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A MONOIDAL VIEW ON FIXPOINT CHECKS

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Abstract. Fixpoints are ubiquitous in computer science as they play a central role in providing a meaning to recursive and cyclic definitions. Bisimilarity, behavioural metrics, termination probabilities for Markov chains and stochastic games are defined in terms of least or greatest fixpoints. Here we show that our recent work which proposes a technique for checking whether the fixpoint of a function is the least (or the largest) admits a natural categorical interpretation in terms of gs-monoidal categories. The technique is based on a construction that maps a function to a suitable approximation. We study the compositionality properties of this mapping and show that under some restrictions it can naturally be interpreted as a (lax) gs-monoidal functor. This guides the development of a tool, called UDEfix that allows us to build functions (and their approximations) like a circuit out of basic building blocks and subsequently perform the fixpoints checks. We also show that a slight generalisation of the theory allows one to treat a new relevant case study: coalgebraic behavioural metrics based on Wasserstein liftings.

1. Introduction

Fixpoints are fundamental in computer science: extremal (least or greatest) fixpoints are commonly used to interpret recursive and cyclic definitions. In a recent work [BEKP21, BEKP23a] we proposed a technique for checking whether the fixpoint of a function is the least (or the largest). In this paper we show that such technique admits a natural categorical interpretation in terms of gs-monoidal categories. This allows us to provide a compositional flavour to our technique and enables the realisation of a corresponding tool UDEfix.

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The theory in [BEKP21, BEKP23a] can be used in a variety of fairly diverse application scenarios, such as bisimilarity [San11], behavioural metrics [DGJP04, vB17, CvBW12, BBKK18], termination probabilities for Markov chains [BK08] and simple stochastic games [Con90]. It applies to non-expansive functions of the kind $f\colon \mathbb{M}^Y \to \mathbb{M}^Y$, where $\mathbb{M}$ is a set of values and Y is a finite set. More precisely, the set of values $\mathbb{M}$ is an MV-chain, i.e., a totally ordered complete lattice endowed with suitable operations of sum and complement, which allow for the definition of a natural notion of non-expansiveness for $\mathbb{M}$-valued functions. A prototypical example of MV-algebra, largely used in the examples, is the interval $[0,1]$ with truncated sum. Roughly, the idea consists in mapping the semantic functions of interest $f\colon \mathbb{M}^Y \to \mathbb{M}^Y$ to corresponding approximations $f^\#$ over (a subset of) $\mathcal{P}(Y)$, the powerset of Y. While $\mathbb{M}$ is in general infinite (or very large), the set Y is typically finite in a way that the fixpoints of $f^\#$ can be computed effectively and provide information on the fixpoints of the original function. In particular they allow us to decide, whether a given fixpoint is indeed the least one (or dually greatest one).

In this paper, we show that the approximation framework and its compositionality properties can be naturally interpreted in categorical terms using gs-monoidal categories. In essence gs-monoidal categories describe graph-like structures with dedicated input and output interfaces, operators for disjoint union (tensor), duplication and termination of wires, quotiented by appropriate axioms. Particularly useful are gs-monoidal functors that preserve such operators and hence naturally describe compositional operations.

More concretely, we introduce two gs-monoidal categories, that we refer to as the concrete category and the category of approximations, where the concrete functions and their approximations, respectively, live as arrows. We then define a mapping # from the concrete category to the category of approximations and show that, whenever, as in [BEKP23a], we assume finiteness of the underlying set Y, the mapping # turns out to be a gs-monoidal functor.

In the general case, for functions $f\colon \mathbb{M}^Y \to \mathbb{M}^Y$, where Y is not necessarily finite, the mapping # is only known to be the union of lax functors. However, we observe that we can characterise it again as a gs-monoidal functor if we restrict it to the subcategory of the concrete category where arrows are reindexings. While this might seem to be a severe restriction, we will see that it is sufficient to cover the intended applications.

We prove a number of properties of # that enable us to give a recipe for finding approximations for a special type of functions: predicate liftings as those introduced for coalgebraic modal logic [Pat03, Sch08]. This allows us to include a new case study in the machinery for fixpoint checking: coalgebraic behavioural metrics, based on Wasserstein liftings.

Besides shedding further light on the theoretical approximation framework of [BEKP23a], the results in the paper guide the realisation of a tool, called UDEfix which realises a number of fixpoints checks (given a post- resp. pre-fixpoint, is below the least resp. above the greatest fixpoint? If it is a fixpoint, is it the least or the greatest?). Leveraging the visual string diagrammatic language of gs-monoidal categories, UDEfix models system functions as (hyper)graphs whose edges represent suitable basic building blocks. This also extends to approximations, thanks to the fact that the mapping # can be seen as a gs-monoidal functor, enabling a compositional construction of the approximation function.

The paper is organised as follows. In Section 2 we provide some high-level motivation. In Section 3 we introduce some preliminaries and, in particular, we review the theory if fixpoint checks from [BEKP23a]. In Section 4 we introduce two (gs-monoidal) categories,

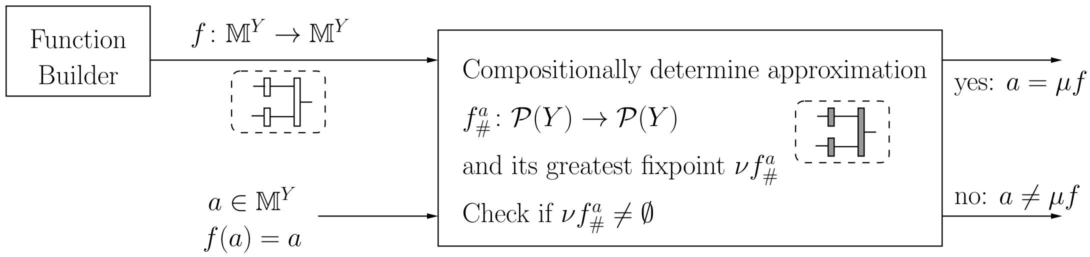


Figure 1: A graphical overview over the fixpoint checking method

$\mathbb{C}$ and $\mathbb{A}$, the so-called concrete category and category of approximations, and investigate the properties of the approximation map $\# : \mathbb{C} \to \mathbb{A}$. In Section 5 we show how to handle predicate liftings and then, in Section 6, we exploit such results to treat behavioural metrics. In Section 7, we show that, in the finitary case, the categories $\mathbb{C}$, $\mathbb{A}$ are indeed gs-monoidal and $\#$ is gs-monoidal functor. The tool UDEfix is discussed in Section 8. Finally, in Section 9 we draw some conclusions.

This article is an extended version of the paper [BEK$^+$23] presented at ICGT 2023. With respect to the conference version, the present paper contains additional explanations and examples and full proofs of the results.

2. Motivation

We start by giving a high-level overview over the approach, followed by a small worked-out example. The general idea follows the upside-down theory of fixpoint checks from [BEKP21]. We are given a monotone function[1] $f\colon [0,1]^Y \to [0,1]^Y$ (later $[0,1]$ will be replaced by a general MV-algebra) and we also assume that f is non-expansive with respect to a suitable norm. By Knaster-Tarski [Tar55] we have a guarantee that f has fixpoints – among them a least fixpoint μf and a greatest fixpoint νf – but there is no guarantee that there is a unique fixpoint, indeed there might be several of them.

