FIXPOINT THEORY - UPSIDE DOWN

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ABSTRACT. Knaster-Tarski's theorem, characterising the greatest fixpoint of a monotone function over a complete lattice as the largest post-fixpoint, naturally leads to the so-called coinduction proof principle for showing that some element is below the greatest fixpoint (e.g., for providing bisimilarity witnesses). The dual principle, used for showing that an element is above the least fixpoint, is related to inductive invariants. In this paper we provide proof rules which are similar in spirit but for showing that an element is above the greatest fixpoint or, dually, below the least fixpoint. The theory is developed for non-expansive monotone functions on suitable lattices of the form \mathbb{M}^Y , where Y is a finite set and \mathbb{M} an MV-algebra, and it is based on the construction of (finitary) approximations of the original functions. We show that our theory applies to a wide range of examples, including termination probabilities, metric transition systems, behavioural distances for probabilistic automata and bisimilarity. Moreover it allows us to determine original algorithms for solving simple stochastic games.

1. INTRODUCTION

Fixpoints are ubiquitous in computer science as they provide a meaning to inductive and coinductive definitions (see, e.g., [San11, NNH10]). A monotone function $f: L \to L$ over a complete lattice (L, \sqsubseteq) , by Knaster-Tarski's theorem [Tar55], admits a least fixpoint μf and greatest fixpoint νf which are characterised as the least pre-fixpoint and the greatest post-fixpoint, respectively. This immediately gives well-known proof principles for showing that a lattice element $l \in L$ is below νf or above μf

$$\frac{l \sqsubseteq f(l)}{l \sqsubseteq \nu f} \qquad \qquad \frac{f(l) \sqsubseteq l}{\mu f \sqsubseteq l}$$

On the other hand, showing that a given element l is above νf or below μf is more difficult. One can think of using the characterisation of least and largest fixpoints via Kleene's iteration. E.g., the largest fixpoint is the least element of the (possibly transfinite)

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descending chain obtained by iterating f from \top . Then showing that $f^i(\top) \sqsubseteq l$ for some i, one concludes that $\nu f \sqsubseteq l$. This proof principle is related to the notion of ranking functions. However, this is a less satisfying notion of witness since f has to be applied i times, and this can be inefficient or unfeasible when i is an infinite ordinal.

The aim of this paper is to present an alternative proof rule for this purpose for functions over lattices of the form $L = \mathbb{M}^Y$ where Y is a finite set and M is an MV-chain, i.e., a totally ordered complete lattice endowed with suitable operations of sum and complement. This allows us to capture several examples, ranging from ordinary relations for dealing with bisimilarity to behavioural metrics, termination probabilities and simple stochastic games.

Assume $f : \mathbb{M}^Y \to \mathbb{M}^Y$ monotone and consider the question of proving that some fixpoint $a : Y \to \mathbb{M}$ is the largest fixpoint νf . The idea is to show that there is no "slack" or "wiggle room" in the fixpoint a that would allow us to further increase it. This is done by associating with every $a : Y \to \mathbb{M}$ a function $f_a^{\#}$ on 2^Y whose greatest fixpoint gives us the elements of Y where we have a potential for increasing a by adding a constant. If no such potential exists, i.e. $\nu f_a^{\#}$ is empty, we conclude that a is νf . A similar function $f_{\#}^a$ (specifying decrease instead of increase) exists for the case of least fixpoints. Note that the premise is $\nu f_{\#}^a = \emptyset$, i.e. the witness remains coinductive. The proof rules are:

$$\frac{f(a) = a}{\nu f_a^{\#} = \emptyset} \qquad \frac{f(a) = a}{\mu f = a} \qquad \frac{f(a) = a}{\mu f = a}$$

For applying the rule we compute a greatest fixpoint on $\mathbf{2}^{Y}$, which is finite, instead of working on the potentially infinite \mathbb{M}^{Y} . The rule does not work for all monotone functions $f: \mathbb{M}^{Y} \to \mathbb{M}^{Y}$, but we show that whenever f is non-expansive the rule is valid. Actually, it is not only sound, but also reversible, i.e., if $a = \nu f$ then $\nu f_{a}^{\#} = \emptyset$, providing an if-and-only-if characterisation of whether a given fixpoint corresponds to the greatest fixpoint.

Quite interestingly, under the same assumptions on f, using a restricted function f_a^* , the rule can be used, more generally, when a is just a *pre-fixpoint* $(f(a) \sqsubseteq a)$ and it allows to conclude that $\nu f \sqsubseteq a$. A dual result holds for *post-fixpoints* in the case of least fixpoints.

$$\frac{f(a) \sqsubseteq a \quad \nu f_a^* = \emptyset}{\nu f \sqsubseteq a} \qquad \frac{a \sqsubseteq f(a) \quad \nu f_*^a = \emptyset}{a \sqsubseteq \mu f}$$

As already mentioned, the theory above applies to many interesting scenarios: witnesses for non-bisimilarity, algorithms for simple stochastic games [Con92], lower bounds for termination probabilities and behavioural metrics in the setting of probabilistic [BBLM17] and metric transition systems [dFS09] and probabilistic automata [BBL⁺19]. In particular we were inspired by, and generalise, the self-closed relations of [Fu12], also used in [BBL⁺19].

Motivating example. Consider a Markov chain (S, T, η) with a finite set of states S, where $T \subseteq S$ are the terminal states and every state $s \in S \setminus T$ is associated with a probability distribution $\eta(s) \in \mathcal{D}(S)$.¹ Intuitively, $\eta(s)(s')$ denotes the probability of state s choosing s' as its successor. Assume that, given a fixed state $s \in S$, we want to determine the termination probability of s, i.e. the probability of eventually reaching any terminal state from s. As a concrete example, take the Markov chain given in Fig. 1, where u is the only terminal state.

The termination probability arises as the least fixpoint of a function \mathcal{T} defined as in Fig. 1. The values of $\mu \mathcal{T}$ are indicated in green (left value).

¹By $\mathcal{D}(S)$ we denote the set of all maps $p: S \to [0,1]$ such that $\sum_{s \in S} p(s) = 1$.

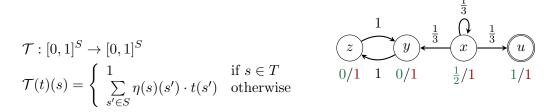


FIGURE 1. Function \mathcal{T} (left) and a Markov chain with two fixpoints of \mathcal{T} (right)

Now consider the function t assigning to each state the termination probability written in red (right value). It is not difficult to see that t is another fixpoint of \mathcal{T} , in which states y and z convince each other incorrectly that they terminate with probability 1, resulting in a vicious cycle that gives "wrong" results. We want to show that $\mu \mathcal{T} \neq t$ without knowing $\mu \mathcal{T}$. Our idea is to compute the set of states that still has some "wiggle room", i.e., those states which could reduce their termination probability by δ if all their successors did the same. This definition has a coinductive flavour and it can be computed as a greatest fixpoint on the finite powerset $\mathbf{2}^S$ of states, instead of on the infinite lattice $[0, 1]^S$. We hence consider a function $\mathcal{T}^t_{\#}: \mathbf{2}^{[S]^t} \to \mathbf{2}^{[S]^t}$, dependent on t, defined as follows. Let

We hence consider a function $\mathcal{T}^t_{\#} : \mathbf{2}^{[S]^t} \to \mathbf{2}^{[S]^t}$, dependent on t, defined as follows. Let $[S]^t$ be the support of t, i.e., the set of all states s such that t(s) > 0, where a reduction in value is in principle possible. Then a state $s \in [S]^t$ is in $\mathcal{T}^t_{\#}(S')$ iff $s \notin T$ and for all s' for which $\eta(s)(s') > 0$ it holds that $s' \in S'$, i.e. all successors of s are in S'.

The greatest fixpoint of $\mathcal{T}^t_{\#}$ is $\{y, z\}$. The fact that it is not empty means that there is some "wiggle room", i.e., the value of t can be reduced on the elements $\{y, z\}$ and thus t cannot be the least fixpoint of f. Moreover, the intuition that t can be improved on $\{y, z\}$ can be made precise, leading to the possibility of performing the improvement and search for the least fixpoint from there.

Contributions. In the paper we formalise the theory outlined above, showing that the proof rules work for non-expansive monotone functions f on lattices of the form \mathbb{M}^Y , where Y is a finite set and \mathbb{M} a (potentially infinite) MV-algebra (Section 3 and Section 4). Additionally, given a decomposition of f we show how to obtain the corresponding approximation compositionally (Section 5). Then, in order to show that our approach covers a wide range of examples and allows us to derive useful and original algorithms, we discuss various applications: termination probability, behavioural distances for metric transition systems and probabilistic automata, bisimilarity (Section 6) and simple stochastic games (Section 7).

Further proofs and material can be found in the appendix.

2. Lattices and MV-algebras

In this section, we review some basic notions used in the paper, concerning complete lattices and MV-algebras [Mun07].

A preordered or partially ordered set (P, \sqsubseteq) is often denoted simply as P, omitting the order relation. Given $x, y \in P$, with $x \sqsubseteq y$, we denote by [x, y] the interval $\{z \in P \mid x \sqsubseteq z \sqsubseteq y\}$. The *join* and the *meet* of a subset $X \subseteq P$ (if they exist) are denoted $\bigsqcup X$ and $\bigsqcup X$, respectively.

A complete lattice is a partially ordered set (L, \sqsubseteq) such that each subset $X \subseteq L$ admits a join $\bigsqcup X$ and a meet $\bigsqcup X$. A complete lattice (L, \sqsubseteq) always has a least element $\bot = \bigsqcup \emptyset$ and a greatest element $\top = \bigsqcup \emptyset$.

A function $f: L \to L$ is monotone if for all $l, l' \in L$, if $l \sqsubseteq l'$ then $f(l) \sqsubseteq f(l')$. By Knaster-Tarski's theorem [Tar55, Theorem 1], any monotone function on a complete lattice has a least and a greatest fixpoint, denoted respectively μf and νf , characterised as the meet of all pre-fixpoints, respectively the join of all post-fixpoints: $\mu f = \prod \{l \mid f(l) \sqsubseteq l\}$ and $\nu f = \lfloor |\{l \mid l \sqsubseteq f(l)\}$.

Let (C, \sqsubseteq) , (A, \leq) be complete lattices. A *Galois connection* is a pair of monotone functions $\langle \alpha, \gamma \rangle$ such that $\alpha : C \to A$, $\gamma : A \to C$ and for all $a \in A$ and $c \in C$:

$$\alpha(c) \le a$$
 iff $c \sqsubseteq \gamma(a)$.

Equivalently, for all $a \in A$ and $c \in C$, (i) $c \sqsubseteq \gamma(\alpha(c))$ and (ii) $\alpha(\gamma(a)) \le a$. In this case we will write $\langle \alpha, \gamma \rangle : C \to A$. For a Galois connection $\langle \alpha, \gamma \rangle : C \to A$, the function α is called the left (or lower) adjoint and γ the right (or upper) adjoint.

Galois connections are at the heart of abstract interpretation [CC77, CC00]. In particular, when $\langle \alpha, \gamma \rangle$ is a Galois connection, given $f^C : C \to C$ and $f^A : A \to A$, monotone functions, if $f^C \circ \gamma \sqsubseteq \gamma \circ f^A$, then $\nu f^C \sqsubseteq \gamma(\nu f^A)$. If the equality $f^C \circ \gamma = \gamma \circ f^A$ holds, a condition sometimes referred to as γ -completeness, then greatest fixpoints are preserved along the connection, i.e., $\nu f^C = \gamma(\nu f^A)$.

Given a set Y and a complete lattice L, the set of functions $L^Y = \{f \mid f : Y \to L\}$, endowed with pointwise order, i.e., for $a, b \in L^Y$, $a \sqsubseteq b$ if $a(y) \sqsubseteq b(y)$ for all $y \in Y$, is a complete lattice.

In the paper we will mostly work with lattices of the form \mathbb{M}^{Y} where \mathbb{M} is a special kind of lattice with a rich algebraic structure, i.e. an MV-algebra [Mun07].

Definition 2.1 (MV-algebra). An MV-algebra is a tuple $\mathbb{M} = (M, \oplus, 0, \overline{(\cdot)})$ where $(M, \oplus, 0)$ is a commutative monoid and $\overline{(\cdot)} : M \to M$ maps each element to its *complement*, such that for all $x, y \in M$

(1) $\overline{\overline{x}} = x$

- (2) $x \oplus \overline{0} = \overline{0}$
- (3) $\overline{(\overline{x} \oplus y)} \oplus y = \overline{(\overline{y} \oplus x)} \oplus x.$

We denote $1 = \overline{0}$, multiplication $x \otimes y = \overline{\overline{x} \oplus \overline{y}}$ and subtraction $x \oplus y = x \otimes \overline{y}$.

Note that by using the derived operations, axioms (2) and (3) above can be written as

(2)
$$x \oplus 1 = 1$$

(3) $(y \ominus x) \oplus x = (x \ominus y) \oplus y$

MV-algebras are endowed with a natural order.

Definition 2.2 (natural order). Let $\mathbb{M} = (M, \oplus, 0, \overline{(\cdot)})$ be an MV-algebra. The *natural* order on \mathbb{M} is defined, for $x, y \in M$, by $x \sqsubseteq y$ if $x \oplus z = y$ for some $z \in M$. When \sqsubseteq is total \mathbb{M} is called an *MV-chain*.

The natural order gives an MV-algebra a lattice structure where $\perp = 0, \top = 1$, $x \sqcup y = (x \ominus y) \oplus y$ and $x \sqcap y = \overline{x} \sqcup \overline{y} = x \otimes (\overline{x} \oplus y)$. We call the MV-algebra *complete*, if it is a complete lattice. This is not true in general, e.g., $([0,1] \cap \mathbb{Q}, \leq)$.

Example 2.3. A prototypical example of an MV-algebra is $([0, 1], \oplus, 0, (\cdot))$ where $x \oplus y = \min\{x + y, 1\}$ and $\overline{x} = 1 - x$ for $x, y \in [0, 1]$. This means that $x \otimes y = \max\{x + y - 1, 0\}$

and $x \ominus y = \max\{0, x - y\}$ (truncated subtraction). The operators \oplus and \otimes are also known as strong disjunction and conjunction in Lukasiewicz logic [Mun11]. The natural order is \leq (less or equal) on the reals.

Another example is $(\{0, \ldots, k\}, \oplus, 0, \overline{(\cdot)})$ where $n \oplus m = \min\{n + m, k\}$ and $\overline{n} = k - n$ for $n, m \in \{0, \ldots, k\}$. We are in particular interested in the case k = 1. Both MV-algebras are complete and MV-chains.

Boolean algebras (with disjunction and complement) also form MV-algebras that are complete, but in general not MV-chains.

MV-algebras are the algebraic semantics of Łukasiewicz logic. They can be shown to correspond to intervals of the kind [0, u] in suitable groups, i.e., abelian lattice-ordered groups with a strong unit u [Mun07].

We next review some properties of MV-algebras. They are taken from or easy consequences of properties in [Mun07] and will be used throughout the paper.

Lemma 2.4 (properties of MV-algebras). Let $\mathbb{M} = (M, \oplus, 0, \overline{(\cdot)})$ be an MV-algebra. For all $x, y, z \in M$

(1) $x \oplus \overline{x} = 1$

(2) $x \sqsubseteq y$ iff $\overline{x} \oplus y = 1$ iff $x \otimes \overline{y} = 0$ iff $y = x \oplus (y \ominus x)$

- (3) $x \sqsubseteq y$ iff $\overline{y} \sqsubseteq \overline{x}$
- (4) \oplus , \otimes are monotone in both arguments, \ominus monotone in the first and antitone in the second argument.
- (5) if $x \sqsubset y$ then $0 \sqsubset y \ominus x$;
- (6) $(x \oplus y) \ominus y \sqsubseteq x$
- (7) $z \sqsubseteq x \oplus y$ if and only if $z \ominus x \sqsubseteq y$.
- (8) if $x \sqsubset y$ and $z \sqsubseteq \overline{y}$ then $x \oplus z \sqsubset y \oplus z$;
- (9) $y \sqsubseteq \overline{x}$ if and only if $(x \oplus y) \ominus y = x$;
- (10) $x \ominus (x \ominus y) \sqsubseteq y$ and if $y \sqsubseteq x$ then $x \ominus (x \ominus y) = y$.
- (11) Whenever \mathbb{M} is an MV-chain, $x \sqsubset y$ and $0 \sqsubset z$ imply $(x \oplus z) \ominus y \sqsubset z$

Note that we adhere to the following convention: whenever brackets are missing, we always assume that we associate from left to right. So $a \oplus b \oplus c$ should be read as $(a \oplus b) \oplus c$ and not as $a \oplus (b \oplus c)$, which is in general different.

3. Non-expansive functions and their approximations

As mentioned in the introduction, our interest is for fixpoints of monotone functions $f : \mathbb{M}^Y \to \mathbb{M}^Y$, where \mathbb{M} is an MV-chain and Y is a finite set. We will see that for nonexpansive functions we can over-approximate the sets of points in which a given $a \in \mathbb{M}^Y$ can be increased in a way that is preserved by the application of f. This will be the core of the proof rules outlined earlier.

3.1. Non-expansive functions on MV-algebras. For defining non-expansiveness it is convenient to introduce a norm, which can be seen as an adaptation of the standard l_{∞} norm.

Definition 3.1 (norm). Let \mathbb{M} be an \mathbb{M} -chain and let Y be a finite set. Given $a \in \mathbb{M}^{Y}$ we define its *norm* as $||a|| = \max\{a(y) \mid y \in Y\}$.

Given a finite set Y we extend \oplus and \otimes to \mathbb{M}^Y pointwise. E.g. if $a, b \in \mathbb{M}^Y$, we write $a \oplus b$ for the function defined by $(a \oplus b)(y) = a(y) \oplus b(y)$ for all $y \in Y$. Given $Y' \subseteq Y$ and $\delta \in \mathbb{M}$, we write $\delta_{Y'}$ for the function defined by $\delta_{Y'}(y) = \delta$ if $y \in Y'$ and $\delta_{Y'}(y) = 0$, otherwise. Whenever this does not generate confusion, we write δ instead of δ_Y .

As shown in the lemma below, $\|\cdot\|$ has the standard properties of a norm. Moreover, it is clearly monotone, i.e., if $a \sqsubseteq b$ then $\|a\| \sqsubseteq \|b\|$.

Lemma 3.2 (properties of the norm). Let \mathbb{M} be an *MV*-chain and let *Y* be a finite set. Then $\|\cdot\| : \mathbb{M}^Y \to \mathbb{M}$ satisfies, for all $a, b \in \mathbb{M}^Y$, $\delta \in \mathbb{M}$

- (1) $||a \oplus b|| \subseteq ||a|| \oplus ||b||$,
- (2) $\|\delta \otimes a\| = \delta \otimes \|a\|$ and
- (3) ||a|| = 0 implies that a is the constant 0.

We next introduce non-expansiveness. Despite the fact that we will eventually be interested in endo-functions $f : \mathbb{M}^Y \to \mathbb{M}^Y$, in order to allow for a compositional reasoning we work with functions where domain and codomain can be different.

Definition 3.3 (non-expansiveness). Let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a function, where \mathbb{M} is an MV-chain and Y, Z are finite sets. We say that it is *non-expansive* if for all $a, b \in \mathbb{M}^Y$ it holds that $||f(b) \ominus f(a)|| \subseteq ||b \ominus a||$.

Note that $(a, b) \mapsto ||a \ominus b||$ is the supremum lifting of a directed version of Chang's distance [Mun07]. It is easy to see that all non-expansive functions on MV-chains are monotone (see Lemma B.1 in the appendix). Moreover, when $\mathbb{M} = \{0, 1\}$, i.e., \mathbb{M} is the two-point boolean algebra, the two notions coincide.

3.2. Approximating the propagation of increases. Let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a monotone function and take $a, b \in \mathbb{M}^Y$ with $a \sqsubseteq b$. We are interested in the difference $b(y) \ominus a(y)$ for some $y \in Y$ and on how the application of f "propagates" this difference. The reason is that, understanding that no increase can be propagated will be crucial to establish when a fixpoint of a non-expansive function f is actually the largest one, and, more generally, when a (pre-)fixpoint of f is above the largest fixpoint.

In order to formalise the above intuition, we rely on tools from abstract interpretation. In particular, the following pair of functions, which, under a suitable condition, form a Galois connection, will play a major role. For this purpose we fix $a \in \mathbb{M}^Y$, $\delta \in \mathbb{M}$. The left adjoint $\alpha_{a,\delta}$ takes as input a set $Y' \subseteq Y$ and, for $y \in Y'$, it increases the values a(y) by δ , while the right adjoint $\gamma_{a,\delta}$ takes as input a function $b \in \mathbb{M}^Y$, $b \in [a, a \oplus \delta]$ and checks for which parameters $y \in Y$ the value b(y) exceeds a(y) by δ .

We also define $[Y]_a$, the subset of elements in Y where a(y) is not 1 and thus there is a potential to increase, and δ_a , which gives us the least of such increases (i.e., the largest increase that can be used on all elements in $[Y]_a$ without "overflowing").

Definition 3.4 (functions to sets, and vice versa). Let \mathbb{M} be an \mathbb{M} -algebra and let Y be a finite set. Define the set $[Y]_a = \{y \in Y \mid a(y) \neq 1\}$ (support of \overline{a}) and $\delta_a = \min\{\overline{a(y)} \mid y \in [Y]_a\}$ with $\min \emptyset = 1$.

For $0 \sqsubset \delta \in \mathbb{M}$ we consider the functions $\alpha_{a,\delta} : \mathbf{2}^{[Y]_a} \to [a, a \oplus \delta]$ and $\gamma_{a,\delta} : [a, a \oplus \delta] \to \mathbf{2}^{[Y]_a}$, defined, for $Y' \in \mathbf{2}^{[Y]_a}$ and $b \in [a, a \oplus \delta]$, by

$$\alpha_{a,\delta}(Y') = a \oplus \delta_{Y'} \qquad \gamma_{a,\delta}(b) = \{ y \in [Y]_a \mid b(y) \ominus a(y) \sqsupseteq \delta \}.$$

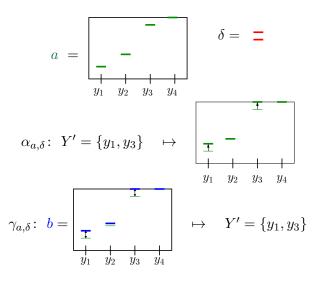


FIGURE 2. Visual representation of $\alpha_{a,\delta}$ and $\gamma_{a,\delta}$

When δ is sufficiently small, the pair $\langle \alpha_{a,\delta}, \gamma_{a,\delta} \rangle$ is a Galois connection.

Lemma 3.5 (Galois connection). Let \mathbb{M} be an MV-algebra and Y be a finite set. For $0 \neq \delta \sqsubseteq \delta_a$, the pair $\langle \alpha_{a,\delta}, \gamma_{a,\delta} \rangle : \mathbf{2}^{[Y]_a} \to [a, a \oplus \delta]$ is a Galois connection.

$$\mathbf{2}^{[Y]_a} \overbrace{\gamma_{a,\delta}}^{lpha_{a,\delta}} [a, a \oplus \delta]$$

Observe that differently from what normally happens in abstract interpretation, the component α of the Galois connection, i.e., the left adjoint, transforms abstract values (sets) into concrete ones (functions) and thus it plays the role of a concretisation function.

Example 3.6. We illustrate the definitions with small examples whose sole purpose is to get a better intuition. (See Fig. 2 for a visual representation.) Consider the MV-chain $\mathbb{M} = [0, 1]$, a set $Y = \{y_1, y_2, y_3, y_4\}$ and a function $a: Y \to [0, 1]$ with $a(y_1) = 0.2, a(y_2) = 0.4$ $a(y_3) = 0.9, a(y_4) = 1$. In this case $[Y]_a = \{y_1, y_2, y_3\}$ and $\delta_a = 0.1$. Choose $\delta = 0.1$ and $Y' = \{y_1, y_3\}$. Then $\alpha_{a,\delta}(Y')$ is a function that maps $y_1 \mapsto 0.3$,

 $y_2 \mapsto 0.4, y_3 \mapsto 1, y_4 \mapsto 1.$

We keep $\delta = 0.1$ and consider a function $b: Y \to [0,1]$ with $b(y_1) = 0.3$, $b(y_2) = 0.45$, $b(y_3) = b(y_4) = 1$. Then $\gamma_{a,\delta}(b) = \{y_1, y_3\}.$

Whenever f is non-expansive, it is easy to see that it restricts to a function $f : [a, a \oplus \delta] \rightarrow$ $[f(a), f(a) \oplus \delta]$ for all $\delta \in \mathbb{M}$.

As mentioned before, a crucial result shows that for all non-expansive functions, under the assumption that Y, Z are finite and the order on M is total, we can suitably approximate the propagation of increases. In order to state this result, a useful tool is a notion of approximation of a function.

Definition 3.7 ((δ , a)-approximation). Let \mathbb{M} be an MV-chain, let Y, Z be finite sets and let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function. For $a \in \mathbb{M}^Y$ and any $\delta \in \mathbb{M}$ we define $f_{a,\delta}^{\#} : \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[Z]_{f(a)}}$ as $f_{a,\delta}^{\#} = \gamma_{f(a),\delta} \circ f \circ \alpha_{a,\delta}$.

Given $Y' \subseteq [Y]_a$, its image $f_{a,\delta}^{\#}(Y') \subseteq [Z]_{f(a)}$ is the set of points $z \in [Z]_{f(a)}$ such that $\delta \sqsubseteq f(a \oplus \delta_{Y'})(z) \ominus f(a)(z)$, i.e., the points to which f propagates an increase of the function a with value δ on the subset Y'.

Example 3.8. We continue with Example 3.6 and consider the function $f: [0,1]^Y \to [0,1]^Y$ with $f(b) = b \oplus 0.3$ for every $b \in [0,1]^Y$, which can easily be seen to be non-expansive. We again consider $a: Y \to [0,1]$ and $\delta = 0.1$ as in Example 3.6, and $Y' = \{y_1, y_2, y_3\}$. The maps $a, \alpha_{a,\delta}(Y'), f(a)$ and $f(\alpha_{a,\delta}(Y'))$ are given in the table below and we obtain $f_{a,\delta}^{\#}(Y') = \gamma_{f(a),\delta}(f(\alpha_{a,\delta}(Y'))) = \{y_2, y_3\}$, that is only the increase at y_2 and y_3 can be propagated, while the value of y_1 is too low and y_4 is not even contained in $[Y]_a$ (the domain of $f_{a,\delta}^{\#}$), since its value is already 1.0 and there is no slack left. That is, we obtain those elements of Y for which the last two lines in the table below differ by 0.1.

	y_1	y_2	y_3	y_4
a	0.2	0.4	0.9	1.0
$\alpha_{a,\delta}(Y')$	0.3	$\begin{array}{c} 0.5 \\ 0.1 \end{array}$	1.0	1.0
f(a)	0.0	0.1	0.6	0.7
$f(\alpha_{a,\delta}(Y'))$	0.0	0.2	0.7	0.7

In general we have $f_{a,\delta}^{\#}(Y') = Y' \cap \{y_2, y_3\}$ if $\delta \leq \delta_a = 0.1$, $f_{a,\delta}^{\#}(Y') = Y' \cap \{y_2\}$ if $0.1 < \delta \leq 0.6$ and $f_{a,\delta}^{\#}(Y') = \emptyset$ if $0.6 < \delta$.

We now show that $f_{a\,\delta}^{\#}$ is antitone in the parameter δ , a non-trivial result.

Lemma 3.9 (antitonicity). Let \mathbb{M} be an MV-chain, let Y, Z be finite sets, let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function and let $a \in \mathbb{M}^Y$. For $\theta, \delta \in \mathbb{M}$, if $\theta \sqsubseteq \delta$ then $f_{a,\delta}^{\#} \subseteq f_{a,\theta}^{\#}$.

Since $f_{a,\delta}^{\#}$ increases when δ decreases and there are only finitely many such functions, there must be a value ι_a^f such that all functions $f_{a,\delta}^{\#}$ for $0 \sqsubset \delta \sqsubseteq \iota_a^f$ are equal. The resulting function will be the approximation of interest.

We next show how ι_a^f can be determined. We start by observing that for each $z \in [Z]_{f(a)}$ and $Y' \subseteq [Y]_a$ there is a largest increase θ such that $z \in f_{a,\theta}^{\#}(Y')$.

Lemma 3.10 (largest increase for a point). Let \mathbb{M} be a complete MV-chain, let Y, Z be finite sets, let $f: \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function and fix $a \in \mathbb{M}^Y$. For all $z \in [Z]_{f(a)}$ and $Y' \subseteq [Y]_a$ the set $\{\theta \in \mathbb{M} \mid z \in f_{a,\theta}^{\#}(Y')\}$ has a maximum, that we denote by $\iota_a^f(Y', z)$.

We can then provide an explicit definition of ι_a^f and of the approximation of a function. **Lemma 3.11** (*a*-approximation for a function). Let \mathbb{M} be a complete MV-chain, let Y, Z be finite sets and let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function. Let

 $\iota_a^f = \min\{\iota_a^f(Y',z) \mid Y' \subseteq [Y]_a \land z \in [Z]_{f(a)} \land \iota_a^f(Y',z) \neq 0\} \cup \{\delta_a\}.$ Then for all $0 \neq \delta \sqsubseteq \iota_a^f$ it holds that $f_{a,\delta}^{\#} = f_{a,\iota_a^f}^{\#}.$

The function $f_{a,f}^{\#}$ is called the a-approximation of f and it is denoted by $f_a^{\#}$.

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In the following, we show that indeed, for all non-expansive functions, the *a*-approximation properly approximates the propagation of increases. Given an MV-chain \mathbb{M} and a finite set Y, we first observe that each function $b \in \mathbb{M}^Y$ can be expressed as a suitable sum of functions of the shape $\delta_{Y'}$.

Lemma 3.12 (standard form). Let \mathbb{M} be an MV-chain and let Y be a finite set. Then for any $b \in \mathbb{M}^Y$ there are $Y_1, \ldots, Y_n \subseteq Y$ with $Y_{i+1} \subseteq Y_i$ for $i \in \{1, \ldots, n-1\}$ and $\delta^i \in \mathbb{M}$, $0 \neq \delta^i \sqsubseteq \overline{\bigoplus_{j=1}^{i-1} \delta^j}$ for $i \in \{1, \ldots, n\}$ such that

$$b = \bigoplus_{i=1}^{n} \delta_{Y_i}^i$$
 and $\|b\| = \bigoplus_{i=1}^{n} \delta^i$

where we assume that an empty sum evaluates to 0.

The above characterisation allows us to show a technical property of the functions in the interval $[a, a \oplus \delta]$ of interest.

Lemma 3.13. Let \mathbb{M} be an MV-chain, let Y, Z be finite sets and let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function. Let $a \in \mathbb{M}^Y$. For $b \in [a, a \oplus \delta]$, let $b \ominus a = \bigoplus_{i=1}^n \delta_{Y_i}^i$ be a standard form for $b \ominus a$. If $\gamma_{f(a),\delta}(f(b)) \neq \emptyset$ then $Y_n = \gamma_{a,\delta}(b)$ and $\gamma_{f(a),\delta}(f(b)) \subseteq f_{a,\delta^n}^{\#}(Y_n)$.

We can finally prove the main result about legitimacy of the approximation.

