolding represent tokens and firing of transitions in the original net. Each item of the unfolding is a copy of an item in the original net, enriched with the corresponding "history". The folding morphism f maps each item of the unfolding to the corresponding item in the original net. The unfolding can be given also an inductive definition, from which uniqueness easily follows.

The two proposed unfolding constructions are tightly related, in the sense that $\mathscr{U}_i^{\mathbf{e}}(N)$ can be obtained from $\mathscr{U}_i(N)$ simply by removing the non executable transitions. This fact is formalized by defining a "pruning" functor $\Pi : \mathbf{O\text{-}IN} \to \mathbf{O\text{-}IN^e}$ which removes the non executable transitions from a general occurrence i-net thus producing an executable occurrence i-net. The functor $\Pi : \mathbf{O\text{-}IN} \to \mathbf{O\text{-}IN^e}$ is right adjoint to the inclusion functor $\mathscr{J}^{\mathbf{e}} : \mathbf{O\text{-}IN^e} \to \mathbf{O\text{-}IN}$, and thus $\mathbf{O\text{-}IN^e}$ is a coreflective subcategory of $\mathbf{O\text{-}IN}$. Then one can formally state the relationship between $\mathscr{U}_i^{\mathbf{e}}(N)$ and $\mathscr{U}_i(N)$, providing also an indirect proof of the universality of the new unfolding construction.

Proposition 3. *For any semi-weighted i-net N, $\mathscr{U}_i^{\mathbf{e}}(N) = \Pi(\mathscr{U}_i(N))$. Therefore $\mathscr{U}_i^{\mathbf{e}}$ is right adjoint to the inclusion functor and they establish a coreflection between* $\mathbf{SW\text{-}IN}$ *and* $\mathbf{O\text{-}IN^e}$.

5 Inhibitor event structure semantics for i-nets

In this section we define an event structure and a domain semantics for i-nets by relating occurrence i-nets and inhibitor event structures. The dependencies arising among transitions in an occurrence i-net can be naturally represented by the DE-relation, and therefore the IES corresponding to an occurrence i-net is obtained by forgetting the places and taking the transitions of the net as events.

Definition 19. *Let N be an occurrence i-net. The pre-*IES *associated to N is defined as $I^p_N = \langle T, \vdash\!\circ^p_N\rangle$, with $\vdash\!\circ_N \subseteq \mathbf{2}^T_1 \times T \times \mathbf{2}^T$, given by: for $t, t' \in T$, $t \neq t'$ and $s \in S$*

1. *if $t^\bullet \cap ({}^\bullet t' \cup \underline{t'}) \neq \emptyset$ then $\vdash\!\circ^p_N(\emptyset, t', \{t\})$*
2. *if $({}^\bullet t \cup \underline{t}) \cap {}^\bullet t' \neq \emptyset$ then $\vdash\!\circ^p_N(\{t'\}, t, \emptyset)$;*
3. *if $s \in {}^{\circledcirc}t$ then $\vdash\!\circ^p_N({}^\bullet s, t, s^\bullet)$.*

The IES *associated to N, denoted by $\mathscr{E}_i(N) = \langle T, \vdash\!\circ_N\rangle$, is obtained by saturating I^p_N, i.e., $\mathscr{E}_i(N) = \overline{I^p_N}$.*

Clauses (1) and (2) encode, by using the DE-relation, the causal dependencies and the asymmetric conflicts induced by the flow and read arcs (we could have written if $t <_r t'$ then $\vdash\!\circ^p_N(\emptyset, t', \{t\})$ and if $t \nearrow_r t'$ then $\vdash\!\circ^p_N(\{t'\}, t, \emptyset)$). The last clause fully exploits the expressiveness of the DE-relation to represent the dependencies induced by inhibitor places.

Since the transition component of an i-net morphism is an IES-morphism between the corresponding IES's we have the following result.

Proposition 4. *The construction which maps each i-net N to the corresponding* IES *$\mathscr{E}_i(N)$ can be extended to a functor $\mathscr{E}_i : \mathbf{O\text{-}IN} \to \mathbf{IES}$ by defining $\mathscr{E}_i(h) = h_T$ for each morphism $h : N_0 \to N_1$.*

One can verify that if N is an executable occurrence i-net then $\mathscr{E}_i(N)$ is an $\mathbf{IES^e}$ object, and thus the functor $\mathscr{E}_i$ restricts to a functor $\mathscr{E}^{\mathbf{e}}_i : \mathbf{O\text{-}IN^e} \to \mathbf{IES^e}$.

The converse step, from IES's to occurrence i-nets, instead, turns out to be very problematic. An object level constructions can be defined, associating to any IES a corresponding i-net. However the problem of finding a functorial construction (if any) is still unsolved. See [2, 3] for a wider discussion suggesting how the difficulties are intimately connected to or-causality.

6 Conclusions

We have defined a functorial concurrent semantics for semi-weighted Petri nets with read and inhibitor arcs. The proposed constructions, which generalize Winskel's work on safe ordinary nets and the work in [4] on contextual nets, are summarized in Fig. 2. Unfortunately, the objective of providing a coreflective semantics for inhibitor nets is partially missed, since the construction mapping each occurrence i-net to an IES is not expressed as a coreflection. Hence the problem of fully extending to i-nets Winskel's chain of coreflections remains open.

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$$
\begin{tikzcd}
 & \mathbf{O\text{-}IN} \arrow[dl, hook] \arrow[r, "\mathscr{E}_i"] \arrow[d, "\Pi"', shift right] \arrow[d, phantom, "\vdash"] & \mathbf{IES} \arrow[dr, "\mathscr{L}_i"'] \arrow[d, "\Psi"', shift right] \arrow[d, phantom, "\vdash"] & & \\
\mathbf{SW\text{-}IN} \arrow[ur, "\mathscr{U}_i"'] \arrow[dr, "\mathscr{U}_i^{\mathbf{e}}"'] \arrow[r, phantom, "\perp"] & & & \mathbf{Dom} \arrow[ul, "\mathscr{P}_i"'] \arrow[dl, "\mathscr{P}_i^{\mathbf{e}}"'] \arrow[r, "\mathscr{P}"', shift right] \arrow[r, phantom, "\sim"] & \mathbf{PES} \arrow[l, "\mathscr{L}"', shift right] \\
 & \mathbf{O\text{-}IN^e} \arrow[ul, hook] \arrow[u, hook, shift right] \arrow[r, "\mathscr{E}_i^{\mathbf{e}}"] & \mathbf{IES^e} \arrow[u, hook, shift right] \arrow[ur, "\mathscr{L}_i^{\mathbf{e}}"'] & &
\end{tikzcd}
$$

Fig. 2. A summary of the constructions in the paper (unnamed functors are inclusions).

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