Now, given a fixpoint $a \in [0,1]^Y$ of f, our aim is to check whether a is the least fixpoint, i.e., $a = \mu f$. (Or dually, whether it is the greatest fixpoint, i.e., $a = \nu f$.)

The check proceeds by determining from f and a a so-called approximation $f^a_\#\colon \mathcal{P}(Y) \to \mathcal{P}(Y)$. Intuitively, the approximation describes how function f propagates decreases of its argument a: for $Y' \subseteq Y$, the set $f^a_\#(Y')$ contains those elements $y \in Y$ where the value of $f(a)$ decreases by some fixed amount δ whenever we decrease a on $Y' \subseteq Y$ by δ. Then one determines the greatest fixpoint of this approximation ($\nu f^a_\#$) and observes that a coincides with μf if and only if $\nu f^a_\#$ is the empty set. Otherwise a is too large and intuitively there is still "wiggle room" and potential for decrease. This procedure works under conditions that will be made precise later in Section 3.2.

The technique is schematised in Figure 1. A feature of this approach is the fact that the function f can be assembled compositionally from basic functions via composition and disjoint union. Such a decomposition can be represented via a string diagram. Furthermore, approximations can be determined compositionally, leading to a string diagram of the same type, but with approximations instead of concrete functions.

[1]The set of functions from X to Y is denoted by Y^X.

Function	c_k $k\colon Z \to \mathbb{M}$	g^* $g\colon Z \to Y$	$\min_u$ $u\colon Y \to Z$	$\max_u$ $u\colon Y \to Z$	$\mathrm{av}_D = \tilde{\mathcal{D}}$ $\mathbb{M} = [0,1]$, $Z = \mathcal{D}(Y)$
Name	constant	reindexing	minimum	maximum	expectation
$a \mapsto \ldots$	k	$a \circ g$	$\lambda z.\ \min\limits_{u(y)=z} a(y)$	$\lambda z.\ \max\limits_{u(y)=z} a(y)$	$\lambda z.\ \sum\limits_{y \in Y} z(y) \cdot a(y)$

Table 1: Basic functions of type $\mathbb{M}^Y \to \mathbb{M}^Z$, $a\colon Y \to \mathbb{M}$.

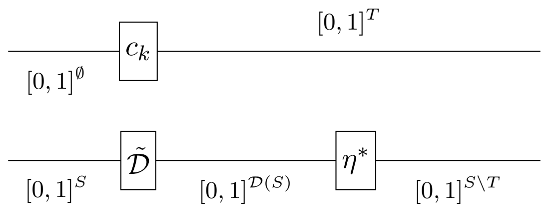


Figure 2: Decomposition of the fixpoint function $\mathcal{T}$ for computing termination probabilities.

We conclude with an example which illustrates the notions and tools discussed above for fixpoint checks. We consider (unlabelled) Markov chains (S, T, η), where S is a finite set of states, $T \subseteq S$ is a set of terminal states and $\eta\colon S\backslash T \to \mathcal{D}(S)$ assigns to each non-terminal state a probability distribution over its successors. We denote by $p_s = \eta(s)$ the probability distribution associated with state s.

We are interested in the termination probability of a given state of the Markov chain, which can be computed by taking the least fixpoint of a function $\mathcal{T}\colon [0,1]^S \to [0,1]^S$:

$$\mathcal{T} : [0,1]^S \to [0,1]^S$$
$$\mathcal{T}(t)(s) = \begin{cases} 1 & \text{if } s \in T \\ \sum\limits_{s' \in S} \eta(s)(s') \cdot t(s') & \text{otherwise} \end{cases}$$

We observe that the function $\mathcal{T}$ can be decomposed as

$$\mathcal{T} = (\eta^* \circ \tilde{D}) \otimes c_k$$

where $\otimes$ stands for disjoint union (over functions) and we use some of the basic functions given in Table 1. We depict this decomposition diagrammatically in Figure 2. For the Markov chain in Figure 3 the parameters instantiate as follows (cf. Table 1):

- $c_k\colon [0,1]^\emptyset \to [0,1]^T$, $k\colon T \to [0,1]$ with $T = \{u\}$ and $k(u) = 1$
- $\tilde{\mathcal{D}}\colon [0,1]^S \to [0,1]^{\mathcal{D}(S)}$
- $\eta^*\colon [0,1]^{\mathcal{D}(S)} \to [0,1]^{S\backslash T}$ where $\eta\colon S\backslash T \to \mathcal{D}(S)$ with $\eta(s) = p_s$, where $p_x(y) = p_x(u) = 1/2$, $p_y(z) = p_z(y) = 1$ and all other values are 0.

Since S is finite we could alternatively restrict to a finite subset D of $\mathcal{D}(S)$.

The function $\mathcal{T}$ is a monotone function on a complete lattice, hence it has a least fixpoint by Knaster-Tarski's fixpoint theorem [Tar55]. Furthermore, it is non-expansive, allowing us to use the fixpoint checking techniques from [BEKP23a].

We will illustrate this on the concrete instance (Markov chain in Figure 3). The state set is $S = \{x, y, u, z\}$ and u is the only terminal state. The least fixpoint $\mu\mathcal{T}$ of $\mathcal{T}$ is given in Figure 3 in green (left) and the greatest fixpoint $\nu\mathcal{T}$ in red (right). These are two of the

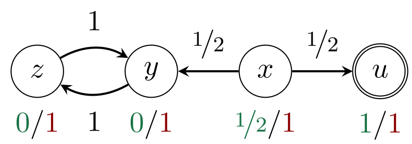


Figure 3: A Markov chain with two fixpoints of $\mathcal{T}$ (right)

infinitely many fixpoints of $\mathcal{T}$. The cycle on the left (including y, z) can be seen as some kind of vicious cycle, where y, z convince each other erroneously that they terminate with a probability that might be too high. Such cycles have a coinductive flavour, hence the computation of the greatest fixpoint of the approximation.

Now let $a = \nu\mathcal{T}$ be the greatest fixpoint of $\mathcal{T}$, i.e., the function $S \to [0,1]$ that maps every state to 1. In this case we can associate $\mathcal{T}$ with an approximation $\mathcal{T}^a_\#$ on subsets of S that records the propagation of decrease. Given $S' \subseteq S$, the approximation $\mathcal{T}^a_\#$ is as follows:

$$\mathcal{T}^a_\#(S') = \{s \in S \mid a(s) > 0, s \notin T, \mathit{supp}(\eta(s)) \subseteq S'\}.$$

where $\mathit{supp}(p)$ denotes the *support* of a probability distribution $p\colon S \to [0,1]$, i.e., the set of states v for which $p(v) > 0$. Intuitively, if we decide to decrease the a-values of all states in S' by some small amount δ, the states in $\mathcal{T}^a_\#(S')$ will also decrease their values by δ after applying $\mathcal{T}$. This is true for all states that have non-zero value, are not terminal and whose successors are *all* included in S'.

In the example $\mathcal{T}^a_\#(\{y,z\}) = \{y,z\}$ and $\{y,z\}$ is indeed the greatest fixpoint of the approximation. Since it is non-empty, we deduce that a is not the least fixpoint. Furthermore we could now subtract a small value (for details on how to obtain this value see [BEKP23a, Proposition 4.5]) from $a(y)$, $a(z)$ to obtain a smaller pre-fixpoint, from where one can continue to iterate to the least fixpoint (see also [BEKP23b]).