Theorem 3.14 (approximation of non-expansive functions). Let \mathbb{M} be a complete MV-chain, let Y, Z be finite sets and let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function. Then for all $0 \sqsubset \delta \in \mathbb{M}$:

a. $\gamma_{f(a),\delta} \circ f \subseteq f_a^{\#} \circ \gamma_{a,\delta}$ b. for $\delta \sqsubseteq \delta_a$: $\delta \sqsubseteq \iota_a^f$ iff $\gamma_{f(a),\delta} \circ f = f_a^{\#} \circ \gamma_{a,\delta}$ $[a, a \oplus \delta] \xrightarrow{\gamma_{a,\delta}} \mathbf{2}^{[Y]_a}$

$$\begin{array}{c} f \\ \downarrow \\ [f(a), f(a) \oplus \delta] \xrightarrow{} \gamma_{f(a), \delta} \end{array} 2^{[Z]_{f(a)}} \end{array}$$

Proof. a. Let $b \in [a, a \oplus \delta]$. First note that whenever $\gamma_{f(a),\delta}(f(b)) = \emptyset$, the desired inclusion obviously holds.

If instead $\gamma_{f(a),\delta}(f(b)) \neq \emptyset$, let $b \ominus a = \bigoplus_{i=1}^n \delta_{Y_i}^i$ be a standard form with $\delta^n \neq 0$. First observe that, by Lemma 3.13, we have $Y_n = \gamma_{a,\delta^n}(b)$ and

$$\gamma_{f(a),\delta}(f(b)) \subseteq f_{a,\delta^n}^{\#}(Y_n). \tag{3.1}$$

For all $z \in f_{a,\delta^n}^{\#}(Y_n)$, by definition of $\iota_a^f(Y_n, z)$ we have that $0 \sqsubset \delta_n \sqsubseteq \iota_a^f(Y_n, z)$, therefore $\iota_a^f \sqsubseteq \iota_a^f(Y_n, z)$. Moreover, $z \in f_{a,\iota_a^f}^{\#}(Y_n, z)$ $(Y_n) \subseteq f_{a,\iota_a^f}^{\#}(Y_n) = f_a^{\#}(Y_n)$, where the last inequality is motivated by Lemma 3.9 since $\iota_a^f \sqsubseteq \iota_a^f(Y_n, z)$. Therefore, $f_{a,\delta^n}^{\#}(Y_n) \subseteq f_a^{\#}(\gamma_{a,\delta}(b))$, which combined with (3.1) gives the desired result.

- b. For (b.), we first show the direction from left to right. Assume that $\delta \sqsubseteq \iota_a^f$. By (a) clearly, $\gamma_{f(a),\delta} \circ f(b) \subseteq f_a^{\#} \circ \gamma_{a,\delta}(b)$. For the converse inclusion, note that:
 - $\begin{aligned} f_a^{\#}(\gamma_{a,\delta}(b)) & \text{[by definition of } f_a^{\#}] \\ &= f_{a,\iota_a^f}^{\#}(\gamma_{a,\delta}(b)) & \text{[by Lemma 3.9, since } \delta \sqsubseteq \iota_a^f] \end{aligned}$

$$\subseteq f_{a,\delta}^{\#}(\gamma_{a,\delta}(b))$$
 [by definition of $f_{a,\delta}^{\#}$]
= $\gamma_{f(a),\delta}(f(\alpha_{a,\delta}(\gamma_{a,\delta}(b))))$ [since $\alpha_{a,\delta} \circ \gamma_{a,\delta}(b) \sqsubseteq b$]
 $\subseteq \gamma_{f(a),\delta}(f(b))$

as desired.

For the other direction, assume $\gamma_{f(a),\delta} \circ f(b) = f_a^{\#} \circ \gamma_{a,\delta}(b)$ holds for all $b \in [a, a \oplus \delta]$. Now, for every $Y' \subseteq [Y]_a$ we have $f_{a,\delta}^{\#}(Y') = \gamma_{f(a),\delta} \circ f \circ \alpha_{a,\delta}(Y') = f_a^{\#} \circ \gamma_{a,\delta} \circ \alpha_{a,\delta}(Y')$. We also have $\gamma_{a,\delta} \circ \alpha_{a,\delta}(Y') = Y'$ (see proof of Lemma 3.5), thus $f_{a,\delta}^{\#}(Y') = f_a^{\#}(Y')$. For any δ with $\iota_a^f \sqsubset \delta \sqsubseteq \delta_a$ there exists $Y' \subseteq [Y]_a$ and $z \in [Z]_{f(a)}$ with $z \in f_a^{\#}(Y')$ but $z \notin f_{a,\delta}^{\#}(Y')$, by definition of ι_a^f . Therefore $\delta \sqsubseteq \iota_a^f$ has to hold.

Note that if Y = Z and a is a fixpoint of f, i.e., a = f(a), then condition (a.) above corresponds exactly to soundness in the sense of abstract interpretation [CC77]. Moreover, when $\delta \equiv \delta_a$ and thus $\langle \alpha_{a,\delta}, \gamma_{a,\delta} \rangle$ is a Galois connection, $f_{a,\delta}^{\#} = \gamma_{a,\delta} \circ f \circ \alpha_{a,\delta}$ is the best correct approximation of f. In particular, when $\delta \equiv \iota_a^f$, such a best correct approximation is $f_a^{\#}$, the *a*-approximation of f, i.e., it becomes independent from δ , and condition (b.) corresponds to (γ -)completeness [GRS00] (see also Section 2).

4. Proof rules

In this section we formalise the proof technique outlined in the introduction for showing that a fixpoint is the largest and, more generally, for checking over-approximations of greatest fixpoints of non-expansive functions.

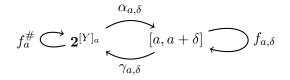
4.1. **Proof rules for fixpoints.** Consider a monotone function $f : \mathbb{M}^Y \to \mathbb{M}^Y$ for some finite set Y. We first focus on the problem of establishing whether some given fixpoint a of f coincides with νf (without explicitly knowing νf), and, in case it does not, finding an "improvement", i.e., a post-fixpoint of f, larger than a. To this aim we need a technical lemma.

Lemma 4.1. Let \mathbb{M} be a complete MV-chain, Y a finite set and $f : \mathbb{M}^Y \to \mathbb{M}^Y$ be a nonexpansive function. Let $a \in \mathbb{M}^Y$ be a pre-fixpoint of f (i.e., $f(a) \sqsubseteq a$), let $f_a^{\#} : \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[Y]_{f(a)}}$ be the a-approximation of f. Assume $\nu f \not\sqsubseteq a$ and let $Y' = \{y \in [Y]_a \mid \nu f(y) \ominus a(y) = \|\nu f \ominus a\|\}$. Then for all $y \in Y'$ it holds a(y) = f(a)(y) and $Y' \subseteq f_a^{\#}(Y')$.

Observe that, when a is a fixpoint, clearly $[Y]_a = [Y]_{f(a)}$, and thus the a-approximation of f (Lemma 3.11) is an endo-function $f_a^{\#} : [Y]_a \to [Y]_a$ and Y' is its post-fixpoint. Then, we have the following result, which relies on the fact that, due to Theorem 3.14 and properties of Galois connections, $\gamma_{a,\delta}$ maps the greatest fixpoint of f to to the greatest fixpoint of $f_a^{\#}$.

Theorem 4.2 (soundness and completeness for fixpoints). Let \mathbb{M} be a complete MV-chain, Y a finite set and $f: \mathbb{M}^Y \to \mathbb{M}^Y$ be a non-expansive function. Let $a \in \mathbb{M}^Y$ be a fixpoint of f. Then $\nu f_a^{\#} = \emptyset$ if and only if $a = \nu f$.

Proof. Let a be a fixpoint of f and assume that $a = \nu f$. For $\delta = \iota_a^f \subseteq \delta_a$, according to Lemma 3.5, we have a Galois connection:



Since a is a fixpoint, then $[Y]_{f(a)} = [Y]_a$ and, by Theorem 3.14(b.), $\gamma_{a,\delta} \circ f = \gamma_{f(a),\delta} \circ f = f_a^{\#} \circ \gamma_{a,\delta}$.

Therefore by [CC00, Proposition 14], $\nu f_a^{\#} = \gamma_{a,\delta}(\nu f)$. Recall that $\gamma_{a,\delta}(\nu f) = \{y \in Y \mid \delta \sqsubseteq \nu f(y) \ominus a(y)\}$. Since $a = \nu f$ and $\delta \sqsupset 0$, we know that $\gamma_{a,\delta}(\nu f) = \emptyset$ and we conclude $\nu f_a^{\#} = \emptyset$, as desired.

Conversely, in order to prove that if $\nu f_a^{\#} = \emptyset$ then $a = \nu f$, we prove the contrapositive. Assume that $a \neq \nu f$. Since a is a fixpoint and νf is the largest, this means that $a \sqsubset \nu f$ and thus $\|\nu f \ominus a\| \neq 0$. Consider $Y' = \{y \in [Y]_a \mid \nu f(y) \ominus a(y) = \|\nu f \ominus a\|\} \neq \emptyset$. By Lemma 4.1, Y' is a post-fixpoint of $f_a^{\#}$, i.e., $Y' \subseteq f_a^{\#}(Y')$, and thus $\nu f_a^{\#} \supseteq Y'$ which implies $\nu f_a^{\#} \neq \emptyset$, as desired.

Whenever a is a fixpoint, but not yet the largest fixpoint of f, from the result above $\nu f_a^{\#} \neq \emptyset$. Intuitively, $\nu f_a^{\#}$ is the set of points where a can still be "improved". More precisely, we can show that a can be increased on the points in $\nu f_a^{\#}$ producing a post-fixpoint of f. In order to determine how much a can be increased we proceed similarly to what we have done for defining ι_a^f (Lemma 3.11), but restricting the attention to $\nu f_a^{\#}$ instead of considering the full $[Y]_a$. While ι_a^f could always be used, by restricting to $\nu f_a^{\#}$ we are able to find a better, that is larger, value which is still correct.

Definition 4.3 (largest increase for a subset). Let \mathbb{M} be a complete MV-chain and let $f : \mathbb{M}^Y \to \mathbb{M}^Y$ be a non-expansive function, where Y is a finite set and let $a \in \mathbb{M}^Y$. For $Y' \subseteq Y$, we define $\delta_a(Y') = \min\{\overline{a(y)} \mid y \in Y'\}$ and $\iota_a^f(Y') = \min\{\iota_a^f(Y', y) \mid y \in Y'\}$.

Example 4.4. We intuitively explain the computation of the values in the definition above. Let $g: [0,1]^Y \to [0,1]^Y$ with $g(b) = b \oplus 0.1$, where the set Y and the function $a \in [0,1]^Y$ are as in Example 3.6.

Let $Y' = \{y_1, y_2\}$. Then $\delta_a(Y') = 0.6$ and $\iota_a^g(Y') = 0.5$, i.e., since g adds 0.1, we can propagate an increase of at most 0.5.

We next prove that when $a \in \mathbb{M}^Y$ is a fixpoint of f and $Y' = \nu f_a^{\#}$, the value $\iota_a^f(Y')$ is the largest increase δ below $\delta_a(Y')$ such that $a \oplus \delta_{Y'}$ is a post-fixpoint of f.

Proposition 4.5 (from a fixpoint to larger post-fixpoint). Let \mathbb{M} be a complete MV-chain, $f: \mathbb{M}^Y \to \mathbb{M}^Y$ a non-expansive function, $a \in \mathbb{M}$ a fixpoint of f, and let $Y' = \nu f_a^{\#}$ be the greatest fixpoint of the corresponding a-approximation. Then $\iota_a^f \sqsubseteq \iota_a^f(Y') \sqsubseteq \delta_a(Y')$. Moreover, for all $\theta \sqsubseteq \iota_a^f(Y')$ the function $a \oplus \theta_{Y'}$ is a post-fixpoint of f, while for $\iota_a^f(Y') \sqsubset \theta \sqsubseteq \delta_a(Y')$ it is not.

Proof. We first show that $\iota_a^f \sqsubseteq \iota_a^f(Y')$. By Lemma 3.11 and since a = f(a), we have that $\iota_a^f = \min\{\iota_a^f(Y'', y) \mid Y'' \subseteq [Y]_a \land y \in [Y]_a \land \iota_a^f(Y'', y) \neq 0\} \cup \{\delta_a\}$. Moreover, we have $Y' = \nu f_a^{\#} \subseteq [Y]_a$ and $\iota_a^f(Y', y) \neq 0$, for every $y \in Y'$, since $\iota_a^f(Y', y) = \max\{\delta \in \mathbb{M} \mid y \in f_{a,\delta}^{\#}(Y')\}$ and $y \in Y' = \nu f_a^{\#} = f_a^{\#}(\nu f_a^{\#}) = f_{a,\iota_a^f}^{\#}(Y')$, hence $\iota_a^f(Y', y) \supseteq \iota_a^f \supseteq 0$. Therefore,

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the minimum in $\iota_a^f(Y')$ is computed on a subset of the values on which the one in ι_a^f is, and so the former must be larger or equal to the latter.

Next, we prove that $\iota_a^f(Y') \sqsubseteq \delta_a(Y')$. Observe that for all $y \in Y'$ and $\delta \in \mathbb{M}$, if $y \in f_{a,\delta}^{\#}(Y')$, by definition of $f_{a,\delta}^{\#}$, it holds that $\delta \sqsubseteq f(a \oplus \delta_{Y'})(y) \oplus f(a)(y) = f(a \oplus \delta_{Y'})(y) \oplus a(y) \sqsubseteq 1 \oplus a(y) = \overline{a(y)}$, where the second equality is motivated by the fact that a is a fixpoint. Therefore for all $y \in Y'$ we have $\max\{\delta \in \mathbb{M} \mid y \in f_{a,\delta}^{\#}(Y')\} \sqsubseteq \overline{a(y)}$ and thus $\iota_a^f(Y') = \min_{y \in Y'} \max\{\delta \in \mathbb{M} \mid y \in f_{a,\delta}^{\#}(Y')\} \sqsubseteq \overline{a(y)} = \delta_a(Y')$, as desired.

Given $\theta \sqsubseteq \iota_a^f(Y')$, let us prove that $a \oplus \theta_{Y'}$ is a post-fixpoint of f, i.e., $a \oplus \theta_{Y'} \sqsubseteq f(a \oplus \theta_{Y'})$. If $y \in Y'$, since $\theta \sqsubseteq \iota_a^f(Y')$, by definition of $\iota_a^f(Y')$, we have $\theta \sqsubseteq \max\{\delta \in \mathbb{M} \mid y \in f_{a,\delta}^\#(Y')\}$ and thus, by antitonicity of $f_{a,\delta}^\#$ with respect to δ , we have $y \in f_{a,\theta}^\#(Y')$. This means that $\theta \sqsubseteq f(a \oplus \theta_{Y'})(y) \ominus f(a)(y) = f(a \oplus \theta_{Y'})(y) \ominus a(y)$, where the last passage uses the fact that a is a fixpoint. Adding a(y) on both sides and using Lemma 2.4(2), we obtain $a(y) \oplus \theta \sqsubseteq (f(a \oplus \theta_{Y'})(y) \ominus a(y)) \oplus a(y) = f(a \oplus \theta_{Y'})(y)$. Since $y \in Y'$, $(a \oplus \theta_{Y'})(y) = a(y) \oplus \theta$ and thus $(a \oplus \theta_{Y'})(y) \sqsubseteq f(a \oplus \theta_{Y'})(y)$, as desired.

If instead, $y \notin Y'$, clearly $(a \oplus \theta_{Y'})(y) = a(y) = f(a)(y) \sqsubseteq f(a \oplus \theta_{Y'})(y)$, where we again use the fact that a is a fixpoint and monotonicity of f.

Lastly, we have to show that if $\iota_a^f(Y') \sqsubset \theta \sqsubseteq \delta_a(Y')$, then $a \oplus \theta_{Y'}$ is a not a postfixpoint of f. By definition of $\iota_a^f(Y')$, from the fact that $\iota_a^f(Y') \sqsubset \theta$, we deduce that $\max\{\delta \in \mathbb{M} \mid y \in f_{a,\delta}^{\#}(Y')\} \sqsubset \theta$ for some $y \in Y'$ and thus $y \notin f_{a,\theta}^{\#}(Y')$.

By definition of $f_{a,\theta}^{\#}$ and totality of \sqsubseteq , the above means $\theta \sqsupset f(a \oplus \theta_{Y'})(y) \ominus f(a)(y) = f(a \oplus \theta_{Y'})(y) \ominus a(y)$, since a is a fixpoint of f. Since $\theta \sqsubseteq \delta_a(Y')$, we can add a(y) on both sides and, by Lemma 2.4(8), we obtain $a(y) \oplus \theta \sqsupset f(a \oplus \theta_{Y'})(y)$. Since $y \in Y'$, the left-hand side is $(a \oplus \theta_{Y'})(y)$. Hence we conclude that indeed $a \oplus \theta_{Y'}$ is not a post fixpoint.

Using these results one can perform an alternative fixpoint iteration where we iterate to the largest fixpoint from below: start with a post-fixpoint $a_0 \sqsubseteq f(a_0)$ (which is clearly below νf) and obtain, by (possibly transfinite) iteration, an ascending chain that in the order converges to a,² the least fixpoint above a_0 . Now, letting $Y' = \nu f_a^{\#}$, check whether $Y' = \emptyset$. If so, by Theorem 4.2 we know we have reached $\nu f = a$. If not, $\alpha_{a,\iota_a^f(Y')}(Y') = a \oplus (\iota_a^f(Y'))_{Y'}$ is again a post-fixpoint (cf. Proposition 4.5) and we continue this procedure until – for some ordinal – we reach the largest fixpoint νf , for which we have $\nu f_{\nu f}^{\#} = \emptyset$.

In order to make the above procedure as efficient as possible, one would like to consider, whenever a fixpoint a is reached, the largest possible increase ι which is valid, i.e. such that $a \oplus \iota$ is again a post-fixpoint of f. Thus the question naturally arises asking whether $\iota_a^f(Y')$ is such largest valid increase. From Proposition 4.5, it immediately follows that $\iota_a^f(Y')$ is the largest valid increase below $\delta_a(Y')$, but it can be seen that there can be larger valid increases above $\delta_a(Y')$ (an explicit example is provided later in Example 6.3, for the dual case of least fixpoints). However, while the set of valid increases below $\delta_a(Y')$ is downward-closed, as proved in Proposition 4.5, this is not the case for those above $\delta_a(Y')$. Hence, we believe that the most efficient approach would be to search for $\iota_a^f(Y')$, or some satisfying approximation, via a binary search bounded by $\delta_a(Y')$.

²Note that throughout the paper the term "convergence" used on complete MV-chains will always implicitly refer to convergence in the natural order.

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4.2. **Proof rules for pre-fixpoints.** Interestingly, the soundness result in Theorem 4.2 can be generalised to the case in which a is a pre-fixpoint instead of a fixpoint. In this case, the *a*-approximation for a function $f : \mathbb{M}^Y \to \mathbb{M}^Y$ is a function $f_a^{\#} : [Y]_a \to [Y]_{f(a)}$ where domain and codomain are different, hence it would not be meaningful to look for fixpoints. However, as explained below, it can be restricted to an endo-function.

Theorem 4.6 (soundness for pre-fixpoints). Let \mathbb{M} be a complete MV-chain, Y a finite set and $f : \mathbb{M}^Y \to \mathbb{M}^Y$ be a non-expansive function. Given a pre-fixpoint $a \in \mathbb{M}^Y$ of f, let $[Y]_{a=f(a)} = \{y \in [Y]_a \mid a(y) = f(a)(y)\}$. Let us define $f_a^* : [Y]_{a=f(a)} \to [Y]_{a=f(a)}$ as $f_a^*(Y') = f_a^{\#}(Y') \cap [Y]_{a=f(a)}$, where $f_a^{\#} : \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[Y]_{f(a)}}$ is the a-approximation of f. If $\nu f_a^* = \emptyset$ then $\nu f \sqsubseteq a$.

Proof. We prove the contrapositive, i.e., we show that $\nu f \not\sqsubseteq a$ allows us to derive that $\nu f_a^* \neq \emptyset$.

Assume $\nu f \not\sqsubseteq a$, i.e., there exists $y \in Y$ such that $\nu f(y) \not\sqsubseteq a(y)$. Since the order is total, this means that $a(y) \sqsubset \nu f(y)$. Hence, by Lemma 2.4(5), $\nu f(y) \ominus a(y) \sqsupset 0$. Then $\delta = \|\nu f \ominus a\| \sqsupseteq 0$.

Consider $Y' = \{y \in Y_a \mid \nu f(y) \ominus a(y) = \|\nu f \ominus a\|\} \neq \emptyset$. By Lemma 4.1, Y' is a post-fixpoint of $f_a^{\#}$, i.e., $Y' \subseteq f_a^{\#}(Y')$, and thus $Y' \subseteq \nu f_a^{\#}$. Moreover, for all $y \in Y'$, a(y) = f(a)(y), i.e., $Y' \subseteq [Y]_{a=f(a)}$. Therefore we conclude $Y' \subseteq f_a^{\#}(Y') \cap [Y]_{a=f(a)} = f_a^*(Y')$, i.e., Y' is a post-fixpoint also for f_a^* , and thus $\nu f_a^* \supseteq Y' \neq \emptyset$, as desired. \Box

The reason why we can limit our attention to the set of points where a(y) = f(a)(y) is as follows. Observe that, since a is a pre-fixpoint and \ominus is antitone in the second argument, $\nu f \ominus a \sqsubseteq \nu f \ominus f(a)$. Thus $\|\nu f \ominus a\| \sqsubseteq \|\nu f \ominus f(a)\| = \|f(\nu f) \ominus f(a)\| \sqsubseteq \|\nu f \ominus a\|$, where the last passage is motivated by non-expansiveness of f. Therefore $\|\nu f \ominus a\| = \|\nu f \ominus f(a)\|$. From this we can deduce that, if νf is strictly larger than a on some points, surely some of these points are in $[Y]_{a=f(a)}$. In particular, all points y_0 such that $\nu f(y_0) \ominus a(y_0) = \|\nu f \ominus a\|$ are necessarily in $[Y]_{a=f(a)}$. Otherwise, we would have $f(a)(y_0) \sqsubset a(y_0)$ and thus $\|\nu f \ominus a\| =$ $\nu f(y_0) \ominus a(y_0) \sqsubset \nu f(y_0) \ominus f(a)(y_0) \sqsubseteq \|\nu f \ominus f(a)\|$ (cf. Lemma 4.1).

Remark 4.7. Completeness does not generalise to pre-fixpoints, i.e., it is not true that if a is a pre-fixpoint of f and $\nu f \sqsubseteq a$, then $\nu f_a^* = \emptyset$. A pre-fixpoint might contain slack even though it is above the greatest fixpoint. A counterexample is in Example 6.16.

4.3. The dual view for least fixpoints. The theory developed so far can be easily dualised to check under-approximations of least fixpoints. Given a complete MV-algebra $\mathbb{M} = (M, \oplus, 0, \overline{(\cdot)})$ and a non-expansive function $f : \mathbb{M}^Y \to \mathbb{M}^Y$, in order to show that a post-fixpoint $a \in \mathbb{M}^Y$ is such that $a \sqsubseteq \mu f$ we can in fact simply work in the dual MV-algebra, $\mathbb{M}^{op} = (M, \sqsupseteq, \otimes, \overline{(\cdot)}, 1, 0).$

Since \oplus could be the "standard" operation on \mathbb{M} , it is convenient to formulate the conditions using \oplus and \ominus and the original order. The notation for the dual case is obtained from that of the original case, referred to as the *primal case* throughout the paper, exchanging subscripts and superscripts.

The pair of functions $\langle \alpha^{a,\theta}, \gamma^{a,\theta} \rangle$ is as follows. Let $a: Y \to \mathbb{M}$ and $0 \sqsubset \theta \in \mathbb{M}$. The set $[Y]^a = \{y \in Y \mid a(y) \neq 0\}$ and $\delta^a = \min\{a(y) \mid y \in [Y]^a\}$

The target of the approximation is $[a, a \otimes \theta]$ in the reverse order, hence $[a \otimes \theta, a]$ in the original order. Recall that $a \otimes \theta = \overline{\overline{a} \oplus \overline{\theta}} = a \oplus \overline{\theta}$. Hence we obtain

$$\mathbf{2}^{[Y]^a} \overbrace{\overset{\gamma^{a, heta}}{\overbrace{\gamma^{a, heta}}}}^{lpha^{a, heta}} [a\ominus\overline{ heta},a]$$

For $Y' \in \mathbf{2}^{[Y]^a}$ we define

$$\alpha^{a,\theta}(Y') = a \otimes \theta_{Y'} = a \ominus \overline{\theta}_{Y'}$$

Instead $\gamma^{a,\theta}(b) = \{y \in Y \mid \theta \supseteq b(y) \oplus a(y)\}$ where \oplus is the subtraction in the dual MV-algebra. Observe that $x \oplus y = \overline{x} \otimes \overline{y} = \overline{x} \oplus \overline{y} = \overline{y} \oplus \overline{x}$. Hence $\theta \supseteq b(y) \oplus a(y)$ iff $a(y) \oplus b(y) \supseteq \overline{\theta}$. Thus for $b \in [a \oplus \overline{\theta}, a]$ we have

$$\gamma^{a,\theta}(b) = \{ y \in Y \mid \theta \sqsupseteq b(y) \oplus a(y) \} = \{ y \in Y \mid a(y) \ominus b(y) \sqsupseteq \overline{\theta} \}.$$

Let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a monotone function. The norm becomes $||a|| = \min\{a(y) | y \in Y\}$. Non-expansiveness (Def. 3.3) in the dual MV-algebra becomes: for all $a, b \in \mathbb{M}^Y$, $||f(b) \oplus f(a)|| \supseteq ||b \oplus a||$, which in turn is

$$\min\{\overline{f(a) \ominus f(b)} \mid y \in Y\} \sqsupseteq \min\{\overline{a(y) \ominus b(y)} \mid y \in Y\}$$

i.e., $||f(a) \ominus f(b)|| \equiv ||a \ominus b||$, which coincides with non-expansiveness in the original MV-algebra.

Observe that, instead of taking a generic $\theta \sqsubset 1$ and then working with $\overline{\theta}$, we can directly take $0 \sqsubset \theta$ and replace everywhere $\overline{\theta}$ with θ .

While the approximation of a function in the primal case are denoted $f_a^{\#}$, the approximations in the dual case will be denoted by $f_{\#}^a$.

We can also dualise Proposition 4.5 and obtain that, whenever a is a fixpoint and $Y' = \nu f^a_{\#} \neq \emptyset$, then $a \ominus \theta_{Y'}$ is a pre-fixpoint, where $\theta = \iota^f_a(Y')$ is suitably defined, dualising Def. 4.3.

5. (DE)COMPOSING FUNCTIONS AND APPROXIMATIONS

Given a non-expansive function f and a (pre/post-)fixpoint a, it is often non-trivial to determine the corresponding approximations. However, non-expansive functions enjoy good closure properties (closure under composition, and closure under disjoint union) and we will see that the same holds for the corresponding approximations. Furthermore, it turns out that the functions needed in the applications can be obtained from just a few templates. This gives us a toolbox for assembling approximations with relative ease.

We start by introducing some basic functions, which will be used as the building blocks for the functions needed in the applications. Note that below we consider distributions on MV-chains of which the probability distributions introduced earlier are a special case.

Definition 5.1 (basic functions). Let \mathbb{M} be an MV-chain and let Y, Z be finite sets. (1) Constant: For a fixed $k \in \mathbb{M}^Z$, we define $c_k : \mathbb{M}^Y \to \mathbb{M}^Z$ by

$$c_k(a) = k$$

(2) Reindexing: For $u: Z \to Y$, we define $u^*: \mathbb{M}^Y \to \mathbb{M}^Z$ by

$$u^*(a) = a \circ u$$

(3) Min/Max: For $\mathcal{R} \subseteq Y \times Z$, we define $\min_{\mathcal{R}}, \max_{\mathcal{R}} : \mathbb{M}^Y \to \mathbb{M}^Z$ by

 $\min_{\mathcal{R}}(a)(z) = \min_{y \mathcal{R}z} a(y) \qquad \max_{\mathcal{R}}(a)(z) = \max_{y \mathcal{R}z} a(y)$

(4) Average: Call a function $p: Y \to \mathbb{M}$ a distribution when for all $y \in Y$, it holds $\overline{p(y)} = \bigoplus_{y' \in Y \setminus \{y\}} p(y')$ and let $\mathcal{D}(Y)$ be the set of distributions. Assume that \mathbb{M} is endowed with an additional operation \odot such that $(\mathbb{M}, \odot, 1)$ is a commutative monoid, for $x, y \in \mathbb{M}, x \odot y \sqsubseteq x$, and $x \odot y = 0$ iff x = 0 or y = 0, and \odot weakly distributes over \oplus , i.e., for all $x, y, z \in \mathbb{M}$ with $y \sqsubseteq \overline{z}, x \odot (y \oplus z) = x \odot y \oplus x \odot z$. For a finite set $D \subseteq \mathcal{D}(Y)$, we define $\operatorname{av}_D : \mathbb{M}^Y \to \mathbb{M}^D$ by

$$\operatorname{av}_D(a)(p) = \bigoplus_{y \in Y} p(y) \odot a(y)$$

A particularly interesting subcase of (3) is when we take as relation the *belongs to* relation $\in \subseteq Y \times 2^{Y}$. In this way we obtain functions for selecting the minimum and the maximum, respectively, of an input function over a set $Y' \subseteq Y$, that is, the functions $\min_{\in}, \max_{\in} : \mathbb{M}^{Y} \to \mathbb{M}^{2^{Y}}$, defined as

$$\min_{\in}(a)(Y') = \min_{y \in Y'} a(y) \qquad \max_{e}(a)(Y') = \max_{y \in Y'} a(y)$$

The usual probability distributions arise as a special case of $\mathcal{D}(Y)$ in (4) with $\mathbb{M} = [0, 1]$ where \odot is the standard multiplication.

Also note that in the definition of av_D , the operation \odot is necessarily monotone. In fact, if $y \sqsubseteq y'$ then, by Lemma 2.4(2), we have $y' = y \oplus (y' \ominus y)$. Therefore $x \odot y \sqsubseteq x \odot y \oplus x \odot (y' \ominus y) = x \odot (y \oplus (y' \ominus y)) = x \odot y'$, where the second passage holds by weak distributivity.

We can then prove the desired results (non-expansiveness and approximation) for the basic building blocks and their composition (all schematically reported in Table 1).

Theorem 5.2. All basic functions in Def. 5.1 are non-expansive. Furthermore non-expansive functions are closed under composition and disjoint union. The approximations are the ones listed in the third column of Table 1.

6. Applications

6.1. **Termination probability.** We start by making the example from the introduction (Section 1) more formal. Consider a Markov chain (S, T, η) , as defined in the introduction (Fig. 1), where we restrict the codomain of $\eta: S \setminus T \to \mathcal{D}(S)$ to $D \subseteq \mathcal{D}(S)$, where D is finite (to ensure that all involved sets are finite). Furthermore let $\mathcal{T}: [0,1]^S \to [0,1]^S$ be the function (Fig. 1) whose least fixpoint $\mu \mathcal{T}$ assigns to each state its termination probability.

Lemma 6.1. The function \mathcal{T} can be written as

$$\mathcal{T} = (\eta^* \circ \operatorname{av}_D) \uplus c_k$$

where $k: T \to [0,1]$ is the constant function 1 defined only on terminal states.

From this representation and Theorem 5.2 it is obvious that \mathcal{T} is non-expansive.