We anticipate that in our tool UDEfix we can draw a diagram as in Figure 2, from which the approximation and its greatest fixpoint are automatically computed in a compositional way, allowing us to perform such fixpoint checks. A more complex case study in the domain of behavioural metrics will be provided in Section 6.3.

3. Preliminaries

This section reviews some background used throughout the paper. We first introduce some basics of lattices and MV-algebras, the domain where the functions of interest take values. Then we recap some results from [BEKP23a] useful for detecting if a fixpoint of a given function is the least (or greatest).

3.1. **Lattices and MV-algebras.**

A *partially ordered* set $(P, \sqsubseteq)$ is often denoted simply as P, omitting the order relation. For $x, y \in P$, we write $x \sqsubset y$ when $x \sqsubseteq y$ and $x \neq y$. For a function $f : X \to P$, we will write $\arg\min_{x \in X'} f(x)$ to denote the (possibly empty) set of elements where f reaches the minimum on domain X', i.e.,

$$\arg\min_{x \in X'} f(x) = \{x \in X' \mid \forall y \in X.\, f(x) \sqsubseteq f(y)\}.$$

Abusing the notation, we will write $z = \arg\min_{x \in X} f(x)$ instead of $z \in \arg\min_{x \in X} f(x)$.

Definition 3.1 (complete lattice)**.** A *complete lattice* is a partially ordered set $(\mathbb{L}, \sqsubseteq)$ such that each subset $X \subseteq \mathbb{L}$ admits a join $\bigsqcup X$ and a meet $\bigsqcap X$. A complete lattice $(\mathbb{L}, \sqsubseteq)$ always has a least element $\bot = \bigsqcap \mathbb{L}$ and a greatest element $\top = \bigsqcup \mathbb{L}$.

A prototypical example for a complete lattice is the *powerset* $\mathcal{P}(Y)$ of a given set Y, where the partial order is subset inclusion ($\subseteq$) and join and meet are given by union and intersection. The set of finite subsets of X is written $\mathcal{P}_f(X)$.

A function $f : \mathbb{L} \to \mathbb{L}$ is *monotone* if for all $l, l' \in \mathbb{L}$, if $l \sqsubseteq l'$ then $f(l) \sqsubseteq f(l')$. By Knaster-Tarski's theorem [Tar55, Theorem 1], any monotone function on a complete lattice has a least fixpoint μf and a greatest fixpoint νf, characterised as the meet of all pre-fixpoints $\mu f = \bigsqcap\{l \mid f(l) \sqsubseteq l\}$ and, dually, a greatest fixpoint $\nu f = \bigsqcup\{l \mid l \sqsubseteq f(l)\}$, characterised as the join of all post-fixpoints. We denote by $Fix(f)$ the set of all fixpoints of f.

For a set Y and a complete lattice $\mathbb{L}$, the set of functions $\mathbb{L}^Y = \{f \mid f : Y \to \mathbb{L}\}$ with pointwise order (for $a, b \in \mathbb{L}^Y$, $a \sqsubseteq b$ if $a(y) \sqsubseteq b(y)$ for all $y \in Y$), is a complete lattice.

The semantic functions of interest in the paper will take values on special lattices, induced by a suitable class of commutative monoids with a complement operation.

Definition 3.2 (MV-algebra [Mun])**.** An *MV-algebra* is a tuple $\mathbb{M} = (M, \oplus, 0, \overline{(\cdot)})$ where $(M, \oplus, 0)$ is a commutative monoid and $\overline{(\cdot)} : M \to M$ maps each element to its *complement*, such that, if we let $1 = \overline{0}$ and *subtraction* $x \ominus y = \overline{\overline{x} \oplus y}$, then for all $x, y \in M$ it holds that

(1) $\overline{\overline{x}} = x$;
(2) $x \oplus 1 = 1$;
(3) $(x \ominus y) \oplus y = (y \ominus x) \oplus x$.

MV-algebras can be endowed with a partial order, the so-called *natural order*, defined for $x, y \in M$, by $x \sqsubseteq y$ if $x \oplus z = y$ for some $z \in M$. When $\sqsubseteq$ is total, $\mathbb{M}$ is called an *MV-chain*. We will often write $\mathbb{M}$ instead of M.

The natural order gives an MV-algebra a lattice structure where $\bot = 0$, $\top = 1$, $x \sqcup y = (x \ominus y) \oplus y$ and $x \sqcap y = \overline{\overline{x} \sqcup \overline{y}} = x \ominus (x \ominus y)$. We call the MV-algebra *complete* if it is a complete lattice, which is not true in general, e.g., $([0,1] \cap \mathbb{Q}, \leq)$.

Example 3.3. A prototypical MV-algebra is $([0,1], \oplus, 0, \overline{(\cdot)})$ where $x \oplus y = \min\{x + y, 1\}$, $\overline{x} = 1 - x$ and $x \ominus y = \max\{0, x - y\}$ for $x, y \in [0,1]$. The natural order is $\leq$ (less or equal) on the reals. Another example is $K = (\{0, \ldots, k\}, \oplus, 0, \overline{(\cdot)})$ where $n \oplus m = \min\{n + m, k\}$, $\overline{n} = k - n$ and $n \ominus m = \max\{n - m, 0\}$ for $n, m \in \{0, \ldots, k\}$. Both MV-algebras are complete and MV-chains.

3.2. **A Theory of Fixpoint Checks.**

We briefly recap the theory from [BEKP23a]. Given a function $f : \mathbb{M}^Y \to \mathbb{M}^Y$, where $\mathbb{M}$ is an MV-algebra, the theory provides results useful for checking whether a fixpoint of f is the least or the greatest fixpoint. For gaining some intuition, one can think that, as in Section 2, Y is the set of states of a Markov Chain, $\mathbb{M} = [0,1]$ and functions $\mathbb{M}^Y = [0,1]^Y$ provide the probability of termination of each state. Alternatively, Y could be the set of pairs of states of a system, $\mathbb{M} = [0,1]$ and functions $\mathbb{M}^Y = [0,1]^Y$ provide the behavioural distance between states. The technique is for instance useful in a situation where one can obtain a fixpoint for the function of interest, e.g., via strategy iteration, but it is unclear whether the fixpoint is the least (or the largest).

While the theory in [BEKP23a] was restricted to functions over $\mathbb{M}^Y$ where Y is finite, for the purposes of the present paper we actually need a generalisation working for functions with an infinite domain. Hence, hereafter Y and Z denote possibly infinite sets.

Definition 3.4 (norm and non-expansive functions)**.** Given $a \in \mathbb{M}^Y$ we define its *norm* as $\|a\| = \bigsqcup\{a(y) \mid y \in Y\}$. A function $f : \mathbb{M}^Y \to \mathbb{M}^Z$ is *non-expansive* if for all $a, b \in \mathbb{M}^Y$ it holds $\|f(b) \ominus f(a)\| \sqsubseteq \|b \ominus a\|$.

It can be seen that non-expansive functions are monotone. A number of standard operators are non-expansive (e.g., constants, reindexing, max and min over a relation, average in Table 1), and non-expansiveness is preserved by composition and disjoint union (see [BEKP23a, Theorem 5.2]).