Lemma 6.2. Given a function $t: S \to [0,1]$, the t-approximation for \mathcal{T} in the dual sense is $\mathcal{T}^t_{\mu}: \mathbf{2}^{[S]^t} \to \mathbf{2}^{[S]^{\mathcal{T}(t)}}$ with

$$\mathcal{T}^t_{\#}(S') = \{ s \in [S]^{\mathcal{T}(t)} \mid s \notin T \land supp(\eta(s)) \subseteq S' \}.$$

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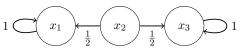
TABLE 1. Basic functions $f: \mathbb{M}^Y \to \mathbb{M}^Z$ (constant, reindexing, minimum, maximum, average), function composition, disjoint union and the corresponding approximations $f_a^{\#}: \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[Z]_{f(a)}}, f_{\#}^a: \mathbf{2}^{[Y]^a} \to \mathbf{2}^{[Z]^{f(a)}}.$

Notation: $\mathcal{R}^{-1}(z) = \{y \in Y \mid y\mathcal{R}z\}, supp(p) = \{y \in Y \mid p(y) \supseteq 0\}$ for $p \in \mathcal{D}(Y)$, arg $\min_{y \in Y'} a(y)$, resp. arg $\max_{y \in Y'} a(y)$, the set of elements where $a|_{Y'}$ reaches the minimum, resp. the maximum, for $Y' \subseteq Y$ and $a \in \mathbb{M}^Y$

function f	definition of f	$f_a^{\#}(Y')$ (above), $f_{\#}^a(Y')$ (below)
c_k	f(a) = k	Ø
$(k \in \mathbb{M}^Z)$		Ø
u^*	$f(a) = a \circ u$	$u^{-1}(Y')$
$(u\colon Z\to Y)$		$u^{-1}(Y')$
$\min_{\mathcal{R}}$	$f(a)(z) = \min_{y \mathcal{R}z} a(y)$	$\{z \in [Z]_{f(a)} \mid \underset{y \in \mathcal{R}^{-1}(z)}{\operatorname{argmin}} a(y) \subseteq Y'\}$
$(\mathcal{R} \subseteq Y \times Z)$		$\{z \in [Z]^{f(a)} \mid \operatorname*{arg min}_{y \in \mathcal{R}^{-1}(z)} a(y) \cap Y' \neq \emptyset\}$
$\max_{\mathcal{R}}$	$f(a)(z) = \max_{y \mathcal{R}z} a(y)$	$\{z \in [Z]_{f(a)} \mid \underset{y \in \mathcal{R}^{-1}(z)}{\operatorname{argmax}} a(y) \cap Y' \neq \emptyset\}$
$(\mathcal{R} \subseteq Y \times Z)$		$\{z \in [Z]^{f(a)} \mid \underset{y \in \mathcal{R}^{-1}(z)}{\operatorname{argmax}} a(y) \subseteq Y'\}$
av_D	$f(a)(p) = \bigoplus_{y \in Y} p(y) \odot a(y)$	$\{p \in [D]_{f(a)} \mid supp(p) \subseteq Y'\}$
$(Z = D \subseteq \mathcal{D}(Y))$	<i>y</i> _1	$\{p \in [D]^{f(a)} \mid supp(p) \subseteq Y'\}$
$h \circ g$	f(a) = h(g(a))	$ \begin{array}{c} h_{g(a)}^{\#} \circ g_{a}^{\#}(Y') \\ h_{\#}^{g(a)} \circ g_{\#}^{a}(Y') \end{array} $
$(g: \mathbb{M}^Y \to \mathbb{M}^W,$		$h^{g(a)}_{\mu} \circ q^a_{\mu}(Y')$
$h: \mathbb{M}^W \to \mathbb{M}^Z$		# 0# ()
$\biguplus_{i \in I} f_i I \text{ finite}$	$f(a)(z) = f_i(a _{Y_i})(z)$	$\biguplus_{i \in I} (f_i)_{a _{Y_i}}^{\#} (Y' \cap Y_i)$
$(f_i \colon \mathbb{M}^{Y_i} \to \mathbb{M}^{Z_i},$	$(z \in Z_i)$	$\biguplus_{i \in I} (f_i)_{\#}^{a _{Y_i}} (Y' \cap Y_i)$
$Y = \bigcup_{i \in I} Y_i, \ Z = \biguplus_{i \in I} Z_i)$		

At this point we have all the ingredients needed to formalise the application presented in the introduction. We refrain from repeating the same example, but rather present a new example that allows us to illustrate the question of the largest decrease for a fixpoint that still guarantees a pre-fixpoint (the dual problem is treated in Proposition 4.3).

Example 6.3. Consider the following Markov chain where $S = \{x_1, x_2, x_3\}$ are non-terminal states. The least fixpoint of the underlying fixpoint function \mathcal{T} is clearly the constant 0, since no state can reach a terminal state.



Now consider the function $t: S \to [0, 1]$ defined by $t(x_1) = 0.1$, $t(x_2) = 0.5$ and $t(x_3) = 0.9$. This is also a fixpoint of \mathcal{T} . Observe that $\mathcal{T}^t_{\#}(S) = S$ and thus, clearly, $\nu \mathcal{T}^t_{\#} = S$. According to (the dual of) Def. 4.3 we have $\delta^t(S) = 0.1$ and thus, by (the dual of) Proposition 4.5, the function $t' = t \ominus (0.1)_S$, with $t'(x_1) = 0$, $t'(x_2) = 0.4$, and $t'(x_3) = 0.8$, is a pre-fixpoint. Indeed, $\mathcal{T}(t')(x_1) = 0$, $\mathcal{T}(t')(x_2) = 0.4$ and $\mathcal{T}(t')(x_3) = 0.8$.

This is not the largest decrease producing a pre-fixpoint. In fact, we can choose $\theta = 0.9$, greater than $\delta^t(S)$ and we have that $t \ominus \theta_S$ is the constant 0, i.e., the least fixpoint of \mathcal{T} . However, if we take $\theta' = 0.5 \sqsubset \theta$, then $t \ominus \theta'_S$ is not a pre-fixpoint. In fact $(t \ominus \theta'_S)(x_2) = 0$, while $\mathcal{T}(t \ominus \theta'_S)(x_2) = 0.2$. This means that the set of decreases (beyond $\delta^t(S)$) producing a pre-fixpoint is not downward-closed and hence the largest decrease cannot be found by binary search, while, as already mentioned, a binary search will work for decreases below $\delta^t(S)$.

It is well-known that the function \mathcal{T} can be tweaked in such a way that it has a unique fixpoint, coinciding with $\mu \mathcal{T}$, by determining all states which cannot reach a terminal state and setting their value to zero [BK08]. Hence fixpoint iteration from above does not really bring us any added value here. It does however make sense to use the proof rule in order to guarantee lower bounds via post-fixpoints.

Furthermore, termination probability is a special case of the considerably more complex stochastic games that will be studied in Section 7, where the trick of modifying the function is not applicable.

6.2. Behavioural metrics for probabilistic automata and metric transition systems. We now consider behavioural metrics for probabilistic automata, which involve both nondeterministic branching as well as probabilistic branching. In addition, state labels can be taken from a metric space in order to capture the fact that there can be a lower bound for the distance related to some intrinsic features of the states. As we will discuss, the model is sufficiently general to capture, as instances, various kinds of probabilistic automata in the literature (e.g., $[BBL^+19]$) as well as metric transition systems [dFS09].

We first consider the Kantorovich and Hausdorff liftings and the corresponding approximations, which will play a major role in the treatment of probabilistic automata.

Kantorovich lifting. The Kantorovich (also known as Wasserstein) lifting converts a metric on X to a metric on probability distributions over X. Actually, as it commonly happens, we will define the lifting for general distance functions on [0, 1], not restricting to metrics.

In order to ensure finiteness of all the sets involved, we restrict to $D \subseteq \mathcal{D}(X)$, some finite set of probability distributions over X. A coupling of $p, q \in D$ is a probability distribution $c \in \mathcal{D}(X \times X)$ whose left and right marginals are p, q, i.e., $p(x_1) = m_c^L(x_1) := \sum_{x_2 \in X} c(x_1, x_2)$ and $q(x_2) = m_c^R(x_2) := \sum_{x_1 \in X} c(x_1, x_2)$. The set of all couplings of p, q, denoted by $\Omega(p, q)$, forms a polytope with finitely many vertices [PC20]. The set of all polytope vertices that are obtained by coupling any $p, q \in D$ is also finite and is denoted by $VP_D \subseteq \mathcal{D}(X \times X)$.

The Kantorovich lifting is defined as $\mathcal{K} : [0,1]^{X \times X} \to [0,1]^{D \times D}$ where

$$\mathcal{K}(d)(p,q) = \min_{c \in \Omega(p,q)} \sum_{(x_1,x_2) \in X \times X} c(x_1,x_2) \cdot d(x_1,x_2).$$

The coupling c can be interpreted as the optimal transport plan to move goods from suppliers to customers [Vil09]. Below we provide an alternative characterisation, which shows non-expansiveness of \mathcal{K} and allows one to derive its approximations.

where as $_{VP_D}$: $[0,1]^{X \times X} \to [0,1]^{VP_D}$, $\min_u : [0,1]^{VP_D} \to [0,1]^{D \times D}$.

We next present the approximation of the Kantorovich lifting in the dual sense. Intuitively, given a distance function d and a relation M on X, it characterises those pairs (p, q)of distributions whose distance in the Kantorovich metric decreases by a constant when we decrease the distance d for all pairs in M by the same constant.

 $\mathcal{K} = \min_{u} \circ \operatorname{av}_{VP_{\mathcal{D}}}$

Lemma 6.5. Let $d: X \times X \to [0,1]$. The approximation for the Kantorovich lifting \mathcal{K} in the dual sense is $\mathcal{K}^d_{\#}: \mathbf{2}^{[X \times X]^d} \to \mathbf{2}^{[D \times D]^{\mathcal{K}(d)}}$ with

$$\begin{aligned} \mathcal{K}^d_{\#}(M) &= \{(p,q) \in [D \times D]^{\mathcal{K}(d)} \mid \exists c \in \Omega(p,q), supp(c) \subseteq M, \\ &\sum_{u,v \in S} d(u,v) \cdot c(u,v) = \mathcal{K}(d)(p,q) \}. \end{aligned}$$

Hausdorff lifting. Given a metric d on a finite set X, the Hausdorff lifting of d provides a metric on the powerset $\mathbf{2}^X$. As for the Kantorovich lifting, we lift distance functions that are not necessarily metrics. The Hausdorff lifting is given by a function $\mathcal{H} : \mathbb{M}^{X \times X} \to \mathbb{M}^{\mathbf{2}^X \times \mathbf{2}^X}$ where

$$\mathcal{H}(d)(X_1, X_2) = \max\{\max_{x_1 \in X_1} \min_{x_2 \in X_2} d(x_1, x_2), \max_{x_2 \in X_2} \min_{x_1 \in X_1} d(x_1, x_2)\}.$$

An alternative characterisation of the Hausdorff lifting due to Mémoli [Mém11], also observed in [BBKK18], is more convenient for our purposes. Let $u : \mathbf{2}^{X \times X} \to \mathbf{2}^X \times \mathbf{2}^X$ be defined by $u(C) = (\pi_1[C], \pi_2[C])$, where π_1, π_2 are the projections $\pi_i : X \times X \to X$ and $\pi_i[C] = {\pi_i(c) \mid c \in C}$. Then $\mathcal{H}(d)(X_1, X_2) = \min{\{\max_{(x_1, x_2) \in C} d(x_1, x_2) \mid C \subseteq X \times X \land u(C) = (X_1, X_2)\}}$, which can be seen to correspond to the Wasserstein distance with C playing the role of couplings. Relying on this characterisation, we can obtain the result below, from which we deduce that \mathcal{H} is non-expansive and construct its approximation as the composition of the corresponding functions from Table 1.

Lemma 6.6. It holds that $\mathcal{H} = \min_u \circ \max_{\in} where \max_{\in} : \mathbb{M}^{X \times X} \to \mathbb{M}^{2^{X \times X}}$, with $\in \subseteq (X \times X) \times 2^{X \times X}$ the "is-element-of"-relation on $X \times X$, and $\min_u : \mathbb{M}^{2^{X \times X}} \to \mathbb{M}^{2^X \times 2^X}$.

We next determine the approximation of the Hausdorff lifting in the dual sense. Intuitively, given a distance function d and a relation R on X, such function characterises those pairs $(X_1, X_2), X_1, X_2 \subseteq X$, whose distance in the Hausdorff metric decreases by a constant when we decrease the distance d for all pairs in R by the same constant.

Lemma 6.7. The approximation for the Hausdorff lifting \mathcal{H} in the dual sense is as follows. Let $d: X \times X \to \mathbb{M}$, then $\mathcal{H}^d_{\#}: \mathbf{2}^{[X \times X]^d} \to \mathbf{2}^{[\mathbf{2}^X \times \mathbf{2}^X]^{\mathcal{H}(d)}}$ with

$$\mathcal{H}_{\#}^{d}(R) = \{ (X_{1}, X_{2}) \in [\mathbf{2}^{X} \times \mathbf{2}^{X}]^{\mathcal{H}(d)} \mid \\ \forall x_{1} \in X_{1} \big(\min_{x_{2}' \in X_{2}} d(x_{1}, x_{2}') = \mathcal{H}(d)(X_{1}, X_{2}) \Rightarrow \exists x_{2} \in X_{2} \colon \\ (x_{1}, x_{2}) \in R \land d(x_{1}, x_{2}) = \mathcal{H}(d)(X_{1}, X_{2}) \big) \land \\ \forall x_{2} \in X_{2} \big(\min_{x_{1}' \in X_{1}} d(x_{1}', x_{2}) = \mathcal{H}(d)(X_{1}, X_{2}) \Rightarrow \exists x_{1} \in X_{1} \colon \\ (x_{1}, x_{2}) \in R \land d(x_{1}, x_{2}) = \mathcal{H}(d)(X_{1}, X_{2}) \big) \}$$

Probabilistic automata. We have now all the tools needed to discuss probabilistic automata. Let (L, d_L) be a fixed metric space, which will be used for labelling states.

A probabilistic automaton is a tuple $\mathcal{A} = (S, \eta, \ell)$, where S is a non-empty finite set of states, $\eta: S \to \mathbf{2}^{\mathcal{D}(S)}$ assigns finite sets of probability distributions to states and $\ell: S \to L$ is a labelling function. (In the following we again replace $\mathcal{D}(S)$ by a finite subset D.)

The probabilistic bisimilarity pseudo-metrics is the least fixpoint of the function \mathcal{M} : $[0,1]^{S\times S} \to [0,1]^{S\times S}$ where for $d: S \times S \to [0,1], s, t \in S$:

$$\mathcal{M}(d)(s,t) = \max\{d_L(\ell(s),\ell(t)), \ \mathcal{H}(\mathcal{K}(d))(\eta(s),\eta(t))\}$$

where \mathcal{H} is the Hausdorff lifting (for $\mathbb{M} = [0, 1]$) and \mathcal{K} is the Kantorovich lifting defined earlier.

The fixpoint function \mathcal{M} can be expressed as the composition of more basic non-expansive functions and thus, by Theorem 5.2, it is non-expansive itself.

Lemma 6.8 (decomposing \mathcal{M}). The fixpoint function \mathcal{M} for probabilistic bisimilarity pseudo-metrics can be written as:

$$\mathcal{M} = \max_{\rho} \circ (((\eta \times \eta)^* \circ \mathcal{H} \circ \mathcal{K}) \uplus ((\ell \times \ell)^* \circ c_{d_L}))$$

where $\rho: (S \times S) \uplus (S \times S) \to (S \times S)$ with $\rho((s,t),i) = (s,t)$.

As discussed below, whenever d_L is discrete, this specializes to the probabilistic automata of [BBL⁺19] and whenever the probability distributions are Dirac distributions we obtain metric transition systems [dFS09].

The above decomposition also helps in determining the approximation of \mathcal{M} .

Lemma 6.9 (approximating \mathcal{M}). Let $d: S \times S \to [0,1]$. The approximation for \mathcal{M} in the dual sense is $\mathcal{M}^d_{\#}: \mathbf{2}^{[S \times S]^d} \to \mathbf{2}^{[S \times S]^{\mathcal{M}(d)}}$ with

$$\mathcal{M}^{d}_{\#}(X) = \{(s,t) \in [S \times S]^{\mathcal{M}(d)} \mid d_{L}(\ell(s),\ell(t)) < \mathcal{H}(\mathcal{K}(d))(\eta(s),\eta(t)) \\ \land (\eta(s),\eta(t)) \in \mathcal{H}^{\mathcal{K}(d)}_{\#} \circ \mathcal{K}^{d}_{\#}(X) \}$$

Comparison with $[BBL^{+}19]$. The paper $[BBL^{+}19]$ describes the first method for computing behavioural distances over probabilistic automata. Although the behavioural distance arises as a least fixpoint, it is in fact better, even the only known method, to iterate from above, in order to reach this least fixpoint. This is done by guessing and improving couplings, similarly to what happens for strategy iteration discussed later in Section 7. A major complication, faced in $[BBL^{+}19]$, is that the procedure can get stuck at a fixpoint which is not the least and one has to determine that this is the case and decrease the current candidate. This is done by relying on an adaptation of the notion of self-closed relation from [Fu12], and next we argue that this is closely related to the theory developed in Section 4. In fact this was our inspiration to generalise this technique to a more general setting.

We next establish a formal correspondence with our results. First note that the probabilistic automata considered in [BBL⁺19] are a special case of those defined above, where the metric on the set of state labels is required to be discrete. Hence states with different labels are necessarily at distance 1.

Let $\mathcal{A} = (S, \eta, \ell)$ be a fixed probabilistic automaton and let us assume that, as in [BBL⁺19], the metric space of labels (L, d_L) is discrete.

Assume that d is a fixpoint of \mathcal{M} , i.e., $d = \mathcal{M}(d)$. In order to check whether $d = \mu f$, [BBL⁺19] adapts the notion of a self-closed relation from [Fu12].

Definition 6.10 [BBL⁺19]. A relation $M \subseteq S \times S$ is *self-closed* with respect to $d = \mathcal{M}(d)$ if, whenever s M t, then

- $\ell(s) = \ell(t)$ and d(s, t) > 0,
- if $p \in \eta(s)$ and $d(s,t) = \min_{q' \in \eta(t)} \mathcal{K}(d)(p,q')$, then there exists $q \in \eta(t)$ and $c \in \Omega(p,q)$ such that $d(s,t) = \sum_{u,v \in S} d(u,v) \cdot c(u,v)$ and $supp(c) \subseteq M$,
- if $q \in \eta(t)$ and $d(s,t) = \min_{p' \in \eta(s)} \mathcal{K}(d)(p',q)$, then there exists $p \in \eta(s)$ and $c \in \Omega(p,q)$ such that $d(s,t) = \sum_{u,v \in S} d(u,v) \cdot c(u,v)$ and $supp(c) \subseteq M$.

The largest self-closed relation, denoted by \approx_d , can be shown to be empty if and only if $d = \mu f$ [BBL⁺19]. This has an immediate correspondence with our results since we can prove an intimate connection between self-closed relations and post-fixpoints of the approximation of \mathcal{M} .

Proposition 6.11. Let $d: S \times S \to [0,1]$ where $d = \mathcal{M}(d)$. Then $\mathcal{M}^d_{\#}: \mathbf{2}^{[S \times S]^d} \to \mathbf{2}^{[S \times S]^d}$, where $[S \times S]^d = \{(s,t) \in S \times S \mid d(s,t) > 0\}.$

Then M is a self-closed relation with respect to d if and only if $M \subseteq [S \times S]^d$ and M is a post-fixpoint of $\mathcal{M}^d_{\#}$.

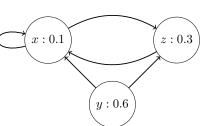
We observe that other kinds of probabilistic automata, e.g., those originally introduced by Rabin [Rab63], where transitions rather than states are labelled and some states are marked as final, i.e., the transition relation is of the kind $\eta: S \to \{0,1\} \times \mathcal{D}(S)^L$ or their non-deterministic variant, can be easily cast in our framework.

Branching Distances for Metric Transition Systems. We observe that also metric transition systems (MTS) and their (symmetrical) branching distances, as studied in [dFS09, FL14], live in our framework. In fact, a *metric transition system* over some metric space (L, d_L) is essentially a probabilistic automaton as defined above, where the probabilistic component is dropped, i.e., $\mathcal{A} = (S, \eta, \ell)$ where S is the set of states, $\eta : S \to \mathbf{2}^S$ is the transition function and $\ell : S \to L$ a labelling function.

Clearly, a metric transition system $\mathcal{A} = (S, \eta, \ell)$ can be formally seen as a special probabilistic automaton. Given a state $s \in S$, let β_s denote the Dirac distribution, assigning probability 1 to s and 0 to all other states. Then we can "transform" the transition relation $\eta: S \to \mathbf{2}^S$ into $\eta': S \to \mathbf{2}^{\mathcal{D}(S)}$, defining $\eta'(s) = \{\beta_t \mid t \in \eta(s)\}$. Using this observation, Lemma 6.9 and the fact that for a distance $d: S \times S \to [0, 1]$ and a pair of states $s, t \in S$, it holds $\mathcal{K}(d)(\beta_s, \beta_t) = d(s, t)$, we obtain the approximation:

$$\mathcal{M}^{d}_{\#}(X) = \{(s,t) \in [S \times S]^{\mathcal{M}(d)} \mid d_{L}(\ell(s),\ell(t)) < \mathcal{H}(d)(\eta(s),\eta(t)) \\ \wedge (\eta(s),\eta(t)) \in \mathcal{H}^{d}_{\#}(X)\}$$

Example 6.12. We consider the MTS depicted below, where the metric space of labels is the real interval [0, 1] with the Euclidean distance $d_L(r, s) = |r - s|$.



Here, $\eta(x) = \{x, z\}, \ \eta(y) = \{x, y, z\}$ and $\eta(z) = \{x\}$. Additionally we have $\ell(x) = 0.1$, $\ell(y) = 0.6$ and $\ell(z) = 0.3$ resulting in $d_L(\ell(x), \ell(y)) = 0.5$, $d_L(\ell(x), \ell(z)) = 0.2$ and $d_L(\ell(y), \ell(z)) = 0.3$. The least fixpoint of \mathcal{M} is a pseudo-metric $\mu \mathcal{M}$ given by $\mu \mathcal{M}(x, y) =$ $\mu \mathcal{M}(y, z) = 0.5$ and $\mu \mathcal{M}(x, z) = 0.3$. (Since $\mu \mathcal{M}$ is a pseudo-metric, the remaining entries are fixed: $\mu \mathcal{M}(u, u) = 0$ and $\mu \mathcal{M}(u, v) = \mu \mathcal{M}(v, u)$ for all $u, v \in \{x, y, z\}$.)

Now consider the pseudo-metric d with d(x, y) = d(x, z) = d(y, z) = 0.5. This is also a fixpoint of \mathcal{M} . Note that $\mathcal{H}(d)(\eta(x), \eta(y)) = \mathcal{H}(d)(\eta(x), \eta(z)) = \mathcal{H}(d)(\eta(y), \eta(z)) = 0.5$. Let us use our technique in order to verify that d is not the least fixpoint of \mathcal{M} , by showing that $\nu \mathcal{M}^d_{\#} \neq \emptyset$.

We start fixpoint iteration with the approximation $\mathcal{M}^d_{\#}$ from the top element $[S \times S]^d$, which is given by the symmetric closure³ of $\{(x, y), (x, z), (y, z)\}$ (since reflexive pairs do not contain slack).

We first observe that the pairs $(x, y), (y, x) \notin \mathcal{M}^d_{\#}(Sym(\{(x, y), (x, z), (y, z)\}))$ since $d_L(\ell(x), \ell(y)) = 0.5 \notin \mathcal{H}(d)(\eta(x), \eta(y)) = 0.5$. Next, $(y, z), (z, y) \notin \mathcal{M}^d_{\#}(Sym(\{(x, z), (y, z)\}))$ since it holds $(\eta(y), \eta(z)) \notin \mathcal{H}^d_{\#}(Sym(\{(x, z), (y, z)\}))$. In order to see this, consider the approximation of the Hausdorff lifting in Lemma 6.7 and note that for $y \in \eta(y)$ we have $\min_{u \in \eta(z)} d(y, u) = 0.5 = \mathcal{H}(d)(\eta(y), \eta(z))$, but $(y, x) \notin Sym(\{(x, z), (y, z)\})$ (where x is the only element in $\eta(z)$).

The pairs (x, z), (z, x) on the other hand satisfy all conditions and hence

$$\mathcal{M}^d_{\#} = Sym(\{(x,z)\}) = \mathcal{M}^d_{\#}(Sym(\{(x,z)\})) \neq \emptyset$$

Thus we conclude that d is not the least fixpoint, but, according to Proposition 4.5, we can decrease the value of d in the positions (x, z), (z, x) and obtain a pre-fixpoint from which we can continue fixpoint iteration.

6.3. **Bisimilarity.** In order to define standard bisimilarity we use a variant \mathcal{G} of the Hausdorff lifting \mathcal{H} defined before, where max and min are swapped. More precisely, $\mathcal{G}: \mathbb{M}^{X \times X} \to \mathbb{M}^{2^X \times 2^X}$ is defined, for $d \in \mathbb{M}^{X \times X}$, by

 $\mathcal{G}(d)(X_1, X_2) = \max\{\min_{(x_1, x_2) \in C} d(x_1, x_2) \mid C \subseteq X \times X \land u(C) = (X_1, X_2)\}.$

Now we can define the fixpoint function for bisimilarity and its corresponding approximation. For simplicity we consider unlabelled transition systems, but it would be straightforward to handle labelled transitions.

Let X be a finite set of states and $\eta: X \to \mathbf{2}^X$ a function that assigns a set of successors $\eta(x)$ to a state $x \in X$. The fixpoint function for bisimilarity $\mathcal{B}: \{0,1\}^{X \times X} \to \{0,1\}^{X \times X}$ can be expressed by using the Hausdorff lifting \mathcal{G} with $\mathbb{M} = \{0,1\}$.

³We denote the symmetric closure of a relation R by Sym(R).

Lemma 6.13. Bisimilarity on η is the greatest fixpoint of $\mathcal{B} = (\eta \times \eta)^* \circ \mathcal{G}$.

Since we are interested in the greatest fixpoint, we are working in the primal sense. Bisimulation relations are represented by their characteristic functions $a: X \times X \to \{0, 1\}$, in fact the corresponding relation can be obtained by taking the complement of $[X \times X]_a = \{(x_1, x_2) \in X_1 \times X_2 \mid a(x_1, x_2) = 0\}.$

Lemma 6.14. Let $a: X \times X \to \{0,1\}$. The approximation for the bisimilarity function \mathcal{B} in the primal sense is $\mathcal{B}_a^{\#}: \mathbf{2}^{[X \times X]_a} \to \mathbf{2}^{[X \times X]_{\mathcal{B}(a)}}$ with

$$\mathcal{B}_{a}^{\#}(R) = \{ (x_{1}, x_{2}) \in [X \times X]_{\mathcal{B}(a)} \mid \\ \forall y_{1} \in \eta(x_{1}) \exists y_{2} \in \eta(x_{2}) ((y_{1}, y_{2}) \notin [X \times X]_{a} \lor (y_{1}, y_{2}) \in R)) \\ \land \forall y_{2} \in \eta(x_{2}) \exists y_{1} \in \eta(x_{1}) ((y_{1}, y_{2}) \notin [X \times X]_{a} \lor (y_{1}, y_{2}) \in R) \}$$

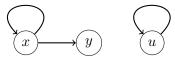
We conclude this section by discussing how this view on bisimilarity can be useful: first, it again opens up the possibility to compute bisimilarity – a greatest fixpoint – by iterating from below, through smaller fixpoints. This could potentially be useful if it is easy to compute the least fixpoint of \mathcal{B} inductively and continue from there.

Furthermore, we obtain a technique for witnessing non-bisimilarity of states. While this can also be done by exhibiting a distinguishing modal formula [HM85, Cle90] or by a winning strategy for the spoiler in the bisimulation game [Sti97], to our knowledge there is no known method that does this directly, based on the definition of bisimilarity.

With our technique we can witness non-bisimilarity of two states $x_1, x_2 \in X$ by presenting a pre-fixpoint a (i.e., $\mathcal{B}(a) \leq a$) such that $a(x_1, x_2) = 0$ (equivalent to $(x_1, x_2) \in [X \times X]_a$) and $\nu \mathcal{B}_a^{\#} = \emptyset$, since this implies $\nu \mathcal{B}(x_1, x_2) \leq a(x_1, x_2) = 0$ by our proof rule.

There are two issues to discuss: first, how can we characterise a pre-fixpoint of \mathcal{B} (which is quite unusual, since bisimulations are post-fixpoints)? In fact, the condition $\mathcal{B}(a) \leq a$ can be rewritten to: for all $(x_1, x_2) \in [X \times X]_a$ there exists $y_1 \in \eta(x_1)$ such that for all $y_2 \in \eta(x_2)$ we have $(y_1, y_2) \in [X \times X]_a$ (or vice versa). Second, at first sight it does not seem as if we gained anything since we still have to do a fixpoint computation on relations. However, the carrier set is $[X \times X]_a$, i.e., a set of non-bisimilarity witnesses and this set can be small even though X might be large, since a might have value 0 only on a small subset of $X \times X$.

Example 6.15. We consider the transition system depicted below.



Our aim is to construct a witness showing that x, u are not bisimilar. This witness is a function $a: X \times X \to \{0, 1\}$ with a(x, u) = 0 = a(y, u) and for all other pairs the value is 1. Hence $[X \times X]_{a=\mathcal{B}(a)} = [X \times X]_a = \{(x, u), (y, u)\}$ and it is easy to check that a is a pre-fixpoint of \mathcal{B} and that $\nu \mathcal{B}_a^* = \emptyset$: we iterate over $\{(x, u), (y, u)\}$ and first remove (y, u) (since y has no successors) and then (x, u). This implies that $\nu \mathcal{B} \leq a$ and hence $\nu \mathcal{B}(x, u) = 0$, which means that x, u are not bisimilar.

Example 6.16. We modify Example 6.15 and consider a function a where a(x, u) = 0 and all other values are 1. Again a is a pre-fixpoint of \mathcal{B} and $\nu \mathcal{B} \leq a$ (since only reflexive pairs are in the bisimilarity). However $\nu \mathcal{B}_a^* \neq \emptyset$, since $\{(x, u)\}$ is a post-fixpoint. This is a counterexample to completeness discussed after Theorem 4.6.

Intuitively speaking, the states y, u over-approximate and claim that they are bisimilar, although they are not. (This is permissible for a pre-fixpoint.) This tricks x, u into thinking that there is some wiggle room and that one can increase the value of (x, u). This is true, but only because of the limited, local view, since the "true" value of (y, u) is 0.

7. SIMPLE STOCHASTIC GAMES

In this section we show how our techniques can be applied to simple stochastic games [Con92, Con90]. In particular, we present two novel algorithms based on strategy iteration and discuss some runtime results.

7.1. Introduction to simple stochastic games. A simple stochastic game is a state-based two-player game where the two players, Min and Max, each own a subset of states they control, for which they can choose the successor. The system also contains sink states with an assigned payoff and averaging states which randomly choose their successor based on a given probability distribution. The goal of Min is to minimise and the goal of Max to maximise the payoff.

Simple stochastic games are an important type of games that subsume parity games and the computation of behavioural distances for probabilistic automata (cf. Section 6.2, [BBL⁺19]). The associated decision problem (if both players use their best strategies, is the expected payoff of Max greater than $\frac{1}{2}$?) is known to lie in NP \cap coNP, but it is an open question whether it is contained in P. There are known randomised subexponential algorithms [BV05].

It has been shown that it is sufficient to consider positional strategies, i.e., strategies where the choice of the player is only dependent on the current state. The expected payoffs for each state form a so-called value vector and can be obtained as the least solution of a fixpoint equation (see below).