Let $f : \mathbb{M}^Y \to \mathbb{M}^Y$, $a \in \mathbb{M}^Y$. For a non-expansive endo-function $f : \mathbb{M}^Y \to \mathbb{M}^Y$ and $a \in \mathbb{M}^Y$, the theory in [BEKP23a] provides a so-called a-approximation $f_\#^a$ of f, which is an endo-function over a suitable subset of $\mathcal{P}(Y)$. Intuitively, given some Y', the set $f_\#^a(Y')$ contains the points where a decrease of the values of a on the points in Y' "propagates" through the function f. Understanding that no decrease can be propagated allows one to establish when a fixpoint of a non-expansive function f is actually the least one, and, more generally, when a (post-)fixpoint of f is above the least fixpoint.

The above intuition is formalised using tools from abstract interpretation [CC77, CC00]. In particular, we define a pair of functions, which, under suitable conditions, form a Galois connection. For $a, b \in \mathbb{M}^Y$, let $[a, b] = \{c \in \mathbb{M}^Y \mid a \sqsubseteq c \sqsubseteq b\}$ and let

$$[Y]^a = \{y \in Y \mid a(y) \neq 0\}.$$

Intuitively $[Y]^a$ contains those elements where we have "wiggle room", i.e., a decrease is in principle feasible. Then for $0 \sqsubset \delta \in \mathbb{M}$ define $\alpha^{a,\delta} : \mathcal{P}([Y]^a) \to [a \ominus \delta, a]$ and $\gamma^{a,\delta} : [a \ominus \delta, a] \to \mathcal{P}([Y]^a)$, as follows: for $Y' \in \mathcal{P}([Y]^a)$ and $b \in [a \ominus \delta, a]$, we have

$$\alpha^{a,\delta}(Y') = a \ominus \delta_{Y'} \qquad\qquad \gamma^{a,\delta}(b) = \{y \in [Y]^a \mid a(y) \ominus b(y) \sqsupseteq \delta\}.$$

where we write $\delta_{Y'}$ for the function defined by $\delta_{Y'}(y) = \delta$ if $y \in Y'$ and $\delta_{Y'}(y) = 0$, otherwise.

$$\begin{tikzcd} \mathcal{P}([Y]_a) \arrow[r, bend left, "\alpha_{a,\delta}"] & {[a, a+\delta]} \arrow[l, bend left, "\gamma_{a,\delta}"] \end{tikzcd}$$

One can see that for sufficiently small δ the pair above is indeed a Galois connection.

Now, in order to allow for a compositional approach, the approximation is defined for non-expansive functions, where domain and codomain are possibly distinct.

Definition 3.5 (approximation)**.** Given $f : \mathbb{M}^Y \to \mathbb{M}^Z$ and $\delta \in \mathbb{M}$, define $f_\#^{a,\delta} : \mathcal{P}([Y]^a) \to \mathcal{P}([Z]^{f(a)})$ as $f_\#^{a,\delta} = \gamma^{f(a),\delta} \circ f \circ \alpha^{a,\delta}$. We then define the *$a$-approximation* of f as

$$f_\#^a = \bigcup_{\delta \sqsupset 0} f_\#^{a,\delta}.$$

Intuitively, for $Y' \subseteq [Y]^a$, we have that $f_\#^{a,\delta}(Y')$ is the set of points to which f propagates a decrease of the function a with value δ on the subset Y'.

For finite sets Y and Z all functions $f_\#^{a,\delta}$, for δ below some bound, are equal. Here, the a-approximation is given by $f_\#^a = f_\#^{a,\delta}$ for such a δ.

As stated above, the set $f^a_\#(Y')$ contains the points where a decrease of the values of a on the points in Y' is propagated by applying the function f. The greatest fixpoint of $f^a_\#$ gives us the subset of Y where such a decrease is propagated in a cycle (a so-called "vicious cycle"). Whenever $\nu f^a_\#$ is non-empty, one can argue that a cannot be the least fixpoint of f since we can decrease the value of a at all elements of $\nu f^a_\#$, obtaining a smaller pre-fixpoint. Interestingly, for non-expansive functions, also the converse holds, i.e., emptiness of the greatest fixpoint of $f^a_\#$ implies that a is the least fixpoint. This is summarised by the following result from [BEKP23a].

Theorem 3.6 (soundness and completeness for fixpoints)**.** *Let $\mathbb{M}$ be a complete MV-chain, Y a finite set and $f : \mathbb{M}^Y \to \mathbb{M}^Y$ be a non-expansive function. Let $a \in \mathbb{M}^Y$ be a fixpoint of f. Then $\nu f^a_\# = \emptyset$ if and only if $a = \mu f$.*

If a is not the least fixpoint and thus $\nu f^a_\# \neq \emptyset$ then there is $0 \sqsubset \delta \in \mathbb{M}$ such that $a \ominus \delta_{\nu f^a_\#}$ is a pre-fixpoint of f.

Using the above theorem we can check whether some fixpoint a of f is the least fixpoint. Whenever a is a fixpoint, but not yet the least fixpoint of f, it can be decreased by a fixed value in $\mathbb{M}$ (see [BEKP23a, Proposition 4.5] for the details) on the points in $\nu f^a_\#$ to obtain a smaller pre-fixpoint. In this way we obtain $a' \sqsubset a$ such that $f(a') \sqsubseteq a'$ and can continue fixpoint iteration from there.

This results in the following biconditional proof rule (where "biconditional" means that it can be used in both directions).

$$\frac{a = f(a) \qquad \nu f^a_\# = \emptyset}{a = \mu f}$$

When $a \in \mathbb{M}^Y$ is not a fixpoint, but a post-fixpoint of f (i.e., $a \sqsubseteq f(a)$), a restriction of the a-approximation of f leading to a sound (but not biconditional) rule.

Lemma 3.7 (soundness for post-fixpoints)**.** *Let $\mathbb{M}$ be a complete MV-chain, Y a finite set and $f : \mathbb{M}^Y \to \mathbb{M}^Y$ a non-expansive function, $a \in \mathbb{M}^Y$ such that $a \sqsubseteq f(a)$. Define*

$$[Y]^{a=f(a)} = \{y \in [Y]^a \mid a(y) = f(a)(y)\}$$

and restrict the approximation $f^a_\# : \mathcal{P}([Y]^a) \to \mathcal{P}([Y]^{f(a)})$ to an endo-function

$$\begin{array}{rcl} f^a_* : [Y]^{a=f(a)} & \to & [Y]^{a=f(a)} \\ f^a_*(Y') & = & f^a_\#(Y') \cap [Y]^{a=f(a)} \end{array}$$

If $\nu f^a_* = \emptyset$ then $a \sqsubseteq \mu f$.

Written more compactly, we obtain the following proof rule:

$$\frac{a \sqsubseteq f(a) \qquad \nu f^a_* = \emptyset}{a \sqsubseteq \mu f}$$

The above theory can be easily be dualised to checking greatest fixpoints.

3.3. **Approximations for Basic Functions.** We next provide the approximation for a number of non-expansive functions which can be used as basic building blocks for constructing the semantics functions of interest (Some of them have already been considered in Table 1.)