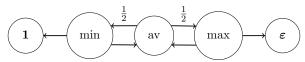
A simple stochastic game is given by a finite set V of nodes, partitioned into MIN, MAX, AV (average) and SINK, and the following data: $\eta_{\min}: MIN \to \mathbf{2}^V, \ \eta_{\max}: MAX \to \mathbf{2}^V$ (successor functions for Min and Max nodes), $\eta_{\rm av}: AV \to D$ (probability distributions, where $D \subseteq \mathcal{D}(V)$ finite) and $w: SINK \to [0,1]$ (weights of sink nodes). The fixpoint function $\mathcal{V}: [0,1]^V \to [0,1]^V$ is defined below for $a: V \to [0,1]$ and $v \in V$:

$$\mathcal{V}(a)(v) = \begin{cases} \min_{v' \in \eta_{\min}(v)} a(v') & v \in MIN\\ \max_{v' \in \eta_{\max}(v)} a(v') & v \in MAX\\ \sum_{v' \in V} \eta_{av}(v)(v') \cdot a(v') & v \in AV\\ \ell(v) & v \in SINK \end{cases}$$

The *least* fixpoint of \mathcal{V} specifies the average payoff for all nodes when Min and Max play optimally. In an infinite game the payoff is 0. In order to avoid infinite games and guarantee uniqueness of the fixpoint, many authors [KH66, Con90, RVAK11] restrict to stopping games, which are guaranteed to terminate for every pair of Min/Max-strategies. Here we deal with general games where more than one fixpoint may exist. Such a scenario has been studied in [KKKW18], which considers value iteration to under- and over-approximate the value vector. The over-approximation faces challenges with cyclic dependencies, similar to the vicious cycles described earlier. Here we focus on strategy iteration, which is usually less efficient than value iteration, but yields a precise result instead of approximating it.

Example 7.1. We consider the game depicted below. Here min is a Min node with $\eta_{\min}(\min) = \{\mathbf{1}, \operatorname{av}\}$, max is a Max node with $\eta_{\max}(\max) = \{\varepsilon, \operatorname{av}\}$, **1** is a sink node with payoff 1, ε is a sink node with some small payoff $\varepsilon \in (0, 1)$ and av is an average node which transitions to both min and max with probability $\frac{1}{2}$. This game is not stopping.

Min should choose av as successor since a payoff of 1 is bad for Min. Given this choice of Min, Max should not declare av as successor since this would create an infinite play and hence the payoff is 0. Therefore Max has to choose ε and be content with a payoff of ε , which is achieved from all nodes different from **1**.



In order to be able to determine the approximation of \mathcal{V} and to apply our techniques, we consider the following equivalent definition.

Lemma 7.2. $\mathcal{V} = (\eta_{\min}^* \circ \min_{\in}) \uplus (\eta_{\max}^* \circ \max_{\in}) \uplus (\eta_{av}^* \circ av_D) \uplus c_w, where \in \subseteq V \times \mathbf{2}^V$ is the "is-element-of"-relation on V.

As a composition of non-expansive functions, \mathcal{V} is non-expansive as well. Since we are interested in the least fixpoint we work in the dual sense and obtain the following approximation, which intuitively says: we can decrease a value at node v by a constant only if, in the case of a Min node, we decrease the value of one successor where the minimum is reached, in the case of a Max node, we decrease the values of all successors where the maximum is reached, and in the case of an average node, we decrease the values of all successors.

Lemma 7.3. Let $a: V \to [0,1]$. The approximation for the value iteration function \mathcal{V} in the dual sense is $\mathcal{V}^a_{\#}: \mathbf{2}^{[V]^a} \to \mathbf{2}^{[V]^{\mathcal{V}(a)}}$ with

$$\mathcal{V}^{a}_{\#}(V') = \{ v \in [V]^{\mathcal{V}(a)} \mid (v \in MIN \land \arg\min_{v' \in \eta_{\min}(v)} a(v') \cap V' \neq \emptyset) \lor \\ (v \in MAX \land \arg\max_{v' \in \eta_{\max}(v)} a(v') \subseteq V') \lor (v \in AV \land supp(\eta_{\mathrm{av}}(v)) \subseteq V') \}$$

7.2. Strategy iteration from above and below. We describe two algorithms based on the idea of strategy iteration, first introduced by Hoffman and Karp in [KH66], that are novel, as far as we know. The first iterates to the least fixpoint from above and uses the techniques described in Section 4 in order not to get stuck at a larger fixpoint. The second iterates from below: the role of our results is not directly visible in the code of the algorithm, but its non-trivial correctness proof is based on the proof rule introduced earlier.

We first recap the underlying notions. A Min-strategy is a mapping $\tau: MIN \to V$ such that $\tau(v) \in \eta_{\min}(v)$ for every $v \in MIN$. Following such a strategy, Min decides to always leave a node v via $\tau(v)$. Analogously, a Max-strategy is a function $\sigma: MAX \to V$. Fixing a strategy for either player induces a modified value function. If τ is a Min-strategy, we obtain \mathcal{V}_{τ} which is defined exactly as \mathcal{V} but for $v \in MIN$ where we set $\mathcal{V}_{\tau}(a)(v) = a(\tau(v))$. Analogously, for σ a Max-strategy, \mathcal{V}_{σ} is obtained by setting $\mathcal{V}_{\sigma}(a)(v) = a(\sigma(v))$ when $v \in MAX$. If both players fix their strategies, the game reduces to a Markov chain. It is easy to see that fixing a strategy for one player produces an under- or over-approximation, depending on the player, of the function \mathcal{V} .

Determine $\mu \mathcal{V}$ (from above)

1. Guess a Min-strategy $\tau^{(0)}, i := 0$ 2. $a^{(i)} := \mu \mathcal{V}_{\tau^{(i)}}$ 3. $\tau^{(i+1)} := sw_{\min}(\tau^{(i)}, a^{(i)})$ 4. If $\tau^{(i+1)} \neq \tau^{(i)}$ then i := i + 1 and goto 2. 5. Compute $V' = \nu \mathcal{V}_{\#}^{a}$, where $a = a^{(i)}$. 6. If $V' = \emptyset$ then stop and return $a^{(i)}$. Otherwise $a^{(i+1)} := a - (\iota_{\mathcal{V}}^{a}(V'))_{V'}, \tau^{(i+2)} := sw_{\min}(\tau^{(i)}, a^{(i+1)}), i := i + 2, goto 2.$

(A) Strategy iteration from above

Determine $\mu \mathcal{V}$ (from below)

1. Guess a Max-strategy $\sigma^{(0)}$, i := 02. $a^{(i)} := \mu \mathcal{V}_{\sigma^{(i)}}$ 3. $\sigma^{(i+1)} := sw_{\max}(\sigma^{(i)}, a^{(i)})$ 4. If $\sigma^{(i+1)} \neq \sigma^{(i)}$ then i := i + 1 and goto 2. Otherwise stop and return $a^{(i)}$.

(B) Strategy iteration from below

FIGURE 3. Strategy iteration from above and below

Lemma 7.4. For every Max-strategy σ and every Min-strategy τ it holds that $\mathcal{V}_{\sigma} \leq \mathcal{V} \leq \mathcal{V}_{\tau}$.

In order to describe our algorithms we also need the notion of a *switch*. Assume that τ is a Min-strategy and let a be a (pre-)fixpoint of \mathcal{V}_{τ} . Min can now potentially improve her strategy for nodes $v \in MIN$ where $\min_{v' \in \eta_{\min}(v)} a(v') < a(\tau(v))$, called *switch nodes*. This results in a Min-strategy $\tau' = sw_{\min}(\tau, a)$, where $\tau'(v) = \arg\min_{v' \in \eta_{\min}(v)} a^{(i)}(v')$ for a switch node v and τ', τ agree otherwise. Also, $sw_{\max}(\sigma, a)$ is defined analogously for Max strategies.

Now strategy iteration from above works as described in Fig. 3a. The computation of $\mu \mathcal{V}_{\tau^{(i)}}$ in the second step intuitively means that Max chooses his best answering strategy and we compute the least fixpoint based on this answering strategy. At some point no further switches are possible and we have reached a fixpoint a, which need not yet be the least fixpoint. Hence we use the techniques from Section 4 to decrease a and obtain a new pre-fixpoint $a^{(i+1)}$, from which we can continue. The correctness of this procedure partially follows from Theorem 4.2 and Proposition 4.5, however we also need to show the following: first, we can compute $a^{(i)} = \mu \mathcal{V}_{\tau^{(i)}}$ efficiently by solving a linear program (cf. Lemma 7.5) by adapting [Con92]. Second, the chain of the $a^{(i)}$ decreases, which means that the algorithm will eventually terminate (cf. Theorem 7.6).

Strategy iteration from below is given in Fig. 3b. At first sight, the algorithm looks simpler than strategy iteration from above, since we do not have to check whether we have already reached $\nu \mathcal{V}$, reduce and continue from there. However, in this case the computation of $\mu \mathcal{V}_{\sigma^{(i)}}$ via a linear program is more involved (cf. Lemma 7.5), since we have to pre-compute (via greatest fixpoint iteration over $\mathbf{2}^V$) the nodes where Min can force a cycle based on the current strategy of Max, thus obtaining payoff 0.

This algorithm does not directly use our technique but we can use our proof rules to prove the correctness of the algorithm (Theorem 7.6). In particular, the proof that the sequence $a^{(i)}$ increases is quite involved: we have to show that $a^{(i)} = \mu \mathcal{V}_{\sigma^{(i)}} \leq \mu \mathcal{V}_{\sigma^{(i+1)}} = a^{(i+1)}$. This could be done by showing that $\mu \mathcal{V}_{\sigma^{(i+1)}}$ is a pre-fixpoint of $\mathcal{V}_{\sigma^{(i)}}$, but there is no straightforward

⁴If the minimum is achieved in several nodes, Min simply chooses one of them. However, she will only switch if this strictly improves the value.

way to do this. Instead, we prove this fact using our proof rules, by showing that $a^{(i)}$ is below the least fixpoint of $\mathcal{V}_{\sigma^{(i+1)}}$.

The algorithm generalises strategy iteration by Hoffman and Karp [KH66]. Note that we cannot simply adapt their proof, since we do not assume that the game is stopping, which is a crucial ingredient. In the case where the game is stopping, the two algorithms coincide, meaning that we also provide an alternative correctness proof in this situation, while other correctness proofs [Con92] are based on linear algebra and inverse matrices.

Lemma 7.5. The least fixpoints of \mathcal{V}_{τ} and \mathcal{V}_{σ} can be determined by solving linear programs.

Theorem 7.6. Strategy iteration from above and below both terminate and compute the least fixpoint of \mathcal{V} .

Proof. Strategy iteration from above:

We start by showing the following: Given any $a^{(i)}$ and a new switched Min-strategy $\tau^{(i+1)}$, i.e., $\tau^{(i+1)} = sw_{\min}(\tau^{(i)}, a^{(i)})$, then $a^{(i)}$ is a pre-fixpoint of $\mathcal{V}_{\tau^{(i+1)}}$. By choice of $\tau^{(i+1)}$ we have

$$\mathcal{V}_{\tau^{(i+1)}}(a^{(i)})(v) = \begin{cases}
a^{(i)}(\tau^{(i+1)}(v)) & v \in MIN \\
\max_{v' \in \eta_{\max}(v)} a^{(i)}(v') & v \in MAX \\
\sum_{v' \in V} a^{(i)}(v') \cdot \eta_{av}(v)(v') & v \in AV \\
w(v) & v \in SINK
\end{cases}$$

$$= \begin{cases}
\min_{v' \in \eta_{\min}(v)} a^{(i)}(v') & v \in MIN \\
\max_{v' \in \eta_{\max}(v)} a^{(i)}(v') & v \in MAX \\
\sum_{v' \in V} a^{(i)}(v') \cdot \eta_{av}(v)(v') & v \in AV \\
w(v) & v \in SINK
\end{cases}$$

$$= \mathcal{V}(a^{(i)})(v)$$

By Lemma 7.4 we know that $\mathcal{V} \leq \mathcal{V}_{\tau^{(i)}}$, and since $a^{(i)}$ is a fixpoint of $\mathcal{V}_{\tau^{(i)}}$ we conclude

$$\mathcal{V}_{\tau^{(i+1)}}(a^{(i)})(v) = \mathcal{V}(a^{(i)})(v) \le \mathcal{V}_{\tau^{(i)}}(a^{(i)})(v) = a^{(i)}(v)$$

Thus we have $a^{(i+1)} \leq a^{(i)}$ (by Knaster-Tarski, since $a^{(i)}$ is a pre-fixpoint of $\mathcal{V}_{\tau^{(i+1)}}$ and $a^{(i+1)}$ is its least fixpoint). Furthermore we know that $a^{(i)}$ is not a fixpoint of $\mathcal{V}_{\tau^{(i+1)}}$ (otherwise we could not have performed a switch) and hence $a^{(i+1)}$ is strictly smaller than $a^{(i)}$ for at least one input. Since there are only finitely many strategies we will eventually stop switching and reach a fixpoint $a = a^{(j)}$ for an index j.

Then, if $V' = \nu \mathcal{V}^a_{\#} = \emptyset$ then *a* is the least fixpoint and we conclude.

Otherwise, we determine $a^{(j+1)} = a - (\iota_{\mathcal{V}}^a(V'))_{V'}$. By Proposition 4.5 (dual version), $a^{(j+1)}$ is a pre-fixpoint of \mathcal{V} . Now Min will choose her best strategy $\tau = \tau^{(j+2)} = sw_{\min}(\tau^{(i)}, a^{(i+1)})$ and we continue computing $a^{(j+2)} = \mu \mathcal{V}_{\tau^{(j+2)}}$. First, observe that since $a^{(j+1)}$ is a pre-fixpoint of \mathcal{V} , it is also a pre-fixpoint of $\mathcal{V}_{\tau^{(j+2)}}$. In fact, \mathcal{V} and $\mathcal{V}_{\tau^{(j+1)}}$ coincide on all nodes $v \notin MIN$. If $v \in MIN$, we have

$$\begin{aligned} \mathcal{V}_{\tau^{(j+2)}}(a^{(j+1)})(v) &= a^{(j+1)}(\tau^{(j+2)}(v)) \\ &= \min_{v' \in \eta_{\min}(v)} a^{(j+1)}(v') = \mathcal{V}(a^{(j+1)})(v) \le a^{(j+1)}(v). \end{aligned}$$

Hence it follows by Knaster-Tarski that $a^{(j+2)} = \mu \mathcal{V}_{\tau^{(j+2)}} \leq a^{(j+1)}$. In turn, $a^{(j+1)} < a^{(j)}$ since V' is non-empty and hence also $a^{(j+2)} < a^{(j)}$ (where < on tuples means means \leq in all components and < in at least one component.)

This means that the chain $a^{(i)}$ is strictly descending. Hence, at each iteration we obtain a new strategy and, since the number of strategies is finite, the iteration will eventually stop.

Hence the algorithm terminates and stops at the least fixpoint of \mathcal{V} .

Strategy iteration from below:

We start as follows: Assume a is the least fixpoint of \mathcal{V}_{σ} , i.e. $a = \mu \mathcal{V}_{\sigma}$ and σ' the new best strategy for Max obtained by switching with respect to a, i.e., $\sigma' = sw_{\max}(\sigma, a)$. We have to show that $a' = \mu \mathcal{V}_{\sigma'}$ lies above a $(a' \ge a)$. Here we use our proof rules (see Theorem 4.6) and show the following:

• First, observe that a is a post-fixpoint of $\mathcal{V}_{\sigma'}$. For any $v \in V$ we have

$$a(v) = \mathcal{V}_{\sigma}(a)(v) = \begin{cases} \min_{v' \in \eta_{\min}(v)} a(v') & v \in MIN\\ a(\sigma(v)) & v \in MAX\\ \sum_{v' \in V} a(v') \cdot \eta_{av}(v)(v') & v \in AV\\ w(v) & v \in SINK \end{cases}$$
$$\leq \begin{cases} \min_{v' \in \eta_{\min}(v)} a(v') & v \in MIN\\ \max_{v' \in \eta_{\max}(v)} a(v') & v \in MAX\\ \sum_{v' \in V} a(v') \cdot \eta_{av}(v)(v') & v \in AV\\ w(v) & v \in SINK \end{cases}$$
$$= \begin{cases} \min_{v' \in \eta_{\min}(v)} a(v') & v \in MIN\\ a(\sigma'(v)) & v \in MIN\\ a(\sigma'(v)) & v \in MAX\\ \sum_{v' \in V} a(v') \cdot \eta_{av}(v)(v') & v \in AV\\ w(v) & v \in SINK \end{cases}$$
$$= \mathcal{V}_{\sigma'_{\max}}(a)(v)$$

• Next we show that $\nu(\mathcal{V}_{\sigma'})^a_* = \emptyset$, thus proving that $a \leq \mu \mathcal{V}_{\sigma'} = a'$ by Theorem 4.6. Note that $(\mathcal{V}_{\sigma'})^a_* \colon [V]^{a = \mathcal{V}_{\sigma'}(a)} \to [V]^{a = \mathcal{V}_{\sigma'}(a)}$, i.e., it restricts to those elements of a where a and $\mathcal{V}_{\sigma'}(a)$ coincide.

Whenever $v \in MAX$ is a node where the strategy has been "switched" with respect to a, we have

$$\mathcal{V}_{\sigma'}(a)(v) = a(\sigma'(v)) > a(\sigma(v)) = a(v).$$

The first equality above is true by the definition of $\mathcal{V}_{\sigma'}$ and the last equality holds since a is a fixpoint of \mathcal{V}_{σ} . So if v is a switch node, it holds that $v \notin [V]^{a=\mathcal{V}_{\sigma}(a)}$. By contraposition if $v \in [V]^{a=\mathcal{V}_{\sigma}(a)}$, v cannot be a switch node.

We next show that $(\mathcal{V}_{\sigma})^a_*$, $(\mathcal{V}_{\sigma'})^a_*$ agree on $[V]^{a=\mathcal{V}_{\sigma'}(a)} \subseteq [V]^a = [V]^{a=\mathcal{V}_{\sigma}(a)}$ (remember that a is a fixpoint of \mathcal{V}_{σ}). It holds that

$$(\mathcal{V}_{\sigma})^{a}_{*}(V') = \gamma^{\mathcal{V}_{\sigma}(a),\iota}(\mathcal{V}_{\sigma}(\alpha^{a,\iota}(V'))) (\mathcal{V}_{\sigma'})^{a}_{*}(V') = \gamma^{\mathcal{V}_{\sigma'}(a),\iota}(\mathcal{V}_{\sigma'}(\alpha^{a,\iota}(V'))) \cap [V]^{a=\mathcal{V}_{\sigma'}(a)}$$

for a suitable constant ι and if we choose ι small enough we can use the same constant in both cases. Now let $v \in [V]^{a=\mathcal{V}_{\sigma'}(a)}$: by definition it holds that $v \in (\mathcal{V}_{\sigma})^{a}_{*}(V') =$ $\gamma^{\mathcal{V}_{\sigma}(a),\iota}(\mathcal{V}_{\sigma}(\alpha^{a,\iota}(V')))$ if and only if $\mathcal{V}_{\sigma}(\alpha^{a,\iota}(V'))(v) \ominus \mathcal{V}_{\sigma}(a)(v) \geq \iota$. Since, by the considerations above, v is not a switch node, $\mathcal{V}_{\sigma}(b)(v) = \mathcal{V}_{\sigma'}(b)(v)$ for all b and we can replace \mathcal{V}_{σ} by $\mathcal{V}_{\sigma'}$, resulting in the equivalent statement $v \in \gamma^{\mathcal{V}_{\sigma'}(a),\iota}(\mathcal{V}_{\sigma'}(\alpha^{a,\iota}(V')))$, also equivalent to $v \in (\mathcal{V}_{\sigma'})^{a}_{*}(V')$.

Thus $\nu(\mathcal{V}_{\sigma'})^a_* \subseteq \nu(\mathcal{V}_{\sigma})^a_* = \emptyset.$

Hence we obtain an ascending sequence $a^{(i)}$. Furthermore, whenever we perform a switch, we know that $a^{(i)}$ is not a fixpoint of $\mathcal{V}_{\sigma^{(i+1)}}$ (otherwise we could not have performed a switch) and hence $a^{(i+1)}$ is strictly larger than $a^{(i)}$ for at least one input. Since there are only finitely many strategies we will eventually stop switching and reach the least fixpoint.

Example 7.7. The previous Example 7.1 is well suited to explain our two algorithms.

Starting with strategy iteration from above, we may guess $\tau^{(0)}(\min) = 1$. In this case, Max would choose av as successor and we would reach a fixpoint, where each node except for ε is associated with a payoff of 1. Next, our algorithm would detect the vicious cycle formed by min, av and max. We can reduce the values in this vicious cycle and reach the correct payoff values for each node.

For strategy iteration from below assume that $\sigma^{(0)}(\max) = av$. Given this strategy of Max, Min can force the play to stay in a cycle formed by min, av and max. Thus, the payoff achieved by the Max strategy $\sigma^{(0)}$ and an optimal play by Min would be 0 for each of these nodes. In the next iteration Max switches and chooses ε as successor, i.e. $\sigma^{(1)}(\max) = \varepsilon$, which results in the correct values.

7.3. Runtime results. We implemented strategy iteration from above and from below – in the following abbreviated by SIA and SIB – and classical Kleene iteration (KI) in MATLAB. In Kleene iteration we terminate with a tolerance of 10^{-14} , i.e., we stop if the change from one iteration to the next is below this value.

In order to test the algorithms we created random stochastic games with n nodes, where each Max, Min respectively average node has a maximal number of m successors. For each node we choose randomly one of the four types of nodes. Sink nodes are given a random weight uniformly in [0, 1]. Max and Min nodes are randomly assigned to successors and for an average nodes we assign a random number to each of its successors, followed by normalisation to obtain a probability distribution.

We performed 1000 runs with different randomly created systems for each value of n and $m = \frac{n}{2}$. Table 2 shows the runtimes in seconds and the number of iterations. Also, for SIB, we display the number of nodes with a payoff of 0 (for an optimal play of Min) and the number of times SIA got stuck at any other fixpoint which is not $\mu \mathcal{V}$ (all numbers – runtime, iterations, etc. – are summed up over all 1000 runs).

Note that SIB always performs slightly better than SIA. Moreover KI neatly beats both of them. Here we need to remember that KI only converges to the solution and it is known that the rate of convergence can be exponentially slow [Con90].

Note that the linear optimisation problems are quite costly to solve, especially for large systems. Thus additional iterations are substantially more costly compared to KI. Observe also that SIA has to perform more iterations than SIB, which explains the slightly higher runtime.

	runti	me (sec	onds)	numbe	r of iter	ations	number nodes	number of
n	KI	SIA	SIB	KI	SIA	SIB	payoff 0	other fp
10	0.59	20.28	18.49	47302	2259	2152	2439	508
20	1.05	31.71	25.96	30275	3620	3018	4714	743
30	2.03	35.98	29.77	27361	3881	3275	7268	771
40	3.77	38.84	32.67	26999	3850	3296	9806	756
50	5.31	38.09	31.85	26604	3799	3215	12573	734
60	7.63	40.33	34.37	26467	3737	3218	15151	727
70	10.77	45.00	37.50	26569	3751	3154	17473	751
80	15.38	54.89	46.72	26179	3713	3105	20031	752
90	16.07	52.21	43.52	26401	3695	3083	22390	777
100	19.46	60.29	50.88	26464	3654	3062	25163	751

TABLE 2. Experimental results for KI, SIA, SIB on randomly generated SSGs with weight of sink nodes in [0,1].

	runtime (sec		onds)	number of iterations		number nodes	number of	
n	KI	SIA	SIB	KI	SIB	SIA	payoff 0	other fp
10	0.36	14.51	14.58	42547	1703	1702	5484	219
20	1.00	19.85	19.98	29515	2385	2478	8168	137
30	1.97	20.45	20.77	27643	2367	2469	11502	33
40	3.30	20.13	20.94	26761	2306	2383	14989	12
50	4.96	20.24	20.94	26562	2253	2306	18821	2
60	6.87	20.57	21.19	26560	2176	2227	22573	0
70	9.14	21.95	22.35	26146	2142	2186	26260	0
80	11.73	24.69	24.94	26235	2084	2131	30136	0
90	14.73	28.90	28.71	26330	2066	2091	33930	0
100	18.22	34.75	34.84	26227	2051	2068	37496	0

TABLE 3. Experimental results for KI, SIA, SIB on randomly generated SSGs with weight of sink nodes in $\{0,1\}$.

The number of nodes with a payoff of 0 seems to grow linearly with the number of nodes in the system. The number of times SIA gets stuck at a fixpoint different from $\mu \mathcal{V}$ however seems to be independent of the system size and comparatively small.

We performed a second comparison (see Table 3), where we assigned to sink nodes a value in 0, 1, which is often done for simple stochastic games.

Here, SIA performs very similar to SIB. The SIA approach seems to suffer, since Max can easily find himself in a situation where he can never reach a 1-sink, since only half of the sink nodes are of this kind. Additionally for these systems a significantly larger number of nodes have a payoff of 0 and SIA is less likely to get stuck at a fixpoint different from $\mu \mathcal{V}$. These factors seem to be correlated since it is now "harder" for Min to choose a bad successor (with a value greater than 0).

8. CONCLUSION

It is well-known that several computations in the context of system verification can be performed by various forms of fixpoint iteration and it is worthwhile to study such methods at a high level of abstraction, typically in the setting of complete lattices and monotone functions. Going beyond the classical results by Tarski [Tar55], combination of fixpoint iteration with approximations [CC00, BKP20] and with up-to techniques [Pou07] has proven to be successful. Here we treated a more specific setting, where the carrier set consists of functions from a finite set into an MV-chain and the fixpoint functions are non-expansive (and hence monotone), and introduced a novel technique to obtain upper bounds for greatest and lower bounds for least fixpoints, also providing associated algorithms. Such techniques are applicable to a wide range of examples and so far they have been studied only in quite specific scenarios, such as in [BBL⁺19, Fu12, KKKW18].

In the future we plan to lift some of the restrictions of our approach. First, an extension to an infinite domain Y would of course be desirable, but since several of our results currently depend on finiteness, such a generalisation does not seem to be easy. The restriction to total orders, instead, seems easier to lift: in particular, if the partially ordered MV-algebra $\overline{\mathbb{M}}$ is of the form \mathbb{M}^I where I is a finite index set and M an MV-chain. (E.g., finite Boolean algebras are of this type.) In this case, our function space is $\overline{\mathbb{M}}^Y = (\mathbb{M}^I)^Y \cong \mathbb{M}^{Y \times I}$ and we have reduced to the setting presented in this paper. This will allow us to handle featured transition systems [CCP⁺12] for compactly specifying software product lines in a single transition system. There, transitions are equipped with boolean formulas that specify for which products (or features) a transition can be taken.

There are several other application examples that did not fit into this paper, but that can also be handled by our approach, for instance coalgebraic behavioural metrics [BBKK18]. While here we introduced strategy iteration techniques for simple stochastic games, we also want to check whether we can provide an improvement to value iteration techniques, combining our approach with [KKKW18]. In this context it is also interesting to consider more generic approaches to strategy iteration, which is done for simple stochastic games in [AdMS21] and in a lattice-theoretical setting in [BEKP22]. The latter also considers energy games [BCD⁺11] as a new instance of our framework.

We also plan to study whether some examples can be handled with other types of Galois connections: here we used an additive variant, but looking at multiplicative variants (multiplication by a constant factor) might also be fruitful.

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Appendix A. Proofs and Additional Material for §2 (Lattices and MV-algebras)

Lemma 2.4 (properties of MV-algebras). Let $\mathbb{M} = (M, \oplus, 0, \overline{(\cdot)})$ be an MV-algebra. For all $x, y, z \in M$

- (1) $x \oplus \overline{x} = 1$
- (2) $x \sqsubseteq y$ iff $\overline{x} \oplus y = 1$ iff $x \otimes \overline{y} = 0$ iff $y = x \oplus (y \ominus x)$
- (3) $x \sqsubseteq y$ iff $\overline{y} \sqsubseteq \overline{x}$
- (4) \oplus , \otimes are monotone in both arguments, \ominus monotone in the first and antitone in the second argument.
- (5) if $x \sqsubset y$ then $0 \sqsubset y \ominus x$;
- (6) $(x \oplus y) \ominus y \sqsubseteq x$
- (7) $z \sqsubseteq x \oplus y$ if and only if $z \ominus x \sqsubseteq y$.
- (8) if $x \sqsubset y$ and $z \sqsubseteq \overline{y}$ then $x \oplus z \sqsubset y \oplus z$;
- (9) $y \sqsubseteq \overline{x}$ if and only if $(x \oplus y) \ominus y = x$;
- (10) $x \ominus (x \ominus y) \sqsubseteq y$ and if $y \sqsubseteq x$ then $x \ominus (x \ominus y) = y$.
- (11) Whenever M is an MV-chain, $x \sqsubset y$ and $0 \sqsubset z$ imply $(x \oplus z) \ominus y \sqsubset z$

Proof. The proof of properties (1), (2), (3), (4) can be found directly in [Mun07]. For the rest:

- (5) Immediate consequence of (2). In fact, given $x \sqsubset y$, if we had $y \ominus x = 0$ then by (2), $y = x \oplus (y \ominus x) = x \oplus 0 = x$, contradicting the hypothesis.
- (6) Observe that $(x \oplus y) \ominus y = \overline{(x \oplus y)} \oplus y = \overline{(\overline{x} \ominus y)} \oplus \overline{y} = \overline{(y \ominus \overline{x})} \oplus \overline{x} \sqsubseteq \overline{\overline{x}} = x$, where the last inequality is motivated by the fact that $\overline{x} \sqsubseteq (y \ominus \overline{x}) \oplus \overline{x}$ and point (3).
- (7) The direction from left to right is an immediate consequence of (6). In fact, if $z \sqsubseteq x \oplus y$ then $z \ominus x \sqsubseteq (x \oplus y) \ominus x \sqsubseteq y$.
 - The other direction goes as follows: if $z \ominus x \sqsubseteq y$, then by monotonicity (4) $(z \ominus x) \oplus x \sqsubseteq y \oplus x = x \oplus y$. The left hand side can be rewritten to $(x \ominus z) \oplus z \sqsupseteq z$.
- (8) Assume that $x \sqsubset y$ and $z \sqsubseteq \overline{y}$. We know, by property (4) that $x \oplus z \sqsubseteq y \oplus z$. Assume by contradiction that $x \oplus z = y \oplus z$. Then we have
 - $\overline{x} \sqsubseteq \qquad [by \text{ properties } (3) \text{ and } (6)]$ $\subseteq \overline{(x \oplus z) \ominus z} \qquad [since \ x \oplus z = y \oplus z]$ $\subseteq \overline{(y \oplus z) \ominus z} \qquad [definition \ of \ominus]$ $= (\overline{y} \ominus z) \oplus z \qquad [since \ z \sqsubseteq \overline{y} \text{ and property } (2)]$ $= \overline{y}$

And with point (3) this is a contradiction.

(9) Assume $y \sqsubseteq \overline{x}$. We know $(x \oplus y) \ominus y \sqsubseteq x$. If it were $(x \oplus y) \ominus y \sqsubset x$, then $((x \oplus y) \ominus y) \oplus y \sqsubset x \oplus y$, with (8). Since the left-hand side is equal to $(y \ominus (x \oplus y)) \oplus (x \oplus y) \sqsupseteq x \oplus y$, this is a contradiction.

For the other direction assume that $(x \oplus y) \ominus y = x$. Hence we have $x = (x \oplus y) \ominus y = \overline{\overline{(x \oplus y)} \oplus y}$. By complementing on both sides we obtain $\overline{x} = \overline{(x \oplus y)} \oplus y$ which implies that $y \sqsubseteq \overline{x}$.

(10) Observe that, by (7), we have $\overline{y} \sqsubseteq \overline{x} \oplus (\overline{y} \ominus \overline{x}) = \overline{x} \oplus (x \ominus y) = x \ominus (x \ominus y)$. Therefore, by (3), $x \ominus (x \ominus y) \sqsubseteq y$, as desired.

For the second part, assume $y \sqsubseteq x$ and thus, by (3), $\overline{x} \sqsubseteq \overline{y}$. Using (2), we obtain $\overline{y} = \overline{x} \oplus (\overline{y} \ominus \overline{x}) = \overline{x} \oplus \overline{y} \oplus \overline{\overline{x}} = \overline{x} \oplus (x \ominus y)$. Hence $y = \overline{\overline{x} \oplus (x \ominus y)} = x \oplus (x \ominus y)$.

(11) We first observe that, given $u, v \in \mathbb{M}$, $u \sqsubseteq v \oplus (u \ominus v)$. This is a direct consequence of axiom (3) of MV-algebras and the definition of natural order.

Second, in an MV-chain if $u, v \supseteq 0$, then $u \ominus v \sqsubset u$. In fact, if $u \sqsubseteq v$ and thus $u \ominus v = 0 \sqsubset u$. If instead, $v \sqsubset u$ we have $0 \sqsubset v$ and $u \ominus v \sqsubseteq 1 \ominus v = \overline{v}$, hence by (8) it holds that $0 \oplus (u \ominus v) \sqsubset v \oplus (u \ominus v)$. Recalling that $v \sqsubset u$ and thus by (2), $(u \ominus v) \oplus v = u$, we conclude $u \ominus v \sqsubset u$.

Now

$(x\oplus z)\ominus y$	
$\sqsubseteq (x \oplus (y \ominus x) \oplus (z \ominus (y \ominus x))) \ominus y$	[by first obs. above]
$=(y\oplus(z\ominus(y\ominus x)))\ominus y$	[since $x \sqsubseteq y$, by (2)]
$\sqsubseteq z \ominus (y \ominus x)$	[by (6)]
$\sqsubset z$	[by second obs. above, since $z \sqsupseteq 0$
	and $y \ominus x \sqsupset 0$ by (5)]

Appendix B. Proofs and Additional Material for §3 (Non-expansive functions and their approximations)

Lemma 3.2 (properties of the norm). Let \mathbb{M} be an *MV*-chain and let *Y* be a finite set. Then $\|\cdot\| : \mathbb{M}^Y \to \mathbb{M}$ satisfies, for all $a, b \in \mathbb{M}^Y$, $\delta \in \mathbb{M}$

- (1) $||a \oplus b|| \subseteq ||a|| \oplus ||b||$,
- (2) $\|\delta \otimes a\| = \delta \otimes \|a\|$ and
- (3) ||a|| = 0 implies that a is the constant 0.

Proof. Concerning (1), let $||a \oplus b||$ be realised on some element $y \in Y$, i.e., $||a \oplus b|| = a(y) \oplus b(y)$. Since $a(y) \sqsubseteq ||a||$ and $b(y) \sqsubseteq ||b||$, by monotonicity of \oplus we deduce that $||a \oplus b|| \sqsubseteq ||a|| \oplus ||b||$.

Concerning (2), note that

$$\begin{split} \|\delta \otimes a\| &= \max\{\overline{\overline{\delta} \oplus \overline{a(y)}} \mid y \in Y\} \\ &= \overline{\min\{\overline{\delta} \oplus \overline{a(y)} \mid y \in Y\}} \\ &= \overline{\overline{\delta} \oplus \min\{\overline{a(y)} \mid y \in Y\}} \\ &= \overline{\overline{\delta} \oplus \max\{a(y) \mid y \in Y\}} \\ &= \overline{\overline{\delta} \oplus \overline{\|a\|}} \\ &= \delta \otimes \|a\| \end{split}$$

Finally, point (3) is straightforward, since 0 is the bottom of \mathbb{M} .

Lemma B.1 (non-expansiveness implies monotonicity). Let \mathbb{M} is an *MV*-chain and let Y, Z be finite sets. Every non-expansive function $f : \mathbb{M}^Y \to \mathbb{M}^Z$ is monotone.

Proof. Let $a, b \in \mathbb{M}^Y$ be such that $a \sqsubseteq b$. Therefore, by Lemma 2.4(2), $a(y) \ominus b(y) = 0$ for all $y \in Y$, hence $a \ominus b = 0$. Thus $||f(a) \ominus f(b)|| \sqsubseteq ||a \ominus b|| = 0$. In turn this implies that for all $z \in Z$, $f(a)(z) \ominus f(b)(z) = 0$. Hence Lemma 2.4(2), allows us to conclude $f(a)(z) \sqsubseteq f(b)(z)$ for all $z \in Z$, i.e., $f(a) \sqsubseteq f(b)$, as desired.

The next lemma provides a useful equivalent characterisation of non-expansiveness.

Lemma B.2 (characterisation of non-expansiveness). Let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a monotone function, where \mathbb{M} is an *MV*-chain and *Y*, *Z* are finite sets. Then *f* is non-expansive iff for all $a \in \mathbb{M}^Y$, $\theta \in \mathbb{M}$ and $z \in Z$ it holds $f(a \oplus \theta)(z) \ominus f(a)(z) \sqsubseteq \theta$.

Proof. Let f be non-expansive and let $a \in \mathbb{M}^Y$ and $\theta \in \mathbb{M}$. We have that for all $z \in Z$

$f(a\oplus heta)(z)\ominus f(a)(z)\sqsubseteq$	
$\sqsubseteq \left\ f(a\oplus\theta)\ominus f(a)\right\ $	[by definition of norm]
$\sqsubseteq \ (a \oplus \theta) \ominus a\ $	[by hypothesis]
$\sqsubseteq \ \lambda y.\theta\ $	[by Lemma 2.4(6) and monotonicity of norm]
= heta	[by definition of norm]

Conversely, assume that for all $a \in \mathbb{M}^Y$, $\theta \in \mathbb{M}$ and $z \in Z$ it holds $f(a \oplus \theta)(z) \oplus f(a)(z) \sqsubseteq \theta$. For $a, b \in \mathbb{M}^Y$, first observe that for all $y \in Y$ it holds $b(y) \oplus a(y) \sqsubseteq \|b \oplus a\|$, hence, if we let $\theta = \|b \oplus a\|$, we have $b \sqsubseteq a \oplus \theta$ and thus, by monotonicity, $f(b) \oplus f(a) \sqsubseteq f(a \oplus \theta) \oplus f(a)$. Thus

$$\begin{split} \|f(b) \ominus f(a)\| &\sqsubseteq \\ &\sqsubseteq \|f(a+\theta) \ominus f(a)\| = \\ & \text{[by the observation above and monotonicity of norm]} \\ &= \max\{f(a+\theta)(z) \ominus f(a)(z)|z \in Z\} \\ & \text{[by definition of norm]} \\ &\sqsubseteq \theta \\ & \text{[by hypothesis]} \\ &= \|b \ominus a\| \\ \end{split}$$

Lemma B.3 (composing non-expansive functions). Let \mathbb{M} be an MV-chain and let Y, W, Z be finite sets. If $g : \mathbb{M}^Y \to \mathbb{M}^W$ and $h : \mathbb{M}^W \to \mathbb{M}^Z$ are non-expansive then $h \circ g : \mathbb{M}^Y \to \mathbb{M}^Z$ is non-expansive.

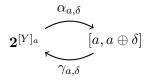
Proof. Straightforward. We have for any $a, b \in \mathbb{M}^Y$ that

$\ h(g(b))\ominus h(g(a))\ \sqsubseteq$		
$\sqsubseteq \ g(b)\ominus g(a)\ $	[by non-expansiveness of h]	
$\sqsubseteq \ b\ominus a\ $	[by non-expansiveness of g]	

Lemma B.4 (well-definedness). The functions $\alpha_{a,\delta}$, $\gamma_{a,\delta}$ from Def. 3.4 are well-defined and monotone.

Proof. The involved functions $\alpha_{a,\delta}$ and $\gamma_{a,\delta}$ are well-defined. In fact, for $Y' \subseteq [Y]_a$, clearly $\alpha_{a,\delta} = a \oplus \delta_{Y'} \in [a, a \oplus \delta]$. Moreover, for $b \in [a, a \oplus \delta]$ we have $\gamma_{a,\delta}(b) \subseteq [Y]_a$. In fact, if $y \notin [Y]_a$ then a(y) = 1, hence b(y) = 1 and thus $b(y) \ominus a(y) = 0 \not\supseteq \delta$, and thus $y \notin \gamma_{a,\delta}(b)$. Moreover, they are clearly monotone.

Lemma 3.5 (Galois connection). Let \mathbb{M} be an MV-algebra and Y be a finite set. For $0 \neq \delta \sqsubseteq \delta_a$, the pair $\langle \alpha_{a,\delta}, \gamma_{a,\delta} \rangle : \mathbf{2}^{[Y]_a} \to [a, a \oplus \delta]$ is a Galois connection.



Proof. For all $Y' \in \mathbf{2}^{[Y]_a}$ it holds

$$\gamma_{a,\delta}(\alpha_{a,\delta}(Y')) = \gamma_{a,\delta}(a \oplus \delta_{Y'}) = Y'.$$

In fact, for all $y \in Y'$, $(a \oplus \delta_{Y'})(y) = a(y) \oplus \delta$. Moreover, and by the choice of δ and definition of $[Y]_a$, we have $\delta \sqsubseteq \delta_a \sqsubseteq \overline{a(y)}$, by Lemma 2.4(9), we have $(a \oplus \delta_{Y'})(y) \ominus a(y) = \delta$ hence $y \in \gamma_{a,\delta}(\alpha_{a,\delta}(Y'))$. Conversely, if $y \notin Y'$, then $(a \oplus \delta_{Y'})(y) = a(y)$, and thus $(a \oplus \delta_{Y'})(y) \ominus a(y) = 0 \not\supseteq \delta$.

Moreover, for all $b \in [a, a \oplus \delta]$ we have

$$\alpha_{a,\delta}(\gamma_{a,\delta}(b)) = a \oplus \delta_{\gamma_{a,\delta}(b)} \sqsubseteq b$$

In fact, for all $y \in Y$, if $y \in \gamma_{a,\delta}(b)$, i.e., $\delta \sqsubseteq b(y) \ominus a(y)$ then $(a \oplus \delta_{\gamma_{a,\delta}(b)})(y) = a(y) \oplus \delta \sqsubseteq a(y) \oplus (b(y) \ominus a(y)) = b(y)$, by Lemma 2.4(2). If instead, $y \notin \gamma_{a,\delta}(b)$, then $(a \oplus \delta_{\gamma_{a,\delta}(b)}(b))(y) = a(y) \sqsubseteq b(y)$.

Lemma B.5 (restricting non-expansive functions to intervals). Let \mathbb{M} be an MV-chain, let Y, Z be finite sets $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function. Then f restricts to a function $f_{a,\delta} : [a, a \oplus \delta] \to [f(a), f(a) \oplus \delta]$, defined by $f_{a,\delta}(b) = f(b)$.

Proof. Given $b \in [a, a \oplus \delta]$, by monotonicity of f we have that $f(a) \sqsubseteq f(b)$. Moreover, $f(b) \sqsubseteq f(a \oplus \delta) \sqsubseteq f(a) \oplus \delta$, where the last passage is motivated by Lemma B.2.

In the following we will simply write f instead of $f_{a,\delta}$.

Lemma 3.9 (antitonicity). Let \mathbb{M} be an MV-chain, let Y, Z be finite sets, let $f: \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function and let $a \in \mathbb{M}^Y$. For $\theta, \delta \in \mathbb{M}$, if $\theta \sqsubseteq \delta$ then $f_{a\,\delta}^{\#} \subseteq f_{a\,\theta}^{\#}$.

Proof. Let $Y' \subseteq [Y]_a$ and let us prove that $f_{a,\delta}^{\#}(Y') \subseteq f_{a,\theta}^{\#}(Y')$. Take $z \in f_{a,\delta}^{\#}(Y')$. This means that $\delta \sqsubseteq f(a \oplus \delta_{Y'})(z) \ominus f(a)(z)$.

We have

$$\begin{split} \delta &\sqsubseteq f(a \oplus \delta_{Y'})(z) \ominus f(a)(z) \\ &[\text{by hypothesis}] \\ &= f(a \oplus \theta_{Y'} \oplus (\delta \ominus \theta)_{Y'})(z) \ominus f(a)(z) \\ &= f(a \oplus \theta_{Y'} \oplus (\delta \ominus \theta)_{Y'})(z) \ominus f(a \oplus \theta_{Y'})(z) \oplus f(a \oplus \theta_{Y'})(z) \ominus f(a)(z) \\ &\sqsubseteq \|f(a \oplus \theta_{Y'} \oplus (\delta \ominus \theta)_{Y'}) \ominus f(a \oplus \theta_{Y'})\| \oplus f(a \oplus \theta_{Y'})(z) \ominus f(a)(z) \\ &[\text{by definition of norm and monotonicity of } \oplus] \\ &\sqsubseteq \|a \oplus \theta_{Y'} \oplus (\delta \ominus \theta)_{Y'} \ominus (a \oplus \theta_{Y'})\| \oplus f(a \oplus \theta_{Y'})(z) \ominus f(a)(z) \\ &[\text{by non-expansiveness of } f \text{ and monotonicity of } \oplus] \\ &\sqsubseteq \|(\delta \ominus \theta)_{Y'}\| \oplus f(a \oplus \theta_{Y'})(z) \ominus f(a)(z) \end{split}$$

If we subtract $\delta \ominus \theta$ on both sides, we get $\delta \ominus (\delta \ominus \theta) \sqsubseteq f(a \oplus \theta_{Y'})(z) \ominus f(a)(z)$, and, as above, since, by Lemma 2.4(10), $\delta \ominus (\delta \ominus \theta) = \theta$ we conclude

$$\theta \sqsubseteq f(a \oplus \theta_{Y'})(z) \ominus f(a)(z)$$

which means $z \in f_{a,\theta}^{\#}(Y')$.

Lemma 3.10 (largest increase for a point). Let \mathbb{M} be a complete MV-chain, let Y, Z be finite sets, let $f: \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function and fix $a \in \mathbb{M}^Y$. For all $z \in [Z]_{f(a)}$ and $Y' \subseteq [Y]_a$ the set $\{\theta \in \mathbb{M} \mid z \in f_{a,\theta}^{\#}(Y')\}$ has a maximum, that we denote by $\iota_a^f(Y', z)$.

Proof. Let $V = \{ \theta \in \mathbb{M} \mid z \in f_{a,\theta}^{\#}(Y') \}$. Expanding the definition we have that

 $V = \{ \theta \in \mathbb{M} \mid \theta \sqsubseteq f(a \oplus \theta_{Y'})(z) \ominus f(a)(z) \}.$

If we let $\eta = \sup V$, for all $\theta \in V$, since $\theta_{Y'} \sqsubseteq \eta_{Y'}$, clearly, by monotonicity

$$\theta \sqsubseteq f(a \oplus \eta_{Y'})(z) \ominus f(a)(z)$$

and therefore, by definition of supremum, $\eta \sqsubseteq f(a \oplus \eta_{Y'})(z) \ominus f(a)(z)$, i.e., $\eta \in V$ is a maximum, as desired.

Lemma 3.11 (a-approximation for a function). Let \mathbb{M} be a complete MV-chain, let Y, Z be finite sets and let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function. Let

$$\iota_a^f = \min\{\iota_a^f(Y', z) \mid Y' \subseteq [Y]_a \land z \in [Z]_{f(a)} \land \iota_a^f(Y', z) \neq 0\} \cup \{\delta_a\}.$$

Then for all $0 \neq \delta \sqsubseteq \iota_a^f$ it holds that $f_{a,\delta}^{\#} = f_{a,\iota_a^f}^{\#}$.

The function $f_{a,t_{a}}^{\#}$ is called the a-approximation of f and it is denoted by $f_{a}^{\#}$.

Proof. Since $\delta \sqsubseteq \iota_a^f$, by Lemma 3.9 we have $f_{a,\delta}^{\#} \supseteq f_{a,\iota_a^f}^{\#}$. For the other inclusion let $Y' \subseteq [Y]_a$. We have

$$f_{a,\delta}^{\#}(Y') = \{ z \in [Z]_{f(a)} \mid f(a \oplus \delta_{Y'})(z) \ominus f(a)(z) \sqsupseteq \delta \}$$

by definition. Assume that there exists $z \in f_{a,\delta}^{\#}(Y')$ where $f(a \oplus (\iota_a^f)_{Y'})(z) \ominus f(a)(z) \not\supseteq \iota_a^f$. But this is a contradiction, since ι_a^f is the minimum of all such non-zero values.

Lemma 3.12 (standard form). Let \mathbb{M} be an MV-chain and let Y be a finite set. Then for any $b \in \mathbb{M}^Y$ there are $Y_1, \ldots, Y_n \subseteq Y$ with $Y_{i+1} \subseteq Y_i$ for $i \in \{1, \ldots, n-1\}$ and $\delta^i \in \mathbb{M}$, $0 \neq \delta^i \sqsubseteq \overline{\bigoplus_{j=1}^{i-1} \delta^j}$ for $i \in \{1, \ldots, n\}$ such that

$$b = \bigoplus_{i=1}^n \delta^i_{Y_i}$$
 and $||b|| = \bigoplus_{i=1}^n \delta^i$.

where we assume that an empty sum evaluates to 0.

Proof. Given $b \in \mathbb{M}^Y$, consider $V = \{b(y) \mid y \in Y\}$. If V is empty, then Y is empty and thus $b = 1_Y$, i.e., we can take n = 1, $\delta^1 = 1$ and $Y_1 = Y$. Otherwise, if $Y \neq \emptyset$, then V is a finite non-empty set. Let $V = \{v_1, \ldots, v_n\}$, with $v_i \sqsubseteq v_{i+1}$ for $i \in \{1, \ldots, n-1\}$. For $i \in \{1, \ldots, n\}$ define $Y_i = \{y \in Y \mid v_i \sqsubseteq b(y)\}$. Clearly, $Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_n$. Moreover let $\delta^1 = v_1$ and $\delta^{i+1} = v_{i+1} \ominus v_i$ for $i \in \{1, \ldots, n-1\}$.

Observe that for each *i*, we have $v_i = \bigoplus_{j=1}^i \delta^i$, as it can easily shown by induction. Hence $\delta^{i+1} = v_{i+1} \ominus v_i = v_{i+1} \ominus \bigoplus_{j=1}^i \delta^i \subseteq 1 \ominus \bigoplus_{j=1}^i \delta^i = \overline{\bigoplus_{j=1}^i \delta^i}$.

We now show that $b = \bigoplus_{i=1}^{n} \delta_{Y_i}^{i}$ by induction on n.

- If n = 1 then $V = \{v_1\}$ and thus b is a constant function $b(y) = v_1$ for all $y \in Y$. Hence $Y_1 = Y$ and thus $b = \delta_Y^1 = \delta_{Y_1}^1$, as desired.
- If n > 1, let $b' \in \mathbb{M}^Y$ defined by b'(y) = b(y) for $y \in Y \setminus Y_n$ and $b'(y) = v_{n-1}$ for $y \in Y_n$. Note that $\{b'(y) \mid y \in Y\} = \{v_1, \ldots, v_{n-1}\}$. Hence, by inductive hypothesis, $b' = \bigoplus_{i=1}^{n-1} \delta_{Y_i}^i$. Moreover, $b'(y) = b \oplus \delta_{Y_n}^n$, and thus we conclude.

Finally observe that the statement requires $\delta^i \neq 0$ for all *i*. We can enjoy this property by just omitting the first summand when $v_1 = 0$.

Lemma 3.13. Let \mathbb{M} be an MV-chain, let Y, Z be finite sets and let $f : \mathbb{M}^Y \to \mathbb{M}^Z$ be a non-expansive function. Let $a \in \mathbb{M}^Y$. For $b \in [a, a \oplus \delta]$, let $b \ominus a = \bigoplus_{i=1}^n \delta_{Y_i}^i$ be a standard form for $b \ominus a$. If $\gamma_{f(a),\delta}(f(b)) \neq \emptyset$ then $Y_n = \gamma_{a,\delta}(b)$ and $\gamma_{f(a),\delta}(f(b)) \subseteq f_{a,\delta^n}^{\#}(Y_n)$.

Proof. By hypothesis $\gamma_{f(a),\delta}(f(b)) \neq \emptyset$. Let $z \in \gamma_{f(a),\delta}(f(b))$. This means that $\delta \sqsubseteq f(b)(z) \ominus f(a)(z)$. First observe that

$$\begin{split} \delta &\sqsubseteq f(b)(z) \ominus f(a)(z) & \text{[by hypothesis]} \\ &\sqsubseteq \|f(b) \ominus f(a)\| & \text{[by definition of norm]} \\ &\sqsubseteq \|b \ominus a\| & \text{[by non-expansiveness of } f] \\ &\sqsubseteq \delta & \text{[since } b \in [a, a \oplus \delta]] \end{split}$$

Hence

$$||f(b) \ominus f(a)|| = \delta = ||b \ominus a|| = \bigoplus_{i=1}^n \delta^i.$$

Also observe that, since $\delta^n \neq 0$, we have $(b \ominus a)(z) = \delta$ iff $z \in Y_n$. In fact, if $z \in Y_n$ then $z \in Y_i$ for all $i \in \{1, \ldots, n\}$ and thus $(b \ominus a)(z) = \bigoplus_{i=1}^n \delta_{Y_i}^i(z) = \bigoplus_{i=1}^n \delta^i = \delta$. Conversely, if $z \notin Y_n$, then $(b \ominus a)(z) \sqsubseteq \bigoplus_{i=1}^{n-1} \delta^i \sqsubset \delta$. In fact, $0 \sqsubset \delta^n$ and $\bigoplus_{i=1}^{n-1} \delta^i \sqsubseteq \overline{\delta^n}$. Thus by Lemma 2.4(8), $\bigoplus_{i=1}^{n-1} \delta^i \sqsubset \delta^n \oplus \bigoplus_{i=1}^{n-1} \delta^i = \bigoplus_{i=1}^n \delta^i = \delta$. Hence $Y_n = \gamma_{a,\delta}(b)$.

Let us now show that $\gamma_{f(a),\delta}(f(b)) \subseteq f_{a,\delta^n}^{\#}(Y_n)$. Given $z \in \gamma_{f(a),\delta}(f(b))$, we show that $z \in f_{a,\delta^n}^{\#}(Y_n)$. Observe that

$$\begin{split} \delta &\sqsubseteq f(b)(z) \ominus f(a)(z) = \\ & [\text{by hypothesis}] \\ = & f(a \oplus (b \ominus a))(z) \ominus f(a)(z) = \\ & [\text{by Lemma 2.4(2), since } a \sqsubseteq b] \\ = & f(a \oplus \bigoplus_{i=1}^{n} \delta_{Y_{i}}^{i})(z) \ominus f(a)(z) = \\ & [\text{by construction}] \\ = & f(a \oplus \bigoplus_{i=1}^{n} \delta_{Y_{i}}^{i}))(z) \ominus f(a \oplus \delta_{Y_{n}}^{n})(z) \oplus f(a \oplus \delta_{Y_{n}}^{n})(z) \ominus f(a)(z) \\ & [\text{by Lemma 2.4(2), since } f(a \oplus \delta_{Y_{n}}^{n})(z) \sqsubseteq f(a \oplus \bigoplus_{i=1}^{n} \delta_{Y_{i}}^{i})(z)] \end{split}$$

$$\begin{split} & \sqsubseteq \|f(a \oplus \bigoplus_{i=1}^{n} \delta_{Y_{i}}^{i}) \ominus f(a \oplus \delta_{Y_{n}}^{n})\| \oplus f(a \oplus \delta_{Y_{n}}^{n})(z) \ominus f(a)(z) \\ & \text{[by definition of norm and monotonicity of } \oplus] \\ & \sqsubseteq \|a \oplus \bigoplus_{i=1}^{n} \delta_{Y_{i}}^{i} \ominus (a \oplus \delta_{Y_{n}}^{n})\| \oplus f(a \oplus \delta_{Y_{n}}^{n})(z) \ominus f(a)(z) \\ & \text{[by non-expansiveness of } f \text{ and monotonicity of } \oplus] \\ & = \|a \oplus \delta_{Y_{n}}^{n} \oplus \bigoplus_{i=1}^{n-1} \delta_{Y_{i}}^{i} \ominus (a \oplus \delta_{Y_{n}}^{n})\| \oplus f(a \oplus \delta_{Y_{n}}^{n})(z) \ominus f(a)(z) \\ & \text{[by algebraic manipulation]} \\ & \sqsubseteq \|\bigoplus_{i=1}^{n-1} \delta_{Y_{i}}^{i}\| \oplus f(a \oplus \delta_{Y_{n}}^{n})(z) \ominus f(a)(z) \\ & \text{[by Lemma 2.4(6) and monotonicity of norm]} \\ & \sqsubseteq \bigoplus_{i=1}^{n-1} \delta^{i} \oplus f(a \oplus \delta_{Y_{n}}^{n})(z) \ominus f(a)(z) \\ & \text{[by Lemma 3.2(1) and the fact that } \|\delta_{Y_{i}}^{i}\| = \delta^{i}] \\ & = (\delta \ominus \delta^{n}) \oplus f(a \oplus \delta_{Y_{n}}^{n})(z) \ominus f(a)(z) \\ & \text{[by construction, since } \delta^{n} = \overline{\bigoplus_{i=1}^{n-1} \delta^{i}}] \end{split}$$

If we subtract $\delta \ominus \delta^n$ on both sides, we get $\delta \ominus (\delta \ominus \delta^n) \sqsubseteq f(a \oplus \delta_{Y_n}^n)(z) \ominus f(a)(z)$, i.e., since, by Lemma 2.4(10), $\delta \ominus (\delta \ominus \delta^n) = \delta^n$ we conclude

$$\delta^n \sqsubseteq f(a \oplus \delta^n_{Y_n})(z) \ominus f(a)(z).$$

Hence $z \in \gamma_{f(a),\delta^n}(f(\alpha_{a,\delta^n}(Y_n)) = f_{a,\delta^n}^{\#}(Y_n)$, which is the desired result.

Appendix C. Proofs and Additional Material for §4 (Proof rules)

Lemma 4.1. Let \mathbb{M} be a complete MV-chain, Y a finite set and $f : \mathbb{M}^Y \to \mathbb{M}^Y$ be a nonexpansive function. Let $a \in \mathbb{M}^Y$ be a pre-fixpoint of f (i.e., $f(a) \sqsubseteq a$), let $f_a^{\#} : \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[Y]_{f(a)}}$ be the a-approximation of f. Assume $\nu f \not\sqsubseteq a$ and let $Y' = \{y \in [Y]_a \mid \nu f(y) \ominus a(y) = \|\nu f \ominus a\|\}$. Then for all $y \in Y'$ it holds a(y) = f(a)(y) and $Y' \subseteq f_a^{\#}(Y')$.

Proof. Let $\delta = \|\nu f \ominus a\|$. Assume $\nu f \not\sqsubseteq a$, i.e., there exists $y \in Y$ such that $\nu f(y) \not\sqsubseteq a(y)$. Since the order is total, this means that $a(y) \sqsubset \nu f(y)$. Hence, by Lemma 2.4(5), $\nu f(y) \ominus a(y) \sqsupseteq 0$. Then $\delta = \|\nu f \ominus a\| \sqsupseteq 0$. Moreover, for all $y \in Y'$, $\overline{a(y)} = 1 \ominus a(y) \sqsupseteq \nu f(y) \ominus a(y) = \delta$.

First, observe that

$$\nu f \sqsubseteq a \oplus \delta, \tag{C.1}$$

since for all $y \in Y \ \nu f(y) \ominus a(y) \sqsubseteq \delta$ by definition of δ and then (C.1) follows from Lemma 2.4(7).

Concerning the first part, let $y \in Y'$. Since a is a pre-fixpoint, $f(a)(y) \sqsubseteq a(y)$. Assume by contradiction that $f(a)(y) \sqsubset a(y)$. Then we have

 $f(a \oplus \delta)(y) = [by \text{ Lemma } 2.4(2), \text{ since } f \text{ is monotone and thus } f(a) \sqsubseteq f(a \oplus \delta)] = f(a)(y) \oplus (f(a \oplus \delta)(y) \ominus f(a)(y)) \\ = f(a)(y) \oplus (f(a \oplus \delta)(y) \ominus f(a)(y)) \subseteq f(a)(y) \sqsubseteq \delta] \\ \sqsubseteq f(a)(y) \oplus \delta \\ [by f(a)(y) \sqsubset a(y), \delta \sqsubseteq \overline{a(y)} \text{ and Lemma } 2.4(6)] \\ \sqsubset a(y) \oplus \delta \\ [by \text{ Lemma } 2.4(2) \text{ since } a(y) \sqsubseteq \nu f(y) \text{ and } \delta = \nu f(y) \ominus a(y)] \\ = \nu f(y) \\ = f(\nu f)(y) \\ [since \nu f \sqsubseteq a \oplus \delta \text{ (C.1) and } f \text{ monotone}] \\ \sqsubseteq f(a \oplus \delta)(y) \end{cases}$

i.e., a contradiction. Hence it must be a(y) = f(a)(y).

For the second part, in order to show $Y' \subseteq f_a^{\#}(Y')$, we let $b = \nu f \sqcup a$. By using (C.1) we immediately have that $b \in [a, a \oplus \delta]$.

We next prove that

$$Y' = \gamma_{a,\delta}(b).$$

We show separately the two inclusions. If $y \in Y'$ then $a(y) \sqsubset \nu f(y)$ and thus $b(y) = a(y) \sqcup \nu f(y) = \nu f(y)$ and thus $b(y) \ominus a(y) = \nu f(y) \ominus a(y) = \delta$. Hence $y \in \gamma_{a,\delta}(b)$. Conversely, if $y \in \gamma_{a,\delta}(b)$, then $a(y) \sqsubset \nu f(y)$. In fact, if it were $a(y) \sqsupseteq \nu f(y)$, then, by definition of b we would have b(y) = a(y) and $b(y) \ominus a(y) = 0 \not\supseteq \delta$. Therefore, $b(y) = \nu f(y)$ and thus $\nu f(y) \ominus a(y) \equiv b(y) \ominus a(y) \sqsupseteq \delta$, whence $y \in Y'$.

We can now conclude. In fact, since f is non-expansive, by Theorem 3.14(a.), we have

$$\gamma_{f(a),\delta}(f(b)) \subseteq f_a^{\#}(Y').$$

Moreover $Y' \subseteq \gamma_{f(a),\delta}(f(b))$. In fact, let $y \in Y'$, i.e., $y \in [Y]_a$ and $\delta \sqsubseteq b(y) \ominus a(y)$. Since a(y) = f(a)(y), we have that $y \in [Y]_{f(a)}$. In order to conclude that $y \in \gamma_{f(a),\delta}(f(b))$ it is left to show that $\delta \sqsubseteq f(b)(y) \ominus f(a)(y)$. We have

$$\begin{aligned} f(b)(y) \ominus f(a)(y) &= f(b)(y) \ominus a(y) & [since \ y \in Y'] \\ &= f(\nu f \sqcup a)(y) \ominus a(y) & [definition \ of \ b] \\ &\supseteq (f(\nu f)(y) \sqcup f(a)(y)) \ominus a(y) & [properties \ of \ \Box] \\ &= (\nu f(y) \sqcup a(y)) \ominus a(y) & [since \ \nu f \ fixpoint \ and \ y \in Y'] \\ &= b(y) \ominus a(y) & [definition \ of \ b] \\ &\supseteq \delta & [since \ y \in Y'] \end{aligned}$$

Combining the two inclusions, we have $Y' \subseteq f_a^{\#}(Y')$, as desired.

Appendix D. Proofs and Additional Material for §5 ((De)Composing functions and approximations)

The basic functions can be shown to be non-expansive.

Proposition D.1. The basic functions from Def. 5.1 are all non-expansive.