In Table 2 these basic non-expansive functions are listed together with their approximations. In particular, $\mathcal{D}(Y) \subseteq [0,1]^Y$ denotes the set of finitely supported probability distributions, i.e., functions $\beta : Y \to [0,1]$ with finite support such that $\sum_{y \in Y} \beta(y) = 1$.

Note that functions min and max are slightly generalised with respect to Table 1 as they can be parameterised by relations instead of functions. We stress, in particular, that the approximation of reindexing is given by the inverse image, a fact that will be extensively used in the paper.

function f	definition of f	$f^a_\#(Y')$
c_k $(k \in \mathbb{M}^Z)$	$f(a) = k$	$\emptyset$
u^* $(u \colon Z \to Y)$	$f(a) = a \circ u$	$u^{-1}(Y')$
$\min_\mathcal{R}$ $(\mathcal{R} \subseteq Y \times Z)$	$f(a)(z) = \min\limits_{y \mathcal{R} z} a(y)$	$\{z \in [Z]^{f(a)} \mid \arg \min\limits_{y \in \mathcal{R}^{-1}(z)} a(y) \cap Y' \neq \emptyset\}$
$\max_\mathcal{R}$ $(\mathcal{R} \subseteq Y \times Z)$	$f(a)(z) = \max\limits_{y \mathcal{R} z} a(y)$	$\{z \in [Z]^{f(a)} \mid \arg \max\limits_{y \in \mathcal{R}^{-1}(z)} a(y) \subseteq Y'\}$
$\tilde{\mathcal{D}}$ $(\mathbb{M} = [0,1],$ $Z = D \subseteq \mathcal{D}(Y))$	$f(a)(p) = \sum\limits_{y \in Y} p(y) \cdot a(y)$	$\{p \in [D]^{f(a)} \mid supp(p) \subseteq Y'\}$

Table 2: Basic functions $f \colon \mathbb{M}^Y \to \mathbb{M}^Z$ (constant, reindexing, minimum, maximum, average) and their approximations $f^a_\# \colon \mathcal{P}([Y]^a) \to \mathcal{P}([Z]^{f(a)})$.

4. A Categorical View of the Approximation Framework

In this section we argue that the framework from [BEKP23a], summarised in the previous section, can be naturally reformulated in a categorical setting. In particular, here we study the compositionality properties of the operation mapping a function f to its approximation $f^a_\#$ (for a given fixpoint a of f). We show that, under some constraints, it can be characterised as a functor and, in general, as a union of (lax) functors.

We will use some standard notions from category theory, in particular categories, functors and natural transformations. The definition of (strict) gs-monoidal categories will be spelled out in detail later in Definition 7.1.

We first define a concrete category $\mathbb{C}$ whose arrows are the non-expansive functions for which we seek the least (or greatest) fixpoint and a category $\mathbb{A}$ whose arrows are the corresponding approximations. Recall that, as discussed in the previous section, given a non-expansive function $f : \mathbb{M}^Y \to \mathbb{M}^Z$, the approximation of f is relative to a fixed map $a \in \mathbb{M}^Y$. Hence objects in $\mathbb{C}$ are intuitively pairs $\langle \mathbb{M}^Y, a \in \mathbb{M}^Y \rangle$ or $\langle Y, a \in \mathbb{M}^Y \rangle$. Since $a \in \mathbb{M}^Y$ determines $\mathbb{M}^Y$ as its domain, for simplifying the notation, we leave the first component implicit and let objects in $\mathbb{C}$ be elements $a \in \mathbb{M}^Y$ and an arrow from $a \in \mathbb{M}^Y$

to $b \in \mathbb{M}^Z$ is a non-expansive function $f : \mathbb{M}^Y \to \mathbb{M}^Z$ required to map a into b ($f(a) = b$). The approximations instead live in a different category $\mathbb{A}$. Recall that the approximation $f^a_\#$ is of type $\mathcal{P}([Y]^a) \to \mathcal{P}([Z]^b)$. Since the domain and codomain are again dependent on maps a and b, we still employ elements of $\mathbb{M}^Y$ as objects, but arrows are functions between powersets.

Definition 4.1 (concrete category and category of approximations)**.** We define two categories, the *concrete category* $\mathbb{C}$ and the *category of approximations* $\mathbb{A}$ and a mapping $\#\colon \mathbb{C} \to \mathbb{A}$.

- The concrete category $\mathbb{C}$ has as objects maps $a \in \mathbb{M}^Y$ where Y is a (possibly infinite) set. Given $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ an arrow $f : a \dashrightarrow b$ is a non-expansive function $f\colon \mathbb{M}^Y \to \mathbb{M}^Z$, such that $f(a) = b$.
- The category of approximations $\mathbb{A}$ has again maps $a \in \mathbb{M}^Y$ as objects. Given $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ an arrow $g : a \dashrightarrow b$ is a monotone (with respect to inclusion) function $g\colon \mathcal{P}([Y]^a) \to \mathcal{P}([Z]^b)$. Arrow composition and identities are the obvious ones.
- The approximation maps $\#^\delta\colon \mathbb{C} \to \mathbb{A}$ (for $\delta \sqsupset 0$) and $\#\colon \mathbb{C} \to \mathbb{A}$ are defined as follows: for an object $a \in \mathbb{M}^Y$, we let $\#(a) = \#^\delta(a) = a$ and, given an arrow $f : a \dashrightarrow b$, we let $\#^\delta(f) = f^{a,\delta}_\#$ and $\#(f) = \bigcup_{\delta \sqsupset 0} \#^\delta(f) = f^a_\#$.

Note that categorical arrows are represented as dashed ($\dashrightarrow$). By definition each such categorical arrow consists of an underlying function whose domain and codomain are sets. The underlying functions, being set-valued, are represented as usual (using the notation $\to$).

Lemma 4.2 (well-definedness)**.** *The categories $\mathbb{C}$ and $\mathbb{A}$ are well-defined and the $\#^\delta$ are lax functors, i.e., identities are preserved and $\#^\delta(f) \circ \#^\delta(g) \subseteq \#^\delta(f \circ g)$ for composable arrows f, g in $\mathbb{C}$.*

Note that while $\#$ clearly also preserves identities, the question whether it is a lax functor (or even a proper functor) is currently open. It is however the union of lax functors.

We next observe that $\#$ is a functor if we restrict to suitable subcategories of $\mathbb{C}$, i.e., the subcategory where arrows are reindexings and the one where objects are maps on finite sets. This allows to recover compositionality in those cases in which it is required by the intended applications (see Sections 5-7).

Definition 4.3 (reindexing subcategory)**.** We denote by $\mathbb{C}^*$ the sub-category of $\mathbb{C}$ that contains all objects and where arrows are restricted to reindexings, i.e., given objects $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ we consider only arrows $f : a \dashrightarrow b$ such that $f = g^*$ for some $g\colon Z \to Y$ (hence, in particular, $b = g^*(a) = a \circ g$).

We prove the following auxiliary lemma that basically shows that reindexings are preserved by α, γ:

Lemma 4.4. *Given $a \in \mathbb{M}^Y$, $g\colon Z \to Y$ and $0 \sqsubset \delta \in \mathbb{M}$, then we have*

(1) $\alpha^{a\circ g,\delta} \circ g^{-1} = g^* \circ \alpha^{a,\delta}$

(2) $\gamma^{a\circ g,\delta} \circ g^* = g^{-1} \circ \gamma^{a,\delta}$

This implies that for two $\mathbb{C}$-arrows $f\colon a \dashrightarrow b$, $h\colon b \dashrightarrow c$, it holds that $\#(h\circ f) = \#(h)\circ\#(f)$ whenever f or h is a reindexing, i.e., is contained in $\mathbb{C}^*$.