Proof.

- Constant functions: immediate.
- Reindexing: Let $u: Z \to Y$. For all $a, b \in \mathbb{M}^Y$, we have

$$\begin{split} \|u^*(b) \ominus u^*(a)\| \\ &= \max_{z \in Z} (b(u(z)) \ominus a(u(z))) \\ &\sqsubseteq \max_{y \in Y} (b(y) \ominus a(y)) \qquad \qquad [\text{since } u(Z) \subseteq Y] \\ &= \|b \ominus a\| \qquad \qquad [\text{by def. of norm}] \end{split}$$

• Minimum: Let $\mathcal{R} \subseteq Y \times Z$ be a relation. For all $a, b \in \mathbb{M}^Y$, we have

$$\|\min_{\mathcal{R}}(b) \ominus \min_{\mathcal{R}}(a)\| = \max_{z \in Z} (\min_{y \in Z} b(y) \ominus \min_{y \in Z} a(y))$$

Observe that

$$\max_{z \in Z} (\min_{y \mathcal{R}z} b(y) \ominus \min_{y \mathcal{R}z} a(y)) = \max_{z \in Z'} (\min_{y \mathcal{R}z} b(y) \ominus \min_{y \mathcal{R}z} a(y))$$

where $Z' = \{z \in Z \mid \exists y \in Y. y \mathcal{R}z\}$, since on every other $z \in Z \setminus Z'$ the difference would be 0. Now, for every $z \in Z'$, take $y_z \in Y$ such that $y_z \mathcal{R}z$ and $a(y_z) = \min_{y \mathcal{R}z} a(y)$. Such a y_z is

guaranteed to exist whenever Y is finite. Then, we have

$$\max_{z \in Z'} (\min_{y \in Z} b(y) \ominus \min_{y \in Z} a(y))$$

$$\sqsubseteq \max_{z \in Z'} (b(y_z) \ominus a(y_z)) \qquad [\ominus \text{ monotone in first arg.}]$$

$$\sqsubseteq \max_{z \in Z'} \|b \ominus a\| \qquad [by \text{ def. of norm}]$$

$$= \|b \ominus a\| \qquad [\|b \ominus a\| \text{ is independent from } z]$$

• Maximum: Let $\mathcal{R} \subseteq Y \times Z$ be a relation. For all $a, b \in \mathbb{M}^Y$ we have

$$\begin{aligned} &|\max_{\mathcal{R}}(b) \ominus \max_{\mathcal{R}}(a)|| \\ &= \max_{z \in Z} (\max_{y \mathcal{R}z} b(y) \ominus \max_{y \mathcal{R}z} a(y)) \\ &\sqsubseteq \max_{z \in Z} (\max_{y \mathcal{R}z} ((b(y) \ominus a(y)) \oplus a(y)) \ominus \max_{y \mathcal{R}z} a(y)) \\ &[\text{since } (b(y) \ominus a(y)) \oplus a(y) = a(y) \sqcup b(y) \text{ and } \ominus \text{ monotone in first arg.}] \\ &\sqsubseteq \max_{z \in Z} ((\max_{y \mathcal{R}z} (b(y) \ominus a(y)) \oplus \max_{y \mathcal{R}z} a(y)) \ominus \max_{y \mathcal{R}z} a(y)) \\ &\quad \text{[by def. of max and monotonicity of } \oplus] \\ &\sqsubseteq \max_{z \in Z} \max_{y \mathcal{R}z} (b(y) \ominus a(y)) \qquad \text{[by Lemma 2.4(6)]} \\ &\sqsubseteq \max_{z \in Z} \max_{y \mathcal{R}z} \|b \ominus a\| \qquad \text{[by def. of norm]} \\ &= \|b \ominus a\| \qquad \text{[since } \|b \ominus a\| \text{ is independent]} \end{aligned}$$

• Average: We first note that, when $p: Y \to \mathbb{M}$, with Y finite, is a distribution, then an inductive argument based on weak distributivity, allows one to show that for all $x \in \mathbb{M}$, $Y' \subseteq Y, x \odot \bigoplus_{y \in Y'} p(y) = \bigoplus_{y \in Y'} x \odot p(y)$.

For all $a, b \in \mathbb{M}^Y$ we have

$$\begin{split} \|av_D(o) \oplus av_D(a)\| \\ &= \max_{p \in D} \left(\bigoplus_{y \in Y} p(y) \odot b(y) \oplus \bigoplus_{y \in Y} p(y) \odot a(y) \right) \\ &\subseteq \max_{p \in D} \left(\bigoplus_{y \in Y} p(y) \odot ((b(y) \oplus a(y)) \oplus a(y)) \oplus \bigoplus_{y \in Y} p(y) \odot a(y) \right) \\ &[by monotonicity of \odot, \oplus, \ominus and (b(y) \ominus a(y)) \oplus a(y) = a(y) \sqcup b(y)] \\ &= \max_{p \in D} \left(\bigoplus_{y \in Y} (p(y) \odot (b(y) \oplus a(y)) \right) \oplus (p(y) \odot a(y)) \oplus \bigoplus_{y \in Y} p(y) \odot a(y)) \\ &[since b(y) \oplus a(y) \sqsubseteq 1 \oplus a(y) = \overline{a(y)}, and \odot weakly distributes over \oplus] \\ &= \max_{p \in D} \left(\left(\bigoplus_{y \in Y} p(y) \odot (b(y) \oplus a(y) \right) \oplus \bigoplus_{y \in Y} p(y) \odot a(y) \right) \oplus \bigoplus_{y \in Y} p(y) \odot a(y)) \\ &\subseteq \max_{p \in D} \bigoplus_{y \in Y} p(y) \odot (b(y) \oplus a(y)) \\ &[\max_{p \in D} \bigoplus_{y \in Y} p(y) \odot (b(y) \oplus a(y)) \\ &= \max_{p \in D} \bigoplus_{y \in Y} p(y) \odot (b(y) \oplus a(y)) \\ &[by def. of norm and monotonicity of \odot] \\ &= \max_{p \in D} \|b \oplus a\| \odot \bigoplus_{y \in Y} p(y) \\ &[since p is a distribution and \odot weakly distributes over \oplus] \\ &= \max_{p \in D} (\|b \oplus a\| \odot 1) \\ &= \max_{p \in D} \|b \oplus a\| \\ &[since p is a distribution over Y] \\ &= \|b \oplus a\| \\ &[since \|b \oplus a\| \ is independent from p] \\ \end{split}$$

The next result determines the approximations associated with the basic functions.

Proposition D.2 (approximations of basic functions). Let \mathbb{M} be an *MV*-chain, *Y*, *Z* be finite sets and let $a \in \mathbb{M}^{Y}$.

• Constant: for $k : \mathbb{M}^Z$, the approximations $(c_k)_a^{\#} : \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[Z]_{c_k(a)}}, (c_k)_{\#}^a : \mathbf{2}^{[Y]^a} \to \mathbf{2}^{[Z]^{c_k(a)}}$ are

$$(c_k)_a^{\#}(Y') = \emptyset = (c_k)_{\#}^a(Y')$$

• Reindexing: for $u: Z \to Y$, the approximations $(u^*)_a^{\#} : \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[Z]_{u^*(a)}}, \ (u^*)_{\#}^a : \mathbf{2}^{[Y]^a} \to \mathbf{2}^{[Z]_{u^*(a)}}$ are

$$(u^*)^{\#}_a(Y') = (u^*)^{a}_{\#}(Y') = u^{-1}(Y') = \{ z \in [Z]_{u^*(a)} \mid u(z) \in Y' \}$$

• Min: for $\mathcal{R} \subseteq Y \times Z$, the approximations $(\min_{\mathcal{R}})_a^{\#} : \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[Z]_{\min_{\mathcal{R}}(a)}}, \ (\min_{\mathcal{R}})_{\#}^a : \mathbf{2}^{[Y]^a} \to \mathbf{2}^{[Z]^{\min_{\mathcal{R}}(a)}}$ are given below, where $\mathcal{R}^{-1}(z) = \{y \in Y \mid y \mathcal{R}z\}$:

$$(\min_{\mathcal{R}})_a^{\#}(Y') = \{ z \in [Z]_{\min_{\mathcal{R}}(a)} \mid \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y) \subseteq Y' \}$$
$$(\min_{\mathcal{R}})_{\#}^a(Y') = \{ z \in [Z]^{\min_{\mathcal{R}}(a)} \mid \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y) \cap Y' \neq \emptyset \}$$

• Max: for $\mathcal{R} \subseteq Y \times Z$, the approximations $(\max_{\mathcal{R}})_a^{\#} : \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[Z]_{\max_{\mathcal{R}}(a)}}, \ (\max_{\mathcal{R}})_{\#}^a : \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[Z]_{\max_{\mathcal{R}}(a)}}$ are

$$(\max_{\mathcal{R}})_a^{\#}(Y') = \{ z \in [Z]_{\max_{\mathcal{R}}(a)} \mid \arg\max_{y \in \mathcal{R}^{-1}(z)} a(y) \cap Y' \neq \emptyset \}$$
$$(\max_{\mathcal{R}})_{\#}^a(Y') = \{ z \in [Z]^{\max_{\mathcal{R}}(a)} \mid \arg\max_{y \in \mathcal{R}^{-1}(z)} a(y) \subseteq Y' \}$$

• Average: for a finite $D \subseteq \mathcal{D}(Y)$, the approximations $(av_D)_a^{\#} : \mathbf{2}^{[Y]_a} \to \mathbf{2}^{[D]_{av_D(a)}}$, $(av_D)_{\#}^a : \mathbf{2}^{[Y]^a} \to \mathbf{2}^{[D]^{av_D(a)}}$ are

$$(av_D)_a^{\#}(Y') = \{ p \in [D]_{av_D(a)} \mid supp(p) \subseteq Y' \}$$
$$(av_D)_{\#}^{a}(Y') = \{ p \in [D]^{av_D(a)} \mid supp(p) \subseteq Y' \},$$

where $supp(p) = \{y \in Y \mid p(y) > 0\}$ for $p \in \mathcal{D}(Y)$.

- *Proof.* We only consider the primal cases, the dual ones are analogous. Let $a \in \mathbb{M}^{Y}$.
- Constant: for all $0 \sqsubset \theta \sqsubseteq \delta_a$ and $Y' \subseteq [Y]_a$ we have

$$(c_k)_{a,\theta}^{\#}(Y') = \gamma_{c_k(a),\theta} \circ c_k \circ \alpha_{a,\theta}(Y')$$

= { $z \in [Z]_{c_k(a)} \mid \theta \sqsubseteq c_k(a \oplus \theta_{Y'})(z) \ominus c_k(a)(z)$ }
= { $z \in [Z]_{c_k(a)} \mid \theta \sqsubseteq k \ominus k$ } = { $z \in Z \mid \theta \sqsubseteq 0$ } = \emptyset

Hence all values $\iota_a^f(Y', z)$ are equal to 0 and we have $\iota_a^f = \delta_a$. Replacing θ by ι_a^f we obtain $(c_k)_a^{\#}(Y') = \emptyset$.

• Reindexing: for all $0 \sqsubset \theta \sqsubseteq \delta_a$ and $Y' \subseteq [Y]_a$ we have

$$(u^*)_{a,\theta}^{\#}(Y') = \gamma_{u^*(a),\theta} \circ u^* \circ \alpha_{a,\theta}(Y')$$

= $\{z \in [Z]_{u^*(a)} \mid \theta \sqsubseteq (a \oplus \theta_{Y'})(u(z)) \ominus a(u(z))\}.$

We show that this corresponds to $u^{-1}(Y') = \{z \in Z \mid u(z) \in Y'\}$. It is easy to see that for all $z \in u^{-1}(Y')$, we have

$$(a \oplus \theta_{Y'})(u(z)) \ominus a(u(z)) = \theta = a(u(z)) \ominus (a \ominus \theta_{Y'})(u(z))$$

since $u(z) \in Y'$ and $\theta \sqsubseteq \delta_a$. Since $u(z) \in Y' \subseteq [Y]_a$, we have $u^*(a)(z) = a(u(z)) \neq 1$ and hence $z \in [Z]_{u^*(a)}$. On the other hand, for all $z \notin u^{-1}(Y')$, we have

$$(a \oplus \theta_{Y'})(u(z)) = a(u(z)) = (a \ominus \theta_{Y'})(u(z))$$

since $u(z) \notin Y'$, and so

$$(a \oplus heta_{Y'})(u(z)) \ominus a(u(z)) = a(u(z)) \ominus (a \ominus heta_{Y'})(u(z)) = 0 \sqsubset heta.$$

Therefore $(u^*)_{a,\theta}^{\#}(Y') = u^{-1}(Y').$

We observe that for $Y' \subseteq [Y]_a$, $z \in [Z]_{u^*(a)}$ either $u^*(a \oplus \theta_{Y'})(z) \ominus u^*(a)(z) \sqsubset \theta$ for all $0 \sqsubset \theta \sqsubseteq \delta_a$ – and in this case $\iota_a^{u^*}(Y', z) = 0$ – or $u^*(a \oplus \theta_{Y'})(z) \ominus u^*(a)(z) = \theta$ for all $0 \sqsubset \theta \sqsubseteq \delta_a$ – and in this case $\iota_a^{u^*}(Y', z) = \delta_a$. By taking the minimum over all non-zero values, we get $\iota_a^{u^*} = \delta_a$.

And finally we observe that $(u^*)_a^{\#}(Y') = (u^*)_{a,u^{u^*}}^{\#}(Y') = u^{-1}(Y').$

• Minimum: let $0 \sqsubset \theta \sqsubseteq \delta_a$. For all $Y' \subseteq [Y]_a$ we have

$$(\min_{\mathcal{R}})_{a,\theta}^{\#}(Y') = \gamma_{\min_{\mathcal{R}}(a),\theta} \circ \min_{\mathcal{R}} \circ \alpha_{a,\theta}(Y')$$
$$= \{ z \in [Z]_{\min_{\mathcal{R}}(a)} \mid \theta \sqsubseteq \min_{y \not\in Z} (a \oplus \theta_{Y'})(y) \ominus \min_{y \not\in Z} a(y) \}$$

We compute the value $V = \min_{y \in \mathcal{R}_z} (a \oplus \theta_{Y'})(y) \ominus \min_{y \in \mathcal{R}_z} a(y)$ and consider the following cases:

- Assume that there exists $\hat{y} \in \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y)$ where $\hat{y} \notin Y'$.

Then $(a \oplus \theta_{Y'})(\hat{y}) = a(\hat{y}) \sqsubseteq a(y) \sqsubseteq (a \oplus \theta_{Y'})(y)$ for all $y \in \mathcal{R}^{-1}(z)$, which implies that $\min_{y \in \mathcal{R}} (a \oplus \theta_{Y'})(y) = a(\hat{y})$. We also have $\min_{y \in \mathcal{R}} a(y) = a(\hat{y})$ and hence V = 0.

- Assume that $\arg\min_{y\in\mathcal{R}^{-1}(z)}a(y)\subseteq Y'$ and $\hat{\theta}\sqsubseteq a(y)\ominus a(\hat{y})$ in all cases where $\hat{y}\in \arg\min_{y\in\mathcal{R}^{-1}(z)}a(y), y\notin Y'$ and $y\mathcal{R}z$.

Since $\arg\min_{y\in\mathcal{R}^{-1}(z)}a(y)\subseteq Y'$ we observe that

$$\min_{y \mathcal{R}z} (a \oplus \theta_{Y'})(y) = \min\{\min_{y \in \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y)} (a(y) \oplus \theta), \min_{y \mathcal{R}z, y \notin Y'} a(y)\}$$

We can omit the values of all y with $y \mathcal{R}z$, $y \notin \arg \min_{y \in \mathcal{R}^{-1}(z)} a(y)$, $y \in Y'$, since we will never attain the minimum there.

Now let $\hat{y} \in \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y)$ and y with $y\mathcal{R}z$ and $y \notin Y'$. Then $\theta \sqsubseteq a(y) \ominus a(\hat{y})$ by assumption, which implies $a(\hat{y}) \oplus \theta \sqsubseteq a(y)$, since $a(\hat{y}) \sqsubseteq a(y)$ and Lemma 2.4(2) holds. From this we can deduce $\min_{y\mathcal{R}z}(a \oplus \theta_{Y'})(y) = a(\hat{y}) \oplus \theta$. We also have $\min_{y\mathcal{R}z} a(y) = a(\hat{y})$ and hence – since $a(\hat{y}) \sqsubseteq \overline{\theta}$ (due to $\theta \sqsubseteq \delta_a \sqsubseteq \overline{a(\hat{y})}$) and Lemma 2.4(9) holds – $V = (a(\hat{y}) \oplus \theta) \ominus a(\hat{y}) = \theta$.

- In the remaining case $\arg\min_{y\in\mathcal{R}^{-1}(z)}a(y)\subseteq Y'$ and there exist $\hat{y}\in \arg\min_{y\in\mathcal{R}^{-1}(z)}a(y)$, $y\notin Y', \ y\mathcal{R}z$ such that $a(y)\ominus a(\hat{y})\sqsubset\theta$.

This implies $a(y) \sqsubseteq (a(y) \ominus a(\hat{y})) \oplus a(\hat{y}) \sqsubset \theta \oplus a(\hat{y})$ since again $a(\hat{y}) \sqsubseteq \overline{\theta}$ and Lemma 2.4(8) holds. Hence $\min_{y \not\in \mathcal{R}} (a \oplus \theta_{Y'})(y) \sqsubseteq a(y)$, which means that $V \sqsubseteq a(y) \ominus a(\hat{y}) \sqsubset \theta$.

Summarizing, for $\theta \sqsubseteq \delta_a$ we observe that $V = \theta$ if and only if $\arg\min_{y \in \mathcal{R}^{-1}(z)} a(y) \subseteq Y'$ and $\theta \sqsubseteq a(y) \ominus a(\hat{y})$ whenever $\hat{y} \in \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y), y \notin Y'$ and $y\mathcal{R}z$.

Hence if $\arg \min_{y \in \mathcal{R}^{-1}(z)} a(y) \subseteq Y'$ we have

$$\iota_a^{\min_{\mathcal{R}}}(Y',z) = \min\{a(y) \ominus a(\hat{y}) \mid \hat{y} \in \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y), y \notin Y', y\mathcal{R}z\} \cup \{\delta_a\}$$

otherwise $\iota_a^{\min_{\mathcal{R}}}(Y', z) = 0.$

The values above are minimal whenever $Y' = \arg \min_{y \in \mathcal{R}^{-1}(z)} a(y)$ and thus we have:

$$\iota_a^{\min_{\mathcal{R}}} = \min_{z \in [Z]_{\min_{\mathcal{R}}(a)}} \{ a(y) \ominus a(\hat{y}) \mid y\mathcal{R}z, \hat{y} \in \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y), y \notin \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y) \} \cup \{\delta_a\}.$$

Finally we deduce that

$$(\min_{\mathcal{R}})_a^{\#}(Y') = (\min_{\mathcal{R}})_{a,\iota_a}^{\#}(Y') = \{z \in [Z]_{\min_{\mathcal{R}}(a)} \mid \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y) \subseteq Y'\}.$$

• Maximum: let $0 \sqsubset \theta \sqsubseteq \delta_a$. For all $Y' \subseteq [Y]_a$ we have

$$(\max_{\mathcal{R}})_{a,\theta}^{\#}(Y') = \gamma_{\max_{\mathcal{R}}(a),\theta} \circ \max_{\mathcal{R}} \circ \alpha_{a,\theta}(Y')$$
$$= \{ z \in [Z]_{\max_{\mathcal{R}}(a)} \mid \theta \sqsubseteq \max_{y \not\in Z} (a \oplus \theta_{Y'})(y) \ominus \max_{y \not\in Z} a(y) \}$$

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We observe that

$$\max_{y\mathcal{R}z}(a\oplus\theta_{Y'})(y) = \max\{\max_{y\in\arg\max_{y\in\mathcal{R}^{-1}(z)}a(y)}(a\oplus\theta_{Y'})(y), \max_{\substack{y\mathcal{R}z,y\in Y'\\y\notin\arg\max_{y\in\mathcal{R}^{-1}(z)}a(y)}}(a(y)\oplus\theta)\}$$

We can omit the values of all y with $y\mathcal{R}z$, $y \notin \arg \max_{y\in \mathcal{R}^{-1}(z)} a(y)$, $y \notin Y'$, since we will never attain the maximum there.

We now compute the value $V = \max_{y \in \mathcal{R}^z} (a \oplus \theta_{Y'})(y) \oplus \max_{y \in \mathcal{R}^z} a(y)$ and consider the following cases:

- Assume that there exists $\hat{y} \in \arg \max_{y \in \mathcal{R}^{-1}(z)} a(y)$ where $\hat{y} \in Y'$.
 - Then $(a \oplus \theta_{Y'})(\hat{y}) = a(\hat{y}) \oplus \theta \sqsupseteq (a \oplus \theta_{Y'})(y) \sqsupseteq a(y)$ for all $y \in \mathcal{R}^{-1}(z)$, which implies that $\max_{y \not\in z} (a \oplus \theta_{Y'})(y) = a(\hat{y}) \oplus \theta$. We also have $\max_{y \not\in z} a(y) = a(\hat{y})$ and hence – since $a(\hat{y}) \sqsubseteq \overline{\theta}$ and Lemma 2.4(9) holds – $V = (a(\hat{y}) \oplus \theta) \ominus a(\hat{y}) = \theta$.
- Assume that $\arg \max_{y \in \mathcal{R}^{-1}(z)} a(y) \cap Y' = \emptyset$. Now let $\hat{y} \in \arg \max_{y \in \mathcal{R}^{-1}(z)} a(y)$ and $y \notin \arg \max_{y \in \mathcal{R}^{-1}(z)} a(y)$ with $y \mathcal{R}z$ and $y \in Y'$. Then

$$\max_{\substack{y \in \arg\max_{y \in \mathcal{R}^{-1}(z)} a(y)}} (a \oplus \theta_{Y'})(y) = a(\hat{y})$$
$$\max_{\substack{y \mathcal{R}z, y \in Y'\\ y \notin \arg\max_{y \in \mathcal{R}^{-1}(z)} a(y)}} (a(y) \oplus \theta) = a(y') \oplus \theta$$

for some value y' with $y'\mathcal{R}z, y' \in Y', y' \notin \arg \max_{y \in \mathcal{R}^{-1}(z)} a(y)$, that is $a(y') \sqsubset a(\hat{y})$. So then either $\max_{y\mathcal{R}z}(a \oplus \theta_{Y'})(y) = a(\hat{y})$ and $V = a(\hat{y}) \ominus a(\hat{y}) = 0$. Or $\max_{y\mathcal{R}z}(a \oplus \theta_{Y'})(y) = a(y') \oplus \theta$ and by Lemma 2.4(11) $V = (a(y') \oplus \theta) \ominus a(\hat{y}) \sqsubset \theta$. Summarizing, for $\theta \sqsubseteq \delta_a$ we observe that $V = \theta$ if and only if $\arg \max_{y \in \mathcal{R}^{-1}(z)} a(y) \cap Y' \neq \emptyset$, where the latter condition is independent of θ .

Hence, as in the case of reindexing, we have $\iota_a^{\max_{\mathcal{R}}} = \delta_a$. Finally we have

$$(\max_{\mathcal{R}})_a^{\#}(Y') = (\max_{\mathcal{R}})_{a,\iota_a}^{\#}(Y') = \{z \in [Z]_{\max_{\mathcal{R}}(a)} \mid \arg\max_{y \in \mathcal{R}^{-1}(z)} a(y) \cap Y' \neq \emptyset\}.$$

• Average: for all $0 \sqsubset \theta \sqsubseteq \delta_a$ and $Y' \subseteq [Y]_a$ by definition

$$(\operatorname{av}_D)_{a,\theta}^{\#}(Y') = \gamma_{\operatorname{av}_D(a),\theta} \circ \operatorname{av}_D \circ \alpha_{a,\theta}(Y')$$
$$= \{ p \in [D]_{\operatorname{av}_D(a)} \mid \theta \sqsubseteq \bigoplus_{y \in Y} p(y) \odot (a \oplus \theta_{Y'})(y) \ominus \bigoplus_{y \in Y} p(y) \odot a(y) \}$$

We show that this set corresponds to $\{p \in [D]_{\operatorname{av}_D(a)} \mid supp(p) \subseteq Y'\}$.

Consider $p \in [D]_{\operatorname{av}_D(a)}$ such that $supp(p) \subseteq Y'$. Note that clearly $\bigoplus_{y \in Y'} p(y) = 1$. Now we have

$$\begin{split} &\bigoplus_{y \in Y} p(y) \odot (a \oplus \theta_{Y'})(y) \ominus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} p(y) \odot (a(y) \oplus \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} (p(y) \odot a(y) \oplus p(y) \odot \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} (p(y) \odot a(y) \oplus p(y) \odot \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} (p(y) \odot a(y) \oplus p(y) \odot \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) = \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} (p(y) \odot a(y) \oplus p(y) \odot \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) = \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} (p(y) \odot a(y) \oplus p(y) \odot \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} (p(y) \odot a(y) \oplus p(y) \odot \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} (p(y) \odot a(y) \oplus p(y) \odot \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} (p(y) \odot a(y) \oplus p(y) \odot \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} (p(y) \odot a(y) \oplus p(y) \oplus p(y) \odot \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \oplus \bigoplus_{y \in Y} p(y) \oplus \bigoplus_{y \in Y} p(y) \oplus p(y) \oplus \bigoplus_{y \in Y} p($$

[by weak distributivity, since for $y \in Y' \subseteq [Y]_a$, $a(y) \sqsubseteq \overline{\delta_a}$]

$$\begin{split} &= \bigoplus_{y \in Y'} p(y) \odot \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} p(y) \odot \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ & \text{[since, for } y \notin Y' \supseteq supp(p), \ p(y) = 0 \text{ and thus } p(y) \odot a(y) = 0] \\ &= (\bigoplus_{y \in Y'} p(y)) \odot \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ & \text{[by weak distributivity, since } p \text{ is a distribution]} \\ &= 1 \odot \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ & \text{[since } p \text{ is a distribution]} \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \oplus \bigoplus_{y \in Y'} p(y) \oplus a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \oplus \bigoplus_{y \in Y'} p(y) \oplus a(y) \\ &= \theta \oplus \bigoplus_{y \in Y'} p(y) \oplus_{y \in Y'} p(y) \oplus_$$

In order to motivate the last passage, observe that for all $y \in Y' \subseteq [Y]_a$, we have $a(y) \sqsubseteq \overline{\delta_a}$, and thus $\bigoplus_{y \in Y'} p(y) \odot a(y) \sqsubseteq \bigoplus_{y \in Y'} p(y) \odot \overline{\delta_a} = (\bigoplus_{y \in Y'} p(y)) \odot \overline{\delta_a} = 1 \odot \overline{\delta_a} = \overline{\delta_a}$, where the third last passage is motivated by weak distributivity. Since $\theta \sqsubseteq \delta_a$, by Lemma 2.4(3), we have $\overline{\delta_a} \sqsubseteq \overline{\theta}$ and thus $\bigoplus_{y \in Y'} p(y) \odot a(y) \sqsubseteq \overline{\theta}$. In turn, using this fact, Lemma 2.4(9) motivates the last equality in the chain above, i.e., $\theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y'} p(y) \odot a(y) = \theta$.

On the other hand, for all $p \in [D]_{\operatorname{av}_D(a)}$ such that $\operatorname{supp}(p) \not\subseteq Y'$, there exists $y' \in Y \setminus Y'$ such that $p(y') \neq 0$. Then, we have

$$\begin{split} &\bigoplus_{y \in Y} p(y) \odot (a \oplus \theta_{Y'})(y) \ominus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} p(y) \odot (a(y) \oplus \theta) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &= \bigoplus_{y \in Y'} p(y) \odot \theta \oplus \bigoplus_{y \in Y'} p(y) \odot a(y) \oplus \bigoplus_{y \in Y \setminus Y'} p(y) \odot a(y) \ominus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &\quad \text{[by weak distributivity, since for } y \in Y' \subseteq [Y]_a, a(y) \sqsubseteq \overline{\delta_a}] \\ &= \bigoplus_{y \in Y'} p(y) \odot \theta \oplus \bigoplus_{y \in Y} p(y) \odot a(y) \ominus \bigoplus_{y \in Y} p(y) \odot a(y) \\ &\sqsubseteq \bigoplus_{y \in Y'} p(y) \odot \theta \\ &\quad \text{[by Lemma 2.4(6)]} \\ &= \theta \odot \bigoplus_{y \in Y'} p(y) \\ &\quad \text{[by weak distributivity, since } p \text{ is a distribution]} \\ & \Box \theta \end{split}$$

In order to motivate the last inequality, we proceed as follows. We have that $supp(p) \not\subseteq Y'$. Let $y_0 \in supp(p) \setminus Y'$. We know that $\overline{p(y_0)} \sqsubseteq \bigoplus_{y \in Y \setminus \{y\}} p(y) \sqsubseteq \bigoplus_{y \in Y'} p(y)$. Therefore $\overline{\bigoplus_{y \in Y'} p(y)} \sqsubseteq p(y_0) \neq 0$. Hence $\bigoplus_{y \in Y'} p(y) \sqsubset 1$.

The strict inequality above now follows, if we further show that given an $x \in \mathbb{M}, x \neq 1$ then $\theta \odot x \sqsubset \theta$. Note that $\overline{x} \neq 0$. Therefore $\theta = \theta \odot 1 = \theta \odot (x \oplus \overline{x}) = \theta \odot x \oplus \theta \odot \overline{x}$, where the last equality follows by weak distributivity. Now $\theta \odot \overline{x} \sqsubseteq \overline{x} \sqsubseteq \overline{\theta} \odot \overline{x}$, and thus, by Lemma 2.4(9), we obtain $\theta \odot x = \theta \odot x \oplus \theta \odot \overline{x} \ominus \theta \odot \overline{x} = \theta \ominus \theta \odot \overline{x} \sqsubset \theta$, as desired. The last passage follows by the fact that $\theta, \overline{x} \neq 0$ and thus $\theta \odot \overline{x} \neq 0$.

Since these results hold for all $\theta \sqsubseteq \delta_a$, we have $\iota_a^{\operatorname{av}_D} = \delta^a$.

And finally
$$(\operatorname{av}_D)_{a,\theta}^{\#}(Y') = (\operatorname{av}_D)_{\#}^{a,\theta}(Y') = \{p \in [D]_{\operatorname{av}_D(a)} \mid supp(p) \subseteq Y'\}.$$

When a non-expansive function arises as the composition of simpler ones (see Lemma B.3) we can obtain the corresponding approximation by just composing the approximations of the simpler functions.

Proposition D.3 (composing approximations). Let $g : \mathbb{M}^Y \to \mathbb{M}^W$ and $h : \mathbb{M}^W \to \mathbb{M}^Z$ be non-expansive functions. For all $a \in \mathbb{M}^Y$ we have that $(h \circ g)_a^{\#} = h_{g(a)}^{\#} \circ g_a^{\#}$. Analogously $(h \circ g)_{\#}^a = h_{\#}^{g(a)} \circ g_{\#}^a$ for the dual case.

Proof. Here we only consider the primal case, the dual case for $(h \circ g)^a_{\#}$ is analogous.

Let $0 \sqsubset \theta \sqsubseteq \min\{\iota_a^g, \iota_{a(a)}^h\}$. Then, by Theorem 3.14(b.) we know that

$$g_a^{\#} = g_{a,\theta}^{\#} = \gamma_{g(a),\theta} \circ g \circ \alpha_{a,\theta}$$
$$h_{g(a)}^{\#} = h_{g(a),\theta}^{\#} = \gamma_{h(g(a)),\theta} \circ h \circ \alpha_{g(a),\theta}$$

Now we will prove that

$$(h \circ g)_{a,\theta}^{\#} = h_{g(a),\theta}^{\#} \circ g_{a,\theta}^{\#}$$

First observe that $g(\alpha_{a,\theta}(Y')) \in [g(a), g(a \oplus \theta)] \subseteq [g(a), g(a) \oplus \theta]$ for all $Y' \subseteq [Y]_a$ by Lemma B.2. Applying Theorem 3.14(b.) on h we obtain

$$(h \circ g)_{a,\theta}^{\#} = \gamma_{h(g(a)),\theta} \circ h \circ g \circ \alpha_{a,\theta}(Y') = h_{g(a),\theta}^{\#} \circ \gamma_{g(a),\theta} \circ g \circ \alpha_{a,\theta}(Y')$$
$$= h_{g(a),\theta}^{\#} \circ g_{a,\theta}^{\#}(Y') = h_{g(a)}^{\#} \circ g_{a}^{\#}(Y')$$

Hence all functions $(h \circ g)_{a,\theta}^{\#}$ are equal and independent of θ and so it must hold that $(h \circ g)_{a,\theta}^{\#} = (h \circ g)_a^{\#}$. Then from Theorem 3.14 we can conclude $\min\{\iota_a^g, \iota_{g(a)}^h\} \sqsubseteq \iota_a^{h \circ g}$.

Furthermore functions can be combined via disjoint union, preserving non-expansiveness, as follows.

Proposition D.4 (disjoint union of non-expansive functions). Let $f_i : \mathbb{M}^{Y_i} \to \mathbb{M}^{Z_i}$, for $i \in I$, be non-expansive and such that the sets Z_i are pairwise disjoint. The function $\biguplus_{i \in I} f_i : \mathbb{M}^{\bigcup_{i \in I} Y_i} \to \mathbb{M}^{\biguplus_{i \in I} Z_i}$ defined by

$$\biguplus_{i \in I} f_i(a)(z) = f_i(a|_{Y_i})(z) \qquad \text{if } z \in Z_i$$

is non-expansive.

Proof. For all $a, b \in \mathbb{M}^{\bigcup_{i \in I} Y_i}$ we have

$$\begin{split} & \left\| \bigoplus_{i \in I} f_i(b) \ominus \bigoplus_{i \in I} f_i(a) \right\| \\ &= \max_{z \in \bigcup_{i \in I} Z_i} (\bigoplus_{i \in I} f_i(b)(z) \ominus \bigoplus_{i \in I} f_i(a)(z)) \\ &= \max_{i \in I} \max_{z \in Z_i} (f_i(b|_{Y_i})(z) \ominus f_i(a|_{Y_i})(z)) \qquad [\text{since all } Z_i \text{ are disjoint}] \\ &= \max_{i \in I} \| f_i(b|_{Y_i}) \ominus f_i(a|_{Y_i}) \| \qquad [\text{by def. of norm}] \\ &\sqsubseteq \max_{i \in I} \| b|_{Y_i} \ominus a|_{Y_i} \| \qquad [\text{since all } f_i \text{ are non-expansive}] \\ &= \max_{i \in I} \max_{y \in Y_i} (b(y) \ominus a(y)) \\ &= \max_{y \in \bigcup_{i \in I} Y_i} (b(y) \ominus a(y)) \\ &= \| b \ominus a \| \qquad [\text{by def. of norm}] \end{split}$$

Also, the corresponding approximation of a disjoint union can be conveniently assembled from the approximations of its components.

Proposition D.5 (disjoint union and approximations). The approximations for $\biguplus_{i \in I} f_i$, where $f_i : \mathbb{M}^{Y_i} \to \mathbb{M}^{Z_i}$ are non-expansive and Z_i are pairwise disjoint, have the following form. For all $a: \bigcup_{i \in I} Y_i \to \mathbb{M}$ and $Y' \subseteq \bigcup_{i \in I} Y_i$:

$$\left(\bigcup_{i\in I}f_i\right)_a^{\#}(Y') = \bigcup_{i\in I}(f_i)_{a|_{Y_i}}^{\#}(Y'\cap Y_i) \qquad \left(\bigcup_{i\in I}f_i\right)_{\#}^a(Y') = \bigcup_{i\in I}(f_i)_{\#}^{a|_{Y_i}}(Y'\cap Y_i)$$

Proof. We just show the statement for the primal case, the dual case is analogous. We abbreviate $Y = \bigcup_{i \in I} Y_i$.

Let $0 \sqsubset \theta \sqsubseteq \delta_a$. According to the definition of *a*-approximation (Lemma 3.11) we have for $Y' \subseteq [Y]_a$:

$$\left(\bigcup_{i\in I} f_i\right)_{a,\theta}^{\#}(Y') = \gamma_{\bigcup_{i\in I} f_i(a),\theta} \circ \bigcup_{i\in I} f_i \circ \alpha_{a,\theta}$$
$$(f_i)_{a|_{Y_i},\theta}^{\#} = \gamma_{f_i(a|_{Y_i}),\theta} \circ f_i \circ \alpha_{a|_{Y_i},\theta}$$

for all $i \in I$. Our first step is prove that

$$\gamma_{\underset{i\in I}{\biguplus}f_i(a),\theta} \circ \underset{i\in I}{\biguplus} f_i \circ \alpha_{a,\theta}(Y') = \underset{i\in I}{\biguplus} \gamma_{f_i(a|_{Y_i}),\theta} \circ f_i \circ \alpha_{a|_{Y_i},\theta}(Y' \cap Y_i)$$

By simply expanding the functions we obtain

$$\gamma_{\underset{i\in I}{\biguplus}}f_{i(a),\theta}\circ\underset{i\in I}{\biguplus}f_{i}\circ\alpha_{a,\theta}(Y') = \{z\in Z_{i}\mid i\in I \land \theta\sqsubseteq f_{i}((a\oplus\theta_{Y'})|_{Y_{i}})(z)\ominus f_{i}(a|_{Y_{i}})(z)\}$$

$$\biguplus_{i\in I}\gamma_{f_i(a|_{Y_i}),\theta}\circ f_i\circ\alpha_{a|_{Y_i},\theta}(Y'\cap Y_i)=\biguplus_{i\in I}\{z\in Z_i\mid \theta\sqsubseteq f_i(a|_{Y_i}\oplus\theta_{Y'\cap Y_i})(z)\ominus f_i(a|_{Y_i})(z)\}$$

which are the same set, since for all $i \in I$ clearly $(a \oplus \theta_{Y'})|_{Y_i} = a|_{Y_i} \oplus \theta_{Y' \cap Y_i}$.

This implies

$$\left(\bigcup_{i\in I}f_i\right)_{a,\theta}^{\#}(Y')=\bigcup_{i\in I}(f_i)_{a|_{Y_i},\theta}^{\#}(Y'\cap Y_i).$$

Whenever $\theta \sqsubseteq \min_{i \in I} \iota_a^{f_i}$, this can be rewritten to

$$\left(\bigcup_{i\in I}f_i\right)_{a,\theta}^{\#}(Y')=\bigcup_{i\in I}(f_i)_{a|_{Y_i}}^{\#}(Y'\cap Y_i).$$

All functions $(\biguplus_{i \in I} f_i)_{a,\theta}^{\#}$ are equal and independent of θ and so it must hold that $(\biguplus_{i \in I} f_i)_{a,\theta}^{\#}$ = $(\biguplus_{i \in I} f_i)_a^{\#}$. Then with Theorem 3.14 we can also conclude $\min_{i \in I} \iota_a^{f_i} \sqsubseteq \iota_a^{\biguplus_{i \in I} f_i}$.

Theorem 5.2. All basic functions in Def. 5.1 are non-expansive. Furthermore non-expansive functions are closed under composition and disjoint union. The approximations are the ones listed in the third column of Table 1.

Proof. Follows directly from Propositions D.1, D.2, D.3, D.4, D.5 and Lemma B.3.

We can also specify the maximal decrease respectively increase that is propagated (here we are using the notation of Lemma 3.11).

Corollary D.6. Let $f: \mathbb{M}^Y \to \mathbb{M}^Z$, $a \in \mathbb{M}^Y$ and ι_a^f be defined as in Lemma 3.11. In the dual view we have $\iota_f^a = \min\{\iota_f^a(Y', z) \mid Y' \subseteq Y \land z \in Z \land \iota_f^a(Y', z) \neq 0\} \cup \{\delta^a\}$, where the set $\{\theta \sqsubseteq \delta^a \mid z \in f_{\#}^{a,\theta}(Y')\}$ has a maximum for each $z \in [Z]^{f(a)}$ and $Y' \subseteq [Y]^a$, that we denote by $\iota_f^a(Y', z)$.

We consider the basic functions from Def. 5.1, function composition as in Lemma B.3 and disjoint union as in Proposition D.4 and give the corresponding values for ι_a^f and ι_f^a .

For greatest fixpoints (primal case) we obtain:

•
$$\iota_a^{c_k} = \iota_a^{u^*} = \iota_a^{\max_{\mathcal{R}}} = \iota_a^{\operatorname{av}_D} = \delta^a$$

• $\iota_a^{\min_{\mathcal{R}}} = \min_{z \in [Z]_{\min_{\mathcal{R}}(a)}} \{a(y) \ominus a(\hat{y}) \mid u^{\mathcal{R}} \geq u^{\mathcal{A}} a(\hat{y}) \}$

 $y\mathcal{R}z, y \notin \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y), \hat{y} \in \arg\min_{y \in \mathcal{R}^{-1}(z)} a(y) \} \cup \{\delta^a\}$

• $\iota_a^{g \circ f} \supseteq \min{\{\iota_a^f, \iota_{f(a)}^g\}}$ • $\iota_a^{\biguplus_{i \in I} f_i} = \min_{i \in I} \iota_{a|_{Y_i}}^{f_i}$

For least fixpoints (dual case) we obtain:

•
$$\iota_{c_k}^a = \iota_{u^*}^a = \iota_{\min_{\mathcal{R}}}^a = \iota_{\operatorname{av}_D}^a = \delta_a$$

• $\iota_{\max_{\mathcal{R}}}^a = \min_{z \in [Z]^{\min_{\mathcal{R}}(a)}} \{a(\hat{y}) \ominus a(y) \mid y \notin \arg\max_{y \in \mathcal{R}^{-1}(z)} a(y), y \notin \arg\max_{y \in \mathcal{R}^{-1}(z)} a(y)\} \cup \{\delta_a\}$
• $\iota_{g \circ f}^a \supseteq \min\{\iota_f^a, \iota_g^{f(a)}\}$

•
$$\iota^a_{\biguplus_{i \in I} f_i} = \min_{i \in I} \iota^{a|_{Y_i}}_{f_i}$$

Proof. The values ι_a^f can be obtained by inspecting the proofs of Propositions D.2, D.3 and D.4.

It only remains to show that $\iota := \iota_a^{\biguplus_{i \in I} f_i} \sqsubseteq \min_{i \in I} \iota_{a|_{Y_i}}^{f_i}$ (cf. Proposition D.4), which means showing $\iota \sqsubseteq \iota_{a|_{Y_i}}^{f_i}$ for every $i \in I$. We abbreviate $\iota_i := \iota_{a|_{Y_i}}^{f_i}$.

If $\iota \sqsupset \iota_i$ for some $i \in I$, we will find a $z \in [Z_i]_{f_i(a)}$ and $Y' \subseteq [Y]_a$, such that $z \in (f_i)^{\#}_{a|_{Y_i},\iota_i}(Y' \cap Y_i) = (f_i)^{\#}_{a|_{Y_i}}(Y' \cap Y_i)$ but $z \notin (f_i)^{\#}_{a|_{Y_i},\iota}(Y' \cap Y_i)$ by definition (cf. Lemma 3.10).

This is a contradiction since

$$z \in \bigoplus_{i \in I} (f_i)_{a|_{Y_i}}^{\#} (Y' \cap Y_i) = \left(\bigcup_{i \in I} f_i \right)_a^{\#} (Y') = \left(\bigcup_{i \in I} f_i \right)_{a,\iota}^{\#} (Y') = \bigcup_{i \in I} (f_i)_{a|_{Y_i},\iota}^{\#} (Y' \cap Y_i)$$

and since $z \in Z_i$, $z \notin (f_i)_{a|_{Y_i},\iota}^{\#}(Y' \cap Y_i)$ and cannot be contained in the union. The arguments for the values ι_f^a in the dual case are analogous.

APPENDIX E. PROOFS AND ADDITIONAL MATERIAL FOR §6 (APPLICATIONS)

Lemma 6.1. The function \mathcal{T} can be written as

$$\mathcal{T} = (\eta^* \circ \operatorname{av}_D) \uplus c_k$$

where $k: T \to [0,1]$ is the constant function 1 defined only on terminal states. *Proof.* Let $t: S \to [0, 1]$. For $s \in T$ we have

$$\begin{aligned} &((\eta^* \circ \operatorname{av}_D) \uplus c_k)(t)(s) \\ &= c_k(t)(s) & [\text{since } s \in T] \\ &= k(s) = 1 & [\text{by definition of } c_k \text{ and } k] \\ &= \mathcal{T}(t)(s) & [\text{since } s \in T] \end{aligned}$$

For $s \notin T$ we have

$$((\eta^* \circ \operatorname{av}_D) \uplus c_k)(t)(s)$$

$$= \eta^* \circ \operatorname{av}_D(t)(s) \qquad [since \ s \notin T]$$

$$= \operatorname{av}_D(t)(\eta(s)) \qquad [by \text{ definition of reindexing}]$$

$$= \sum_{s' \in S} \eta(s)(s') \cdot t(s') \qquad [by \text{ definition of av}_D]$$

$$= \mathcal{T}(t)(s) \qquad [since \ s \notin T]$$

Lemma 6.2. Given a function $t: S \to [0,1]$, the t-approximation for \mathcal{T} in the dual sense is $\mathcal{T}^t_{\#}: \mathbf{2}^{[S]^t} \to \mathbf{2}^{[S]^{\mathcal{T}(t)}}$ with

$$\mathcal{T}^t_{\#}(S') = \{ s \in [S]^{\mathcal{T}(t)} \mid s \notin T \land supp(\eta(s)) \subseteq S' \}.$$

Proof. In the following let $t: S \to [0,1]$ and $S' \subseteq [S]^t$. By Lemma 6.1 we know that $\mathcal{T} = (\eta^* \circ \operatorname{av}_D) \uplus c_k$, then by Propositions D.5, D.3, and D.2 we have

$$\begin{aligned} \mathcal{T}^{t}_{\#}(S') &= ((\eta^{*} \circ \operatorname{av}_{D}) \uplus c_{k})^{t}_{\#}(S') \\ &= (\eta^{*} \circ \operatorname{av}_{D})^{t}_{\#}(S') \cup (c_{k})^{t}_{\#}(S') \\ &= (\eta^{*})^{\operatorname{av}_{D}(t)}_{\#} \circ (\operatorname{av}_{D})^{t}_{\#}(S') \cup (c_{k})^{t}_{\#}(S') \\ &= \{s \in [S \setminus T]^{\eta^{*}(\operatorname{av}_{D}(t))} \mid \eta(s) \in \{q \in [D]^{\operatorname{av}_{D}(t)} \mid supp(q) \subseteq S'\}\} \cup \emptyset \\ &= \{s \in [S \setminus T]^{\eta^{*}(\operatorname{av}_{D}(t))} \mid \eta(s) \in [D]^{\operatorname{av}_{D}(t)} \land supp(\eta(s)) \subseteq S'\} \end{aligned}$$

Observe that actually for all $s \in [S \setminus T]^{\eta^*(\operatorname{av}_D(t))}$ it always holds that $\eta(s) \in [D]^{\operatorname{av}_D(t)}$. In fact, since $s \in [S \setminus T]^{\eta^*(\operatorname{av}_D(t))}$ we must have that $\eta^*(\operatorname{av}_D(t))(s) = \operatorname{av}_D(t)(\eta(s)) \neq 0$, and thus $\eta(s) \in \{q \in D \mid \operatorname{av}_D(t)(q) \neq 0\} = [D]^{\operatorname{av}_D(t)}$. Therefore, we have that

$$\{s \in [S \setminus T]^{\eta^*(\operatorname{av}_D(t))} \mid \eta(s) \in [D]^{\operatorname{av}_D(t)} \land supp(\eta(s)) \subseteq S'\}$$
$$= \{s \in [S \setminus T]^{\eta^*(\operatorname{av}_D(t))} \mid supp(\eta(s)) \subseteq S'\}$$

Finally, the set above is the same as

$$\{s \in [S]^{\mathcal{T}(t)} \mid s \notin T \land supp(\eta(s)) \subseteq S'\} = \{s \in [S \setminus T]^{\mathcal{T}(t)} \mid supp(\eta(s)) \subseteq S'\}$$

because, for all $s \in S \setminus T$, hence $s \notin T$, we have that $\mathcal{T}(t)(s) = \sum_{s' \in S} \eta(s)(s') \cdot t(s') = \eta^*(\operatorname{av}_D(t))(s)$, and so $[S \setminus T]^{\mathcal{T}(t)} = [S \setminus T]^{\eta^*(\operatorname{av}_D(t))}$.

Lemma 6.4. Let $u: VP_D \to D \times D$, $u(c) = (m_c^L, m_c^R)$. Then

 $\mathcal{K} = \min_{u} \operatorname{oav}_{VP_{D}}$ where $\operatorname{av}_{VP_{D}} \colon [0,1]^{X \times X} \to [0,1]^{VP_{D}}, \min_{u} \colon [0,1]^{VP_{D}} \to [0,1]^{D \times D}.$

Proof. It holds that $u^{-1}(p,q) = \Omega(p,q) \cap VP_D$ for $p,q \in D$. Furthermore note it is sufficient to consider as couplings the vertices, i.e., the elements of VP_D , since the minimum is always attained there [PC20].

Hence we obtain for $d: X \times X \to [0, 1], p, q \in D$:

$$\min_{u}(\operatorname{av}_{VP_{D}}(d))(p,q) = \min_{c \in \Omega(p,q) \cap VP_{D}} \operatorname{av}_{VP_{D}}(d)(c)$$
$$= \min_{c \in \Omega(p,q) \cap VP_{D}} \sum_{x_{1},x_{2} \in X \times X} c(x_{1},x_{2}) \cdot d(x_{1},x_{2})$$
$$= \min_{c \in \Omega(p,q)} \sum_{x_{1},x_{2} \in X \times X} c(x_{1},x_{2}) \cdot d(x_{1},x_{2})$$
$$= \mathcal{K}(d)(p,q) \qquad \Box$$

Lemma 6.5. Let $d: X \times X \to [0,1]$. The approximation for the Kantorovich lifting \mathcal{K} in the dual sense is $\mathcal{K}^d_{\#}: \mathbf{2}^{[X \times X]^d} \to \mathbf{2}^{[D \times D]^{\mathcal{K}(d)}}$ with

$$\begin{aligned} \mathcal{K}^d_\#(M) &= \{(p,q) \in [D \times D]^{\mathcal{K}(d)} \mid \exists c \in \Omega(p,q), supp(c) \subseteq M, \\ &\sum_{u,v \in S} d(u,v) \cdot c(u,v) = \mathcal{K}(d)(p,q) \}. \end{aligned}$$

Proof. Let $d: X \times X \to [0,1]$ and $M \subseteq [X \times X]^d$. Then we have:

$$\mathcal{K}^{d}_{\#}(M) = (\min_{u})^{\operatorname{av}_{VP_{D}}(d)}_{\#}((\operatorname{av}_{VP_{D}})^{d}_{\#}(M))$$

where

$$(\operatorname{av}_{VP_D})^d_{\#} \colon \mathbf{2}^{[X \times X]^d} \to \mathbf{2}^{[VP_D]^{\operatorname{av}_{VP_D}(d)}}$$
$$(\operatorname{min}_u)^{\operatorname{av}_{VP_D}(d)}_{\#} \colon \mathbf{2}^{[VP_D]^{\operatorname{av}_{VP_D}(d)}} \to \mathbf{2}^{[D \times D]^{\mathcal{K}(d)}}$$

We are using the approximations associated to non-expansive functions, given in Proposition D.2, and obtain:

$$\mathcal{K}^d_{\#}(M) = \{ (p,q) \in [D \times D]^{\mathcal{K}(d)} \mid \arg\min_{c \in u^{-1}(p,q)} \operatorname{av}_{VP_D}(d)(c) \cap (\operatorname{av}_{VP_D})^d_{\#}(M) \neq \emptyset \}$$

$$= \{(p,q) \in [D \times D]^{\mathcal{K}(d)} \mid \exists c \in \Omega(p,q), c \in (\operatorname{av}_{VP_D})^d_{\#}(M), \\ \operatorname{av}_{VP_D}(d)(c) = \min_{c' \in \Omega(p,q)} \operatorname{av}_{VP_D}(d)(c')\} \\ = \{(p,q) \in [D \times D]^{\mathcal{K}(d)} \mid \exists c \in \Omega(p,q), c \in (\operatorname{av}_{VP_D})^d_{\#}(M), \\ \operatorname{av}_{VP_D}(d)(c) = \mathcal{K}(d)(p,q)\} \\ = \{(p,q) \in [D \times D]^{\mathcal{K}(d)} \mid \exists c \in \Omega(p,q), supp(c) \subseteq M, \\ \sum_{u,v \in S} d(u,v) \cdot c(u,v) = \mathcal{K}(d)(p,q)\}$$

Lemma 6.6. It holds that $\mathcal{H} = \min_u \circ \max_{\in} where \max_{\in} : \mathbb{M}^{X \times X} \to \mathbb{M}^{2^{X \times X}}$, with $\in \subseteq (X \times X) \times 2^{X \times X}$ the "is-element-of"-relation on $X \times X$, and $\min_u : \mathbb{M}^{2^{X \times X}} \to \mathbb{M}^{2^X \times 2^X}$.

Proof. Let for $d: X \times X \to \mathbb{M}, X_1, X_2 \subseteq X$. Then we have

$$\min_{u}(\max_{\in}(d))(X_1, X_2) = \min_{u(C)=(X_1, X_2)} \max_{(x_1, X_2)} (\max_{(x_1, X_2)} (x_1, x_2) = 0) = \min_{u(C)=(X_1, X_2)} \max_{(x_1, X_2) \in C} a(x_1, x_2)$$

which is exactly the definition of the Hausdorff lifting $\mathcal{H}(d)(X_1, X_2)$ via couplings, due to Mémoli [Mém11].

Lemma 6.7. The approximation for the Hausdorff lifting \mathcal{H} in the dual sense is as follows. Let $d: X \times X \to \mathbb{M}$, then $\mathcal{H}^d_{\#}: \mathbf{2}^{[X \times X]^d} \to \mathbf{2}^{[\mathbf{2}^X \times \mathbf{2}^X]^{\mathcal{H}(d)}}$ with

$$\mathcal{H}_{\#}^{d}(R) = \{ (X_{1}, X_{2}) \in [\mathbf{2}^{X} \times \mathbf{2}^{X}]^{\mathcal{H}(d)} \mid \\ \forall x_{1} \in X_{1} \big(\min_{x_{2}' \in X_{2}} d(x_{1}, x_{2}') = \mathcal{H}(d)(X_{1}, X_{2}) \Rightarrow \exists x_{2} \in X_{2} \colon \\ (x_{1}, x_{2}) \in R \land d(x_{1}, x_{2}) = \mathcal{H}(d)(X_{1}, X_{2}) \big) \land \\ \forall x_{2} \in X_{2} \big(\min_{x_{1}' \in X_{1}} d(x_{1}', x_{2}) = \mathcal{H}(d)(X_{1}, X_{2}) \Rightarrow \exists x_{1} \in X_{1} \colon \\ (x_{1}, x_{2}) \in R \land d(x_{1}, x_{2}) = \mathcal{H}(d)(X_{1}, X_{2}) \big) \}$$

Proof. Let $d: X \times X \to \mathbb{M}$ and $R \subseteq [X \times X]^d$. Then we have:

$$\mathcal{H}^{d}_{\#}(R) = (\min_{u})^{\max_{\in}(d)}_{\#}((\max_{\in})^{d}_{\#}(R))$$

where

$$(\max_{\epsilon})_{\#}^{d} \colon \mathbf{2}^{[X \times X]^{d}} \to \mathbf{2}^{[\mathbf{2}^{X \times X}]^{\max_{\epsilon}(d)}}$$
$$(\min_{u})_{\#}^{\max_{\epsilon}(d)} \colon \mathbf{2}^{[\mathbf{2}^{X \times X}]^{\max_{\epsilon}(d)}} \to \mathbf{2}^{[\mathbf{2}^{X} \times \mathbf{2}^{X}]^{\mathcal{H}(d)}}$$

We are using the approximations associated to non-expansive functions, given in Proposition D.2, and obtain:

$$\begin{aligned} \mathcal{H}_{\#}^{d}(R) &= \{ (X_{1}, X_{2}) \in [\mathbf{2}^{X} \times \mathbf{2}^{X}]^{\mathcal{H}(d)} \mid \arg \min_{C \in u^{-1}(X_{1}, X_{2})} \max_{\in} (d)(C) \cap (\max_{\epsilon})_{\#}^{d}(R) \neq \emptyset \} \\ &= \{ (X_{1}, X_{2}) \in [\mathbf{2}^{X} \times \mathbf{2}^{X}]^{\mathcal{H}(d)} \mid \exists C \subseteq X \times X, u(C) = (X_{1}, X_{2}), \\ C \in (\max_{\epsilon})_{\#}^{d}(R), \max_{\epsilon} (d)(C) = \min_{u(C') = (X_{1}, X_{2})} \max_{\epsilon} (d)(C') \} \\ &= \{ (X_{1}, X_{2}) \in [\mathbf{2}^{X} \times \mathbf{2}^{X}]^{\mathcal{H}(d)} \mid \exists C \subseteq X \times X, u(C) = (X_{1}, X_{2}), \end{aligned}$$

$$C \in (\max_{\epsilon})^{a}_{\#}(R), \max d[C] = \min_{u(C')=(X_{1}, X_{2})} \max d[C']\}$$

= $\{(X_{1}, X_{2}) \in [\mathbf{2}^{X} \times \mathbf{2}^{X}]^{\mathcal{H}(d)} \mid \exists C \subseteq X \times X, u(C) = (X_{1}, X_{2}),$
$$\arg \max_{(y_{1}, y_{2}) \in C} d(y_{1}, y_{2}) \subseteq R, \max d[C] = \mathcal{H}(d)(X_{1}, X_{2})\}$$

We show that this is equivalent to the characterisation in the statement of the lemma.

• Assume that for all $x_1 \in X_1$ such that $\min_{x'_2 \in X_2} d(x_1, x'_2) = \mathcal{H}(d)(X_1, X_2)$, there exists $x_2 \in X_2$ such that $(x_1, x_2) \in R$ and $d(x_1, x_2) = \mathcal{H}(d)(X_1, X_2)$ (and vice versa).

We define a set C_m that contains all such pairs (x_1, x_2) , obtained from this guarantee. Now let $x_1 \notin \pi_1[C_m]$. Then necessarily $\min_{x'_2 \in X_2} d(x_1, x'_2) < \mathcal{H}(d)(X_1, X_2)$ (because the minimal distance to an element of X_2 cannot exceed the Hausdorff distance of the two sets). Construct another set C' that contains all such (x_1, x_2) where x_2 is an argument where the minimum is obtained. Also add elements $x_2 \notin \pi_2[C_m]$ and their corresponding partners to C'.

The $C = C_m \cup C'$ is a coupling for X_1, X_2 , i.e., $u(C) = (X_1, X_2)$. Furthermore $\arg \max_{(y_1, y_2) \in C} d(y_1, y_2) = C_m \subseteq R$ and $\max d[C] = \max d[C_m] = \mathcal{H}(d)(X_1, X_2)$.

• Assume that there exists $C \subseteq X \times X$, $u(C) = (X_1, X_2)$, $\arg \max_{(y_1, y_2) \in C} d(y_1, y_2) \subseteq R$, $\max d[C] = \mathcal{H}(d)(X_1, X_2).$

Now let $x_1 \in X_1$ such that $\min_{x'_2 \in X_2} d(x_1, x'_2) = \mathcal{H}(d)(X_1, X_2)$. Since C is a coupling of X_1, X_2 , there exists $x_2 \in X_2$ such that $(x_1, x_2) \in C \subseteq R$. It is left to show that $d(x_1, x_2) = \mathcal{H}(d)(X_1, X_2)$, which can be done as follows:

$$\mathcal{H}(d)(X_1, X_2) = \min_{x_2' \in X_2} d(x_1, x_2') \le d(x_1, x_2) \le \max d[C] = \mathcal{H}(d)(X_1, X_2).$$

For an $x_2 \in X_2$ such that $\min_{x_1 \in X_1} d(x_1', x_2) = \mathcal{H}(d)(X_1, X_2)$ the proof is analogous.

Lemma 6.8 (decomposing \mathcal{M}). The fixpoint function \mathcal{M} for probabilistic bisimilarity pseudo-metrics can be written as:

$$\mathcal{M} = \max_{\rho} \circ (((\eta \times \eta)^* \circ \mathcal{H} \circ \mathcal{K}) \uplus ((\ell \times \ell)^* \circ c_{d_L}))$$

where $\rho \colon (S \times S) \uplus (S \times S) \to (S \times S)$ with $\rho((s,t),i) = (s,t)$.