Then, as an immediate corollary, we obtain the functoriality of $\#$ in the corresponding subcategory.

Corollary 4.5 (approximation functor for reindexing categories)**.** *The approximation map* $\#\colon \mathbb{C} \to \mathbb{A}$ *restricts to* $\#\colon \mathbb{C}^* \to \mathbb{A}$*, which is a (proper) functor.*

We next focus on the subcategory of $\mathbb{C}$ where we consider as objects only maps $a : Y \to \mathbb{M}$ over a finite set Y.

Definition 4.6 (finitary subcategories)**.** We denote by $\mathbb{C}_f$, $\mathbb{A}_f$ the full sub-categories of $\mathbb{C}, \mathbb{A}$ where objects are of the kind $a \in \mathbb{M}^Y$ for a *finite* set Y.

Lemma 4.7 (approximation functor for finitary categories)**.** *The approximation map* $\#\colon \mathbb{C} \to \mathbb{A}$ *restricts to* $\#\colon \mathbb{C}_f \to \mathbb{A}_f$*, which is a (proper) functor.*

We will later show in Theorem 7.4 that $\#$ also respects the monoidal operation $\otimes$ (disjoint union of functions). Using this, we can now apply the framework to an example.

Example 4.8. We revisit the example from Section 2. Remember that the function $\mathcal{T}$, whose least fixpoint is termination probability, can be written as follows:

$$\mathcal{T} = (\eta^* \circ \tilde{\mathcal{D}}) \otimes c_k$$

Let $a = \nu\mathcal{T}$ be the greatest fixpoint of $\mathcal{T}$, i.e., $\mathcal{T}(a) = a$. Hence we can view $\mathcal{T}\colon a \dashrightarrow a$ as a concrete arrow in $\mathbb{C}$ in the sense of Definition 4. Furthermore $\#\mathcal{T} = \mathcal{T}^a_\#\colon a \dashrightarrow a$ is the corresponding approximation, living in the category of approximations $\mathbb{A}$.

We can compute $\mathcal{T}^a_\#$ compositionally. The subfunctions of $\mathcal{T}$ given above are also arrows in $\mathbb{C}_f$ for appropriate finite domains and codomains, which we refrain from spelling out explicitly. Now:

$$\mathcal{T}^a_\# = \#\mathcal{T} = \#((\eta^* \circ \tilde{\mathcal{D}}) \otimes c_k) = \#(\eta^*) \circ \#(\tilde{\mathcal{D}}) \otimes \#(c_k)$$

This view enables us to obtain an approximation $\mathcal{T}^a_\#$ compositionally out of the approximations of the subfunctions.

5. Predicate Liftings

In this section we show how predicate liftings [Pat03, Sch08] can be integrated into our theory. We will characterise predicate liftings which are non-expansive and derive their approximations. This will then be used in Section 6 for treating coalgebraic behavioural metrics.

5.1. **Predicate Liftings and their Properties.**

Roughly speaking, predicates are seen as maps from a set Y to a suitable set of truth values $\mathcal{V}$. Then, given a functor F, a predicate lifting is an operation which transforms predicates over Y to predicates over FY.

One of the simplest examples of a predicate lifting is given by the diamond ($\Diamond$) operator from modal logic. In this case $F = \mathcal{P}$ (powerset functor) and $\mathcal{V} = \{0, 1\}$. Given a predicate $q\colon Y \to \{0, 1\}$, this is mapped to $\Diamond(q)\colon \mathcal{P}(Y) \to \{0, 1\}$ where $\Diamond(q)(Y') = 1$ iff there exists $y \in Y'$ with $q(y) = 1$. Another typical example is expectation where $F = \mathcal{D}$ (distribution functor) and $\mathcal{V} = [0, 1]$ (see also Example 5.3 below). In this case a random variable $r\colon Y \to [0, 1]$ is mapped to a function of type $\mathcal{D}(Y) \to [0, 1]$ where $p \mapsto \mathbb{E}_p[r]$.

Predicate liftings have been studied for predicates valued over arbitrary quantales $\mathcal{V}$ (see, e.g., [BKP18]), i.e., complete lattices with an associative operator that distributes over arbitrary joins. It can be shown that every complete MV-algebra is a quantale with respect to $\oplus$ and the inverse of the natural order. This result can be easily derived from [DNG05].

(See Lemma B.1 in the appendix for an explicit proof.) Hence here we can work with predicates of the kind $Y \to \mathbb{M}$ where $\mathbb{M}$ is a complete MV-algebra.

Definition 5.1 (predicate lifting)**.** Given a functor $F\colon \mathbf{Set} \to \mathbf{Set}$, a *predicate lifting* is a family of functions $\tilde{F}_Y\colon \mathbb{M}^Y \to \mathbb{M}^{FY}$ (where Y is a set), such that for all $g\colon Z \to Y$, $a\colon Y \to \mathbb{M}$ it holds that $(Fg)^*(\tilde{F}_Y(a)) = \tilde{F}_Z(g^*(a))$.

In words, predicate liftings must commute with reindexings. The index Y will be omitted if clear from the context. It can be seen that such predicate liftings are in one-to-one correspondence to so called *evaluation maps* $ev\colon F\mathbb{M} \to \mathbb{M}$. Given ev, we define the corresponding lifting to be $\tilde{F}(a) = ev \circ Fa\colon FY \to \mathbb{M}$, where $a\colon Y \to \mathbb{M}$. Conversely, given a lifting $\tilde{F}$, we obtain $ev = \tilde{F}(id_{\mathbb{M}})$.

A lifting $\tilde{F}$ is well-behaved if (i) $\tilde{F}$ is monotone; (ii) $\tilde{F}(0_Y) = 0_{FY}$ where 0 is the constant 0-function; (iii) $\tilde{F}(a \oplus b) \sqsubseteq \tilde{F}(a) \oplus \tilde{F}(b)$ for $a, b\colon Y \to \mathbb{M}$; (iv) F preserves weak pullbacks. We need well-behavedness in order to prove the next result and in Section 6.

In order to use the theory of fixpoint checks from [BEKP23a] we need to have not only monotone, but non-expansive liftings. We next provide a characterisation of such liftings, following [WS22].

Lemma 5.2 (non-expansive predicate lifting)**.** *Let $ev\colon F\mathbb{M} \to \mathbb{M}$ be an evaluation map and assume that its corresponding lifting $\tilde{F}\colon \mathbb{M}^Y \to \mathbb{M}^{FY}$ is well-behaved. Then $\tilde{F}$ is non-expansive iff for all $\delta \in \mathbb{M}$ it holds that $\tilde{F}\delta_Y \sqsubseteq \delta_{FY}$.*

Example 5.3 (Finitely supported distributions)**.** Consider the (finitely supported) distribution functor $\mathcal{D}$ that maps a set X to all maps $p\colon X \to [0,1]$ that have finite support and satisfy $\sum_{x\in X} p(x) = 1$. (Here $\mathbb{M} = [0,1]$.) A possible evaluation map is $ev\colon \mathcal{D}[0,1] \to [0,1]$ defined by $ev(p) = \sum_{r\in[0,1]} r \cdot p(r)$, where p is a distribution on $[0,1]$. This results in the predicate lifting $\tilde{D}_Y\colon [0,1]^Y \to [0,1]^{\mathcal{D}(Y)}$ with $\tilde{\mathcal{D}}_Y(r)(p) = \mathbb{E}_p[r]$ discussed at the beginning of the section (expectation).