Proof. In fact, given $d: S \times S \to [0,1]$ and $s, t \in S$, we have

$$\begin{aligned} \max_{\rho} ((((\eta \times \eta)^* \circ \mathcal{H} \circ \mathcal{K}) \uplus ((\ell \times \ell)^* \circ c_{d_L}))(d))(s,t) \\ &= \max\{(\eta \times \eta)^* \circ \mathcal{H} \circ \mathcal{K})(d)(s,t), ((\ell \times \ell)^* \circ d_L)(s,t)\} \\ &= \max\{\mathcal{H}(\mathcal{K}(d)(\eta(s),\eta(t)), d_L(\ell(s),\ell(t)))\} \\ &= \mathcal{M}(d)(s,t) \end{aligned}$$

Lemma 6.9 (approximating \mathcal{M}). Let $d: S \times S \to [0,1]$. The approximation for \mathcal{M} in the dual sense is $\mathcal{M}^d_{\#}: \mathbf{2}^{[S \times S]^d} \to \mathbf{2}^{[S \times S]^{\mathcal{M}(d)}}$ with

$$\mathcal{M}^{d}_{\#}(X) = \{(s,t) \in [S \times S]^{\mathcal{M}(d)} \mid d_{L}(\ell(s),\ell(t)) < \mathcal{H}(\mathcal{K}(d))(\eta(s),\eta(t)) \land (\eta(s),\eta(t)) \in \mathcal{H}^{\mathcal{K}(d)}_{\#} \circ \mathcal{K}^{d}_{\#}(X) \}$$

Proof. Let $d: S \times S \to [0,1]$ and $X \subseteq [S \times S]^d$. We abbreviate $g = (\eta \times \eta)^* \circ \mathcal{H} \circ \mathcal{K} \colon [0,1]^{S \times S} \to [0,1]^{S \times S}$ and $j = (\ell \times \ell)^* \circ c_{d_L} \colon [0,1]^{S \times S} \to [0,1]^{S \times S}$, so that $\mathcal{M} = \max_{\rho} \circ (g \uplus j)$. Thus we obtain

$$\mathcal{M}^d_{\#}(X) = (\max_{\rho})^{(g \uplus j)(d)}_{\#} \circ (g \uplus j)^d_{\#}(X)$$

Since $c_{d_L} : [0,1]^{S \times S} \to [0,1]^{L \times L}$ is a constant function and $((\ell \times \ell)^*)^{c_{d_L}(d)}_{\#} = (\ell \times \ell)^{-1}$, we deduce that

$$j_{\#}^{d}(X) = ((\ell \times \ell)^{*})_{\#}^{c_{d_{L}}(d)} \circ (c_{d_{L}})_{\#}^{d}(X) = (\ell \times \ell)^{-1}(\emptyset) = \emptyset$$

On the other hand

$$g^d_{\#} = \left((\eta \times \eta)^* \right)^{\mathcal{H}(\mathcal{K}(d))}_{\#} \circ \mathcal{H}^{\mathcal{K}(d)}_{\#} \circ \mathcal{K}^d_{\#}$$

where

$$\mathcal{H}_{\#}^{\mathcal{K}(d)} \circ \mathcal{K}_{\#}^{d} \colon \mathbf{2}^{[S \times S]^{d}} \to \mathbf{2}^{[\mathbf{2}^{S} \times \mathbf{2}^{S}]^{\mathcal{H}(\mathcal{K}(d))}}$$
$$((\eta \times \eta)^{*})_{\#}^{\mathcal{H}(\mathcal{K}(d))} \colon \mathbf{2}^{[\mathbf{2}^{S} \times \mathbf{2}^{S}]^{\mathcal{H}(\mathcal{K}(d))}} \to \mathbf{2}^{[S \times S]^{g(d)}}$$

We recall that $((\eta \times \eta)^*)_{\#}^{\mathcal{H}(\mathcal{K}(d))} = (\eta \times \eta)^{-1}$, and hence

$$(s,t) \in g^d_{\#}(X) \Leftrightarrow (\eta(s),\eta(t)) \in \mathcal{H}^{\mathcal{K}(d)}_{\#} \circ \mathcal{K}^d_{\#}(X)$$

Lastly, we obtain

$$\mathcal{M}^{d}_{\#}(X) = (\max_{\rho})^{(g \uplus j)(d)}_{\#} \circ (g \uplus j)^{d}_{\#}(X)$$

= $(\max_{\rho})^{(g \uplus j)(d)}_{\#} (g^{d}_{\#}(X) \uplus j^{d}_{\#}(X))$
= $\{(s,t) \in [S \times S]^{\mathcal{M}(d)} \mid \arg\max_{y \in \rho^{-1}(s,t)} (g \uplus j)(d)(y) \subseteq g^{d}_{\#}(X) \times \{0\}\}$

Recalling that $\rho^{-1}(s,t) = \{((s,t),0), ((s,t),1)\}$, the inclusion

$$\arg \max_{y \in \rho^{-1}(s,t)} (g \uplus j)(d)(y) \subseteq g^d_{\#}(X) \times \{0\}$$

can only hold if g(d)(s,t) > j(d)(s,t) (and hence the maximum is achieved by g(d) instead of j(d)) and additionally $((s,t),0) \in g^d_{\#}(X) \times \{0\}$. Thus

$$\mathcal{M}^{d}_{\#}(X) = \{(s,t) \in [S \times S]^{\mathcal{M}(d)} \mid j(d)(s,t) < g(d)(s,t) \land (s,t) \in g^{d}_{\#}(X)\} \\ = \{(s,t) \in [S \times S]^{\mathcal{M}(d)} \mid d_{L}(\ell(s),\ell(t)) < \mathcal{H}(\mathcal{K}(d))(\eta(s),\eta(t)) \\ \land (\eta(s),\eta(t)) \in \mathcal{H}^{\mathcal{K}(d)}_{\#} \circ \mathcal{K}^{d}_{\#}(X)\} \qquad \Box$$

Proposition 6.11. Let $d: S \times S \to [0,1]$ where $d = \mathcal{M}(d)$. Then $\mathcal{M}^d_{\#}: \mathbf{2}^{[S \times S]^d} \to \mathbf{2}^{[S \times S]^d}$, where $[S \times S]^d = \{(s,t) \in S \times S \mid d(s,t) > 0\}.$

Then M is a self-closed relation with respect to d if and only if $M \subseteq [S \times S]^d$ and M is a post-fixpoint of $\mathcal{M}^d_{\#}$.

Proof. First note that whenever M is self-closed, it holds that d(s,t) > 0 for all $(s,t) \in M$ and hence $M \subseteq [S \times S]^d$.

Observe that whenever $\ell(s) \neq \ell(t)$, we would have $d_L(\ell(s), \ell(t)) = 1 \geq \mathcal{H}(\mathcal{K}(d))(\eta(s), \eta(t))$. On the other hand, when $\ell(s) = \ell(t)$, instead, we have $d_L(\ell(s), \ell(t)) = 0 < \mathcal{H}(\mathcal{K}(d))(\eta(s), \eta(t))$, since $\mathcal{H}(\mathcal{K}(d))(\eta(s), \eta(t)) > 0$ for all $(s, t) \in [S \times S]^d$. So, by Lemma 6.9, we obtain that

$$\mathcal{M}^{d}_{\#}(M) = \{(s,t) \in [S \times S]^{d} \mid d_{L}(\ell(s),\ell(t)) < \mathcal{H}(\mathcal{K}(d))(\eta(s),\eta(t)) \\ \wedge (\eta(s),\eta(t)) \in \mathcal{H}^{\mathcal{K}(d)}_{\#} \circ \mathcal{K}^{d}_{\#}(M) \} \\ = \{(s,t) \in [S \times S]^{d} \mid \ell(s) = \ell(t) \land (\eta(s),\eta(t)) \in \mathcal{H}^{\mathcal{K}(d)}_{\#} \circ \mathcal{K}^{d}_{\#}(M) \} \\ = \{(s,t) \in S \times S \mid d(s,t) > 0 \land \ell(s) = \ell(t) \land (\eta(s),\eta(t)) \in \mathcal{H}^{\mathcal{K}(d)}_{\#} \circ \mathcal{K}^{d}_{\#}(M) \}$$

Using the characterisation of the associated approximation of the Hausdorff lifting in Lemma 6.7, we obtain that this is equivalent to

for all $p \in \eta(s)$, whenever $\min_{q' \in \eta(t)} \mathcal{K}(d)(p,q') = \mathcal{H}(\mathcal{K}(d))(\eta(s),\eta(t))$, then there exists $q \in \eta(t)$ such that $(p,q) \in \mathcal{K}^d_{\#}(M)$ and $\mathcal{K}(d)(p,q) = \mathcal{H}(\mathcal{K}(d))(\eta(s),\eta(t))$ (and vice versa),

assuming that $\ell(s) = \ell(t)$ (this is a requirement in the definition of $\mathcal{M}^d_{\#}(M)$), since then we have $\mathcal{H}(\mathcal{K}(d))(\eta(s), \eta(t)) = d(s, t) > 0$ and hence $(\eta(s), \eta(t)) \in [\mathbf{2}^D \times \mathbf{2}^D]^{\mathcal{H}(\mathcal{K}(d))}$. Since also $d = \mathcal{M}(d)$, the condition above can be rewritten to

for all $p \in \eta(s)$, whenever $\min_{q' \in \eta(t)} \mathcal{K}(d)(p,q') = d(s,t)$, then there exists $q \in \eta(t)$ such that $(p,q) \in \mathcal{K}^d_{\#}(M)$ and $\mathcal{K}(d)(p,q) = d(s,t)$ (and vice versa).

From Lemma 6.5 we know that $(p,q) \in \mathcal{K}^d_{\#}(M)$ iff $\mathcal{K}(d)(p,q) > 0$ and there exists $c \in \Omega(p,q)$ such that $supp(c) \subseteq M$ and $\sum_{u,v \in S} c(u,v) \cdot d(u,v) = \mathcal{K}(d)(p,q)$. We instantiate the condition above accordingly and obtain

for all $p \in \eta(s)$, whenever $d(s,t) = \min_{q' \in \eta(t)} \mathcal{K}(d)(p,q')$, then there exists $q \in \eta(t)$ such that there exists $c \in \Omega(p,q)$ with $supp(c) \subseteq M$, $\mathcal{K}(d)(p,q) = \sum_{u,v \in S} c(u,v) \cdot d(u,v)$ and $\mathcal{K}(d)(p,q) = d(s,t)$ (and vice versa).

The two last equalities can be simplified to $d(s,t) = \sum_{u,v \in S} c(u,v) \cdot d(u,v)$, since

$$\mathcal{K}(d)(p,q) \le \sum_{u,v \in S} c(u,v) \cdot d(u,v) = d(s,t) = \min_{q' \in \eta(t)} \mathcal{K}(d)(p,q') \le \mathcal{K}(d)(p,q)$$

and hence $\mathcal{K}(d)(p,q) = d(s,t)$ can be inferred from the remaining conditions.

We finally obtain the following equivalent characterisation:

for all $p \in \eta(s)$, whenever $d(s,t) = \min_{q' \in \eta(t)} \mathcal{K}(d)(p,q')$, then there exists $q \in \eta(t)$ such that there exists $c \in \Omega(p,q)$ with $supp(c) \subseteq M$, $d(s,t) = \sum_{u,v \in S} c(u,v) \cdot d(u,v)$ (and vice versa).

Hence we obtain that $(\eta(s), \eta(t)) \in \mathcal{H}_{\#}^{\mathcal{K}(d)} \circ \mathcal{K}_{\#}^{d}(M)$ is equivalent to the second and third item of Def. 6.10 (under the assumption that $\ell(s) = \ell(t)$), while the first item is covered by the other conditions $(d(s,t) > 0 \text{ and } \ell(s) = \ell(t))$ in the characterisation of $\mathcal{M}_{\#}^{d}(M)$.

Lemma E.1. The approximation for the adapted Hausdorff lifting \mathcal{G} in the primal sense is as follows. Let $a: X \times X \to \{0,1\}$, then $\mathcal{G}_a^{\#}: \mathbf{2}^{[X \times X]_a} \to \mathbf{2}^{[\mathbf{2}^X \times \mathbf{2}^X]_a}$ with

$$\mathcal{G}_{a}^{\#}(R) = \{ (X_{1}, X_{2}) \in [\mathbf{2}^{X} \times \mathbf{2}^{X}]_{\mathcal{H}(a)} \mid \\ \forall x_{1} \in X_{1} \exists x_{2} \in X_{2} \colon ((x_{1}, x_{2}) \notin [X \times X]_{a} \lor (x_{1}, x_{2}) \in R) \\ \land \forall x_{2} \in X_{2} \exists x_{1} \in X_{1} \colon ((x_{1}, x_{2}) \notin [X \times X]_{a} \lor (x_{1}, x_{2}) \in R) \}$$

Proof. We rely on the characterisation of $\mathcal{H}^a_{\#}$ (dual case) of Lemma 6.7 and we examine the case where $\mathbb{M} = \{0, 1\}$. In this case, whenever we have $(X_1, X_2) \in [\mathbf{2}^X \times \mathbf{2}^X]^{\mathcal{H}(a)}$ it must necessarily hold that $\mathcal{H}(a)(X_1, X_2) = 1$. Hence, the first part of the conjunction simplifies to:

$$\forall x_1 \in X_1 \big(\min_{x_2' \in X_2} a(x_1, x_2') = 1 \Rightarrow \exists x_2 \in X_2 \colon (x_1, x_2) \in R \land a(x_1, x_2) = 1 \big),$$

from which we can omit $a(x_1, x_2) = 1$ from the conclusion, since this holds automatically. Furthermore $\min_{x'_2 \in X_2} a(x_1, x'_2) = 1$ can be rewritten to $\forall x_2 \in X_2 : a(x_1, x_2) = 1$. This gives us:

$$\forall x_1 \in X_1 (\neg \forall x_2 \in X_2 : a(x_1, x_2) = 1 \lor \exists x_2 \in X_2 : (x_1, x_2) \in R)$$

$$\equiv \forall x_1 \in X_1 (\exists x_2 \in X_2 : a(x_1, x_2) = 0 \lor \exists x_2 \in X_2 : (x_1, x_2) \in R)$$

$$\equiv \forall x_1 \in X_1 \exists x_2 \in X_2 ((x_1, x_2) \notin [X \times X]^a \lor (x_1, x_2) \in R).$$

Since this characterisation is independent of the order, we can replace $[X \times X]^a$ by $[X \times X]_a$ and obtain a characterizing condition for $\mathcal{G}_a^{\#}$ (primal case).

Lemma 6.13. Bisimilarity on η is the greatest fixpoint of $\mathcal{B} = (\eta \times \eta)^* \circ \mathcal{G}$.

Proof. Let for $a: X \times X \to \{0, 1\}, x, y \in X$. Then we have

$$\begin{aligned} (\eta \times \eta)^* \circ \mathcal{G}(a)(x,y) &= \mathcal{G}(a)(\eta(x),\eta(y)) \\ &= \max_u(\min_{\in}(a))(\eta(x),\eta(y)) \\ &= \max_{u(C)=(\eta(x),\eta(y))}(\min_{\in}^{X \times X}(a))(C) \\ &= \max_{u(C)=(\eta(x),\eta(y))}\min_{(x',y') \in C} a(x',y') \end{aligned}$$

Now we prove that this, indeed, corresponds with the standard bisimulation function, i.e. $\max_{u(C)=(\eta(x),\eta(y))} \min_{(x',y')\in C} a(x',y') = 1$ if and only if for all $x' \in \eta(x)$ there exists $y' \in \eta(y)$ such that a(x',y') = 1 and vice versa. For the first implication, assume that $\max_{u(C)=(\eta(x),\eta(y))} \min_{(x',y')\in C} a(x',y') = 1$. This means that there exists $C \subseteq X \times X$ such that $u(C) = (\pi_1(C), \pi_2(C)) = (\eta(x), \eta(y))$ and $\min_{(x',y')\in C} a(x',y') = 1$. Then we have two cases. Either $C = \emptyset$, which means that $\eta(x) = \eta(y) = \emptyset$, that is, x and y have no successors, and so the bisimulation property vacuously holds. Otherwise, $C \neq \emptyset$, and we must have a(x', y') = 1 for all $(x', y') \in C$. Then, since $(\pi_1(C), \pi_2(C)) = (\eta(x), \eta(y))$, for all $x' \in \eta(x)$ there must exists $y' \in \eta(y)$ such that $(x', y') \in C$, and thus a(x', y') = 1. Vice versa, for all $y' \in \eta(y)$ there must exists $x' \in \eta(x)$ such that $(x', y') \in C$, and thus a(x', y') = 1. So the bisimulation property holds.

For the other implication, assume that for all $x' \in \eta(x)$ there exists $y' \in \eta(y)$ such that a(x', y') = 1 and call $c_1(x')$ such a y'. Vice versa, assume also that for all $y' \in \eta(y)$ there exists $x' \in \eta(x)$ such that a(x', y') = 1 and call $c_2(y')$ such a x'. This means that for all $x' \in \eta(x)$ and $y' \in \eta(y)$, we have $a(x', c_1(x')) = a(c_2(y'), y') = 1$. Now let $C' = \{(x', y') \in \eta(x) \times \eta(y) \mid c_1(x') = y' \lor x' = c_2(y')\}$. Since we assumed that for all $x' \in \eta(x)$ there exists $y' \in \eta(y)$ such that $c_1(x') = y' \lor x' = c_2(y')\}$. Since we assumed that for all $x' \in \eta(x)$ there exists $y' \in \eta(y)$, thus $\pi_2(C') = \eta(y)$. Therefore, we know that $u(C') = (\eta(x), \eta(y))$, and we can conclude by showing that a(x', y') = 1 for all $(x', y') \in C'$, in which case also $\max_{u(C)=(\eta(x),\eta(y))} \min_{(x',y')\in C} a(x', y') = 1$. By definition of C' either $c_1(x') = y'$ or $x' = c_2(y')$, or both, must hold. Assume the first one holds, the other case is similar. Then, we can immediately conclude since by hypothesis we know that $a(x', c_1(x')) = 1$.

Since we proved that the function \mathcal{B} is the same of the standard bisimulation function, then its greatest fixpoint $\nu \mathcal{B}$ is the bisimilarity on η .

Lemma 6.14. Let $a: X \times X \to \{0, 1\}$. The approximation for the bisimilarity function \mathcal{B} in the primal sense is $\mathcal{B}_a^{\#}: \mathbf{2}^{[X \times X]_a} \to \mathbf{2}^{[X \times X]_{\mathcal{B}(a)}}$ with

$$\mathcal{B}_{a}^{\#}(R) = \{(x_{1}, x_{2}) \in [X \times X]_{\mathcal{B}(a)} \mid$$

Proof. From Lemma 6.7 we know that

$$\begin{aligned} \mathcal{G}_a^{\#} \colon [X \times X]_a &\to [\mathbf{2}^X \times \mathbf{2}^X]_{\mathcal{G}(a)} \\ \mathcal{G}_a^{\#}(R) &= \{(X_1, X_2) \in [\mathbf{2}^X \times \mathbf{2}^X]_{\mathcal{G}(a)} \mid \\ &\quad \forall x_1 \in X_1 \exists x_2 \in X_2 \colon ((x_1, x_2) \notin [X \times X]_a \lor (x_1, x_2) \in R) \\ &\quad \land \forall x_2 \in X_2 \exists x_1 \in X_1 \colon ((x_1, x_2) \notin [X \times X]_a \lor (x_1, x_2) \in R) \}. \end{aligned}$$

Furthermore

$$((\eta \times \eta)^*)_{\mathcal{G}(a)}^{\#} \colon [\mathbf{2}^X \times \mathbf{2}^X]_{\mathcal{G}(a)} \to [X \times X]_{\mathcal{B}(a)}$$
$$((\eta \times \eta)^*)_{\mathcal{G}(a)}^{\#}(Q) = (\eta \times \eta)^{-1}(Q)$$

Composing these functions we obtain:

$$\mathcal{B}_{a}^{\#}: [X \times X]_{a} \to [X \times X]_{\mathcal{B}(a)} \\
\mathcal{B}_{a}^{\#}(R) = (\eta \times \eta)^{-1}(\{(Y_{1}, Y_{2}) \in [\mathbf{2}^{X} \times \mathbf{2}^{X}]_{\mathcal{G}(a)} \mid \\
\forall y_{1} \in Y_{1} \exists y_{2} \in Y_{2} \colon ((y_{1}, y_{2}) \notin [X \times X]_{a} \lor (y_{1}, y_{2}) \in R) \\
\land \forall y_{2} \in Y_{2} \exists y_{1} \in Y_{1} \colon ((y_{1}, y_{2}) \notin [X \times X]_{a} \lor (y_{1}, y_{2}) \in R)\}) \\
= \{(x_{1}, x_{2}) \in [X \times X]_{\mathcal{B}(a)} \mid \\
\forall y_{1} \in \eta(x_{1}) \exists y_{2} \in \eta(x_{2}) \colon ((y_{1}, y_{2}) \notin [X \times X]_{a} \lor (y_{1}, y_{2}) \in R) \\
\land \forall y_{2} \in \eta(x_{2}) \exists y_{1} \in \eta(x_{1}) \colon ((y_{1}, y_{2}) \notin [X \times X]_{a} \lor (y_{1}, y_{2}) \in R)\}.$$

APPENDIX F. PROOFS AND ADDITIONAL MATERIAL FOR §7 (SIMPLE STOCHASTIC GAMES) Lemma 7.2. $\mathcal{V} = (\eta_{\min}^* \circ \min_{\epsilon}) \uplus (\eta_{\max}^* \circ \max_{\epsilon}) \uplus (\eta_{av}^* \circ av_D) \uplus c_w$, where $\epsilon \subseteq V \times \mathbf{2}^V$ is the "is-element-of"-relation on V.

Proof. Let $a: V \to [0,1]$. For $v \in MAX$ we have

$$\mathcal{V}(a)(v) = (\eta_{\max}^* \circ \max_{\in})(a)(v) = \max_{\in}(a)(\eta_{\max}(v)) = \max_{v' \in \eta_{\max}(v)} a(v').$$

For $v \in MIN$ we have

$$\mathcal{V}(a)(v) = (\eta_{\min}^* \circ \min_{\in})(a)(v) = \min_{\in}(a)(\eta_{\min}(v)) = \min_{v' \in \eta_{\min}(v)} a(v').$$

For $v \in AV$ we have

$$\mathcal{V}(a)(v) = (\eta_{\mathrm{av}}^* \circ \mathrm{av}_D)(a)(v) = \mathrm{av}_D(a)(\eta_{\mathrm{av}}(v)) = \sum_{v' \in V} \eta_{\mathrm{av}}(v)(v') \cdot a(v').$$

For $v \in SINK$ we have $\mathcal{V}(a)(v) = c_w(a)(v) = w(v)$.

Lemma 7.3. Let $a: V \to [0,1]$. The approximation for the value iteration function \mathcal{V} in the dual sense is $\mathcal{V}^a_{\#}: \mathbf{2}^{[V]^a} \to \mathbf{2}^{[V]^{\mathcal{V}(a)}}$ with

$$\mathcal{V}^{a}_{\#}(V') = \{ v \in [V]^{\mathcal{V}(a)} \mid \left(v \in MIN \land \arg\min_{v' \in \eta_{\min}(v)} a(v') \cap V' \neq \emptyset \right) \lor$$

$$\left(v \in MAX \land \arg\max_{v' \in \eta_{\max}(v)} a(v') \subseteq V'\right) \lor \left(v \in AV \land supp(\eta_{\mathrm{av}}(v)) \subseteq V'\right)\right\}$$

Proof. Let $a: V \to [0,1]$ and $V' \subseteq [V]^a$. By Proposition D.5 we have:

$$\mathcal{V}^{a}_{\#}(V') = \left(MIN \cap (\eta^{*}_{\min} \circ \min_{\epsilon})^{a}_{\#}(V')\right) \cup \left(MAX \cap (\eta^{*}_{\max} \circ \max_{\epsilon})^{a}_{\#}(V')\right) \cup \left(AV \cap (\eta^{*}_{\mathrm{av}} \circ \operatorname{av}_{D})^{a}_{\#}(V')\right) \cup \left(SINK \cap (c_{w})^{a}_{\#}(V')\right)$$

It holds that $(\eta_{\min}^*)_{\#}^{\min \in (v)} = \eta_{\min}^{-1}, \ (\eta_{\max}^*)_{\#}^{\max \in (v)} = \eta_{\max}^{-1} \text{ and } (\eta_{\mathrm{av}}^*)_{\#}^{\mathrm{av}_D(v)} = \eta_{\mathrm{av}}^{-1}$. Using previous results (Proposition D.2) we deduce

$$v \in (\eta_{\min}^* \circ \min_{\varepsilon})^a_{\#}(V') \Leftrightarrow \eta_{\min}(v) \in (\min_{\varepsilon})^a_{\#}(V') \Leftrightarrow \arg\min_{v' \in \eta_{\min}(v)} a(v') \cap V' \neq \emptyset$$
$$v \in (\eta_{\max}^* \circ \max_{\varepsilon})^a_{\#}(V') \Leftrightarrow \eta_{\max}(v) \in (\max_{\varepsilon})^a_{\#}(V') \Leftrightarrow \arg\max_{v' \in \eta_{\max}(v)} a(v') \subseteq V'$$
$$v \in (\eta^*_{\mathrm{av}} \circ \operatorname{av}_D)^a_{\#}(V') \Leftrightarrow \eta_{\mathrm{av}}(v) \in (\operatorname{av}_D)^a_{\#}(V') \Leftrightarrow supp(\eta_{\mathrm{av}}(v)) \subseteq V'$$

Lastly $(c_w)^a_{\#}(V') = \emptyset$ for any $V' \subseteq V$ since c_w is a constant function which concludes the proof.

Lemma 7.4. For every Max-strategy σ and every Min-strategy τ it holds that $\mathcal{V}_{\sigma} \leq \mathcal{V} \leq \mathcal{V}_{\tau}$. Proof. Given any $a: V \to [0, 1]$ and $v \in V$, we have

$$\mathcal{V}_{\sigma}(a)(v) = \begin{cases} \min_{v' \in \eta_{\min}(v)} a(v') & v \in MIN \\ a(\sigma(v')) & v \in MAX \\ \sum_{v' \in V} a(v') \cdot \eta_{av}(v)(v') & v \in AV \\ c_w(v) & v \in SINK \end{cases}$$
$$\leq \begin{cases} \min_{v' \in \eta_{\min}(v)} a(v') & v \in MIN \\ \max_{v' \in \eta_{\max}(v)} a(v') & v \in MAX \\ \sum_{v' \in V} a(v') \cdot \eta_{av}(v)(v') & v \in AV \\ c_w(v) & v \in SINK \end{cases}$$
$$= \mathcal{V}(a)(v)$$

The same proof idea can be applied to show $\mathcal{V} \leq \mathcal{V}_{\tau}$.

Lemma 7.5. The least fixpoints of \mathcal{V}_{τ} and \mathcal{V}_{σ} can be determined by solving linear programs. *Proof.* We adapt the linear programs found in the literature on simple stochastic games (see e.g. [Con92]).

The least fixpoint $a = \mu \mathcal{V}_{\tau}$ can be determined by solving the following linear program:

$$\min \sum_{v \in V} a(v)$$

$$a(v) = a(\tau(v)) \qquad v \in MIN$$

$$a(v) \ge a(v') \qquad \forall v' \in \eta_{\max}(v), v \in MAX$$

$$a(v) = \sum_{v' \in V} a(v') \cdot \eta_{av}(v)(v') \qquad v \in AV$$

$$a(v) = w(v) \qquad v \in SINK$$

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By having $a(v) \ge a(v')$ for all $v' \in \eta_{\max}(v)$ and $v \in MAX$ we guarantee $a(v) = \max_{v' \in \eta_{\max}(v)}$ $a^{(i)}(v')$ since we minimise. The minimisation also guarantees computation of the least fixpoint (in particular, nodes that lie on a cycle will get a value of 0). Hence, the linear program correctly characterises $\mu \mathcal{V}_{\tau}$.

Given a strategy σ for Max, we can determine $a = \mu \mathcal{V}_{\sigma}$ by solving the following linear program:

$$\max \sum_{v \in V} a(v)$$

$$a(v) = 0 \qquad v \in C_{\sigma}$$

$$a(v) \le a(v') \qquad \forall v' \in \eta_{\min}(v), v \in MIN, v \notin C_{\sigma}$$

$$a(v) = a(\sigma(v)) \qquad v \in MAX, v \notin C_{\sigma}$$

$$a(v) = \sum_{v' \in V} a(v') \cdot \eta_{av}(v)(v') \qquad v \in AV, v \notin C_{\sigma}$$

$$a(v) = w(v) \qquad v \in SINK$$

The set C_{σ} contains those nodes which will guarantee a non-terminating play if Min plays optimally, given the fixed Max-strategy σ .

The set C_{σ} can again be computed via fixpoint-iteration by computing the greatest fixpoint of c_{σ} via Kleene iteration on $\mathbf{2}^{V}$ from above:

$$c_{\sigma} \colon \mathbf{2}^{V} \to \mathbf{2}^{V}$$

$$c_{\sigma}(V') = \{ v \in V \mid (v \in MIN \land \eta_{\min}(v) \cap V' \neq \emptyset) \lor (v \in MAX \land \sigma(v) \in V') \land (v \in AV \land supp(\eta_{av}(v)) \subseteq V') \}$$

It is easy to see that $C_{\sigma} = \nu c_{\sigma}$ contains all those nodes from which Min can force a non-terminating play and hence achieve payoff 0. (Note that there are further nodes that guarantee payoff 0 – namely sinks with that payoff and nodes which can reach such sinks – but those will obtain value 0 in any case.)

We now show that this linear program computes $\mu \mathcal{V}_{\sigma}$: first, by requiring $a(v) \leq a(v')$ for all $v \in MIN$, $v' \in \eta_{\min}(v)$, we guarantee $a(v) = \min_{v' \in \eta_{\min}} a(v')$ since we maximise. Hence we obtain the greatest fixpoint of the following function $\mathcal{V}'_{\sigma} : [0,1]^V \to [0,1]^V$:

$$\mathcal{V}'_{\sigma}(a)(v) = \begin{cases} 0 & v \in C_{\sigma} \\ \sum_{v' \in V} a(v') \cdot \eta_{\mathrm{av}}(v)(v') & v \in AV, v \notin C_{\sigma} \\ a(\sigma(v)) & v \in MAX, v \notin C_{\sigma} \\ \min_{v' \in \eta_{\min}(v)} a(v') & v \in MIN, v \notin C_{\sigma} \\ w(v) & v \in SINK \end{cases}$$

It is easy to show that the least fixpoints of \mathcal{V}'_{σ} and \mathcal{V}_{σ} agree, i.e., $\mu \mathcal{V}'_{\sigma}$ and $\mu \mathcal{V}_{\sigma}$:

- $\mu \mathcal{V}'_{\sigma} \leq \mu \mathcal{V}_{\sigma}$ can be shown by observing that $\mathcal{V}'_{\sigma} \leq \mathcal{V}_{\sigma}$. $\mu \mathcal{V}_{\sigma} \leq \mu \mathcal{V}'_{\sigma}$ can be shown by proving that $\mu \mathcal{V}'_{\sigma}$ is a pre-fixpoint of \mathcal{V}_{σ} , which can be done via a straightforward case analysis.
 - We have to show $\mathcal{V}_{\sigma}(\mu \mathcal{V}'_{\sigma})(v) \leq \mu \mathcal{V}'_{\sigma}(v)$ for all $v \in V$. We only spell out the case where $v \in AV$, the other cases are similar. In this case either $v \notin C_{\sigma}$, which means that

$$\mathcal{V}_{\sigma}(\mu \mathcal{V}_{\sigma}')(v) = \mathcal{V}_{\sigma}'(\mu \mathcal{V}_{\sigma}')(v) = \mu \mathcal{V}_{\sigma}'(v).$$

If instead $v \in C_{\sigma}$, we have that $supp(\eta_{av}(v)) \subseteq C_{\sigma}$ and so $\mu \mathcal{V}'_{\sigma}(v') = 0$ for all $v' \in supp(\eta_{av}(v))$. Hence

$$\mathcal{V}_{\sigma}(\mu\mathcal{V}_{\sigma}')(v) = \sum_{v' \in V} \eta_{\mathrm{av}}(v)(v') \cdot \mu\mathcal{V}_{\sigma}'(v') = 0 = \mu\mathcal{V}_{\sigma}'(v)$$

If we can now show that \mathcal{V}'_{σ} has a unique fixpoint, we are done. The argument for this goes as follows: assume that this function has another fixpoint a' different from $\mu \mathcal{V}'_{\sigma}$. Clearly $[V]^{a'} \cap C_{\sigma} = \emptyset$, where $[V]^{a'} = \{v \in V \mid a'(v) \neq 0\}$. Hence, if we compare $(\mathcal{V}'_{\sigma})^a_{\#} : \mathbf{2}^{[V]^a} \to \mathbf{2}^{[V]^{\mathcal{V}'_{\sigma}(a)}}$ (defined analogously to Lemma 7.3) and c_{σ} above, we observe that $(\mathcal{V}'_{\sigma})^{a'}_{\#} \subseteq c_{\sigma}|_{\mathbf{2}^{[V]^{a'}}}$. (Both functions coincide, apart from their treatment of nodes $v \in MIN$, where $c_{\sigma}(V')$ contains v whenever one of its successors is contained in V', whereas $(\mathcal{V}'_{\sigma})^{a'}_{\#}(V')$ additionally requires that the value of this successor is minimal.) Since a' is not the least fixpoint we have by Theorem 4.2 that

$$\emptyset \neq \nu(\mathcal{V}'_{\sigma})^{a'}_{\#} \subseteq \nu(c_{\sigma}|_{\mathbf{2}^{[V]^{a'}}}) \subseteq \nu c_{\sigma} = C_{\sigma}.$$

This is a contradiction, since $[V]^{a'} \cap C_{\sigma} = \emptyset$ as observed above.

This shows that \mathcal{V}'_{σ} has a unique fixpoint and completes the proof. Note that if we do not explicitly require that the values of all nodes in C_{σ} are 0, \mathcal{V}'_{σ} will potentially have several fixpoints and the linear program would not characterise the least fixpoint.