It is easy to see that $\tilde{\mathcal{D}}$ is well-behaved and non-expansive. The latter follows from $\tilde{\mathcal{D}}(\delta_Y) = \delta_{\mathcal{D}Y}$.

Example 5.4 (Finite powerset)**.** Consider the finite powerset functor $\mathcal{P}_f$ with the evaluation map $ev\colon \mathcal{P}_f\mathbb{M} \to \mathbb{M}$, defined for finite $S \subseteq \mathbb{M}$ as $ev(S) = \max S$, where $\max \emptyset = 0$. This results in the predicate lifting $\tilde{P}_Y\colon [0,1]^Y \to [0,1]^{\mathcal{P}(Y)}$, $\mathcal{P}_Y(p)(Y') = \bigsqcup_{y\in Y'} p(y)$.

The lifting $\tilde{\mathcal{P}}_f$ is well-behaved (see [BBKK18]) and non-expansive. To show the latter, observe that $\tilde{\mathcal{P}}_f(\delta_Y) = \delta_{\mathcal{P}_f(Y)\setminus\{\emptyset\}} \sqsubseteq \delta_{\mathcal{P}_f(Y)}$.

Non-expansive predicate liftings can be seen as functors $\tilde{F}\colon \mathbb{C}^* \to \mathbb{C}^*$. To be more precise, $\tilde{F}$ maps an object $a \in \mathbb{M}^Y$ to $\tilde{F}(a) \in \mathbb{M}^{FY}$ and an arrow $g^* : a \dashrightarrow a \circ g$, , where $g\colon Z \to Y$, to $(Fg)^* : \tilde{F}a \dashrightarrow \tilde{F}(a \circ g)$.

5.2. **Approximations of Predicate Liftings.** We now study approximations of predicate liftings. It involves the approximation functor $\#\colon \mathbb{C}^* \to \mathbb{A}$ (restricted to $\mathbb{C}^*$) and the predicate lifting $\tilde{F}\colon \mathbb{C}^* \to \mathbb{C}^*$ introduced before. We start with a result about an auxiliary natural transformation.

Proposition 5.5. *Let $\tilde{F}$ be a (non-expansive) predicate lifting. There is a natural transformation $\beta\colon \# \Rightarrow \#\tilde{F}$ between functors $\#, \#\tilde{F}\colon \mathbb{C}^* \to \mathbb{A}$, whose components, for $a \in \mathbb{M}^Y$, are $\beta_a\colon a \dashrightarrow \tilde{F}(a)$ in $\mathbb{A}$, defined by $\beta_a(U) = \tilde{F}^a_\#(U)$ for $U \subseteq [Y]^a$.*

That is, the following diagrams commute for every $g\colon Z \to Y$ (the diagram on the left indicates the formal arrows, while the one on the right reports the underlying functions).

$$\begin{tikzcd}
\#(a) \arrow[r, dashed, "\#(g^*)"] \arrow[d, dashed, "\beta_a"'] & \#(a \circ g) \arrow[d, dashed, "\beta_{a\circ g}"] \\
\#(\tilde{F}a) \arrow[r, dashed, "\#(\tilde{F}(g^*))"] & \#(\tilde{F}(a \circ g))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}([Y]^a) \arrow[r, "g^{-1}"] \arrow[d, "\tilde{F}^a_\#"'] & \mathcal{P}([Z]^{a\circ g}) \arrow[d, "\tilde{F}^{a\circ g}_\#"] \\
\mathcal{P}([FY]^{\tilde{F}(a)}) \arrow[r, "(Fg)^{-1}"] & \mathcal{P}([FZ]^{\tilde{F}(a\circ g)})
\end{tikzcd}$$

We next characterise $\tilde{F}^d_\#(Y')$. We rely on the fact that d can be decomposed into $d = \pi_1 \circ \bar{d}$, where the projection π_1 is independent of d and $\bar{d}$ is dependent on Y', and exploit the natural transformation in Proposition 5.5.

Proposition 5.6 (Approximations for predicate liftings)**.** *Let $\tilde{F}$ be a predicate lifting. We fix $Y' \subseteq Y$ and let $\chi_{Y'}\colon Y \to \{0,1\}$ be its characteristic function. Furthermore let $a\colon Y \to \mathbb{M}$ be a predicate. Let $\pi_1\colon \mathbb{M} \times \{0,1\} \to \mathbb{M}$, $\pi_2\colon \mathbb{M} \times \{0,1\} \to \{0,1\}$ be the projections and define $\bar{a}\colon Y \to \mathbb{M} \times \{0,1\}$ via $\bar{a}(y) = (a(y), \chi_{Y'}(y))$ as the mediating morphism into the product (see diagram below).*

$$\begin{tikzcd}
\mathbb{M} & Y \arrow[l, "a"'] \arrow[r, "\chi_{Y'}"] \arrow[d, "\bar{a}"] & \{0,1\} \\
& \mathbb{M} \times \{0,1\} \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

Then

$$\tilde{F}^a_\#(Y') = (F\bar{a})^{-1}(\tilde{F}^{\pi_1}_\#((\mathbb{M}\backslash\{0\}) \times \{1\})).$$

Here $\tilde{F}^{\pi_1}_\#((\mathbb{M}\backslash\{0\}) \times \{1\}) \subseteq F(\mathbb{M} \times \{0,1\})$ is independent of a and has to be determined only once for every predicate lifting $\tilde{F}$. We will show how this set looks like for our example functors. We first consider the distribution functor.

Lemma 5.7. *Consider the lifting of the distribution functor presented in Example 5.3 and let $\mathbb{M} = [0,1]$. Then we have*

$$\tilde{\mathcal{D}}^{\pi_1}_\#((0,1] \times \{1\}) = \{p \in \mathcal{D}Z \mid supp(p) \in (0,1] \times \{1\}\}.$$

This means intuitively that a decrease or "slack" can exactly be propagated for elements whose probabilities are strictly larger than 0.

We now turn to the powerset functor.

Lemma 5.8. *Consider the lifting of the finite powerset functor from Example 5.4 with arbitrary $\mathbb{M}$. Then we have*

$$(\tilde{\mathcal{P}}_f)^{\pi_1}_\#((\mathbb{M}\backslash\{0\}) \times \{1\}) = \{S \in [\mathcal{P}_f Z]^{\tilde{\mathcal{P}}_f \pi_1} \mid \exists (s,1) \in S\ \forall (s',0) \in S:\ s \sqsupset s'\}.$$

The idea is that the maximum of a set S decreases if we decrease at least one its values and all values which are not decreased are strictly smaller.

Remark 5.9. Note that $\#$ is a functor on the subcategory $\mathbb{C}_f$ (see Lemma 4.7), while some liftings (e.g., the one for the distribution functor) involve infinite sets, for which we would lose compositionality. In this case, given a finite set Y, we will actually focus on a finite subset $D \subseteq FY$. (This is possible since we work with coalgebras with finite state space that map only into finitely many elements of Y.) Then we consider $\tilde{F}_Y : \mathbb{M}^Y \to \mathbb{M}^{FY}$ and $e \colon D \hookrightarrow FY$ (the embedding of D into FY). We set $f = e^* \circ \tilde{F}_Y : \mathbb{M}^Y \to \mathbb{M}^D$. Given $a \colon Y \to \mathbb{M}$, we view f as an arrow $a \dashrightarrow \tilde{F}_Y(a) \circ e$ in $\mathbb{C}$. The approximation adapts to the "reduced" lifting, which can be seen as follows (cf. Lemma 4.4, which shows that $\#$ preserves composition if one of the arrows is a reindexing):

$$f^a_\# = \#(f) = \#(e^* \circ \tilde{F}_Y) = \#(e^*) \circ \#(\tilde{F}_Y) = e^{-1} \circ \#(\tilde{F}_Y) = \#(\tilde{F}_Y) \cap D.$$

6. Wasserstein Lifting and Behavioural Metrics

In this section we use the results about predicate liftings from the previous section to show how the framework for fixpoint checking can be used to deal with coalgebraic behavioural metrics.

We build on [BBKK18], where an approach is proposed for canonically defining a behavioural pseudo-metric for coalgebras of a functor $F \colon \mathbf{Set} \to \mathbf{Set}$, that is, for functions of the form $\xi \colon X \to FX$ where X is a set. Intuitively ξ specifies a transition system whose branching type is given by F. Our aim is to determine the behavioural distance of two states. Given a coalgebra ξ, the idea is to endow X with a distance function $d_\xi \colon X \times X \to \mathbb{M}$ defined as the least fixpoint of the map $d \mapsto d^F \circ (\xi \times \xi)$ where $\_^F$ lifts a distance function $d \colon X \times X \to \mathbb{M}$ to $d^F \colon FX \times FX \to \mathbb{M}$. Here we focus on a generalisation of the Wasserstein or Kantorovich distances [Vil09] to the categorical setting and then explain how they integrate into the fixpoint checking framework. Such distances are parametric on a suitable notion of predicate lifting, hence we will need the results from the previous section.

We remark that we will use some functors F, for which FY is infinite, even if Y is finite. This is in fact the reason why the categories $\mathbb{C}$ and $\mathbb{A}$ also include infinite sets. However note that the resulting fixpoint function will be always defined for finite sets, although intermediate functions might not conform to this. Hence the restricted compositionality results of Section 4 are sufficient (cf. Remark 5.9 and Definition 6.3).

6.1. **Wasserstein Lifting.**

We first recap the definition of the generalised Wasserstein lifting from [BBKK18]. Hereafter, F denotes a fixed endo-functor on $\mathbf{Set}$ and $\xi \colon X \to FX$ is a coalgebra over a finite set X. We also fix a well-behaved non-expansive predicate lifting $\tilde{F}$. Recall that *pseudo-metrics* are distance functions satisfying: (i) reflexivity: $\forall x \in X \,.\, d(x,x) = 0$; (ii) symmetry: $\forall x, y \in X \,.\, d(x,y) = d(y,x)$; (iii) triangle inequality: $\forall x, y, z \in X \,.\, d(x,z) \sqsubseteq d(x,y) \oplus d(y,z)$. Here we allow arbitrary distance functions $d \colon X \times X \to \mathbb{M}$ and do not restrict to pseudo-metrics.

In order to define a Wasserstein lifting for this functor, a first ingredient is that of a coupling. Given $t_1, t_2 \in FX$ a *coupling* of t_1 and t_2 is an element $t \in F(X \times X)$, such that $F\pi_i(t) = t_i$ for $i = 1, 2$, where $\pi_i \colon X \times X \to X$ are the projections. We write $\Gamma(t_1, t_2)$ for the set of all such couplings.

Definition 6.1 (Wasserstein lifting)**.** The Wasserstein lifting $\_^F \colon \mathbb{M}^{X\times X} \to \mathbb{M}^{FX\times FX}$ is defined for $d \colon X \times X \to \mathbb{M}$ and $t_1, t_2 \in FX$ as

$$d^F(t_1,t_2) = \inf_{t\in\Gamma(t_1,t_2)} \tilde{F}d(t)$$

Example 6.2. We consider the Wasserstein lifting in the concrete case where F equals the distribution functor $\mathcal{D}$. The function $\tilde{\mathcal{D}}$ is obtained as the predicate lifting of $\mathcal{D}$ (see Section 5 and Table 1).

For this instance, the lifting corresponds to the well-known Kantorovich or Wasserstein lifting [Vil09]. In fact, it gives the solution of a transport problem, where we interpret p_1, p_2 as the supply respectively demand at each point $x \in X$. Transporting a unit from x_1 to x_2 costs $d(x_1,x_2)$ and t is a transport plan (= coupling) whose marginals are p_1, p_2. In other words $d^{\mathcal{D}}(p_1,p_2)$ can be seen as the cost of the optimal transport plan, moving the supply p_1 to the demand p_2.

In more detail: given d, we obtain $d^{\mathcal{D}} \colon \mathcal{D}(X) \times \mathcal{D}(X) \to [0,1]$ as

$$\begin{aligned} d^{\mathcal{D}}(p_1,p_2) &= \inf\{\tilde{\mathcal{D}}d(t) \mid t \in \Gamma(p_1,p_2)\} \\ &= \inf\{\sum_{x_1,x_2\in X} d(x_1,x_2)\cdot t(x_1,x_2) \mid t \in \Gamma(p_1,p_2)\} \end{aligned}$$

where $\Gamma(p_1,p_2)$ is the set of couplings of p_1, p_2 (i.e., distributions $t \colon X \times X \to [0,1]$ such that $\sum_{x_2\in X} t(x_1,x_2) = p_1(x_1)$ and $\sum_{x_1\in X} t(x_1,x_2) = p_2(x_2)$).

It can be seen that for well-behaved $\tilde{F}$, the lifting preserves pseudo-metrics (see [BBKK18, BKP18]).

In order to make the theory for fixpoint checks effective we will need to restrict to a subclass of liftings.

Definition 6.3 (finitely coupled lifting)**.** We call a lifting $\tilde{F}$ *finitely coupled* if for all X and $t_1, t_2 \in FX$ there exists a finite $\Gamma'(t_1,t_2) \subseteq \Gamma(t_1,t_2)$, which can be computed given t_1, t_2, such that $\inf_{t\in\Gamma(t_1,t_2)} \tilde{F}d(t) = \min_{t\in\Gamma'(t_1,t_2)} \tilde{F}d(t)$ for all d.

We hence ask that the infimum in Definition 6.1 is actually a minimum. Observe that whenever the infimum above is a minimum, there is trivially such a finite $\Gamma'(t_1,t_2)$. We however ask that $\Gamma'(t_1,t_2)$ is independent of d and there is an effective way to determine it.

The lifting in Example 5.4 (for the finite powerset functor) is obviously finitely coupled. For the lifting $\tilde{\mathcal{D}}$ from Example 5.3 we note that the set of couplings $t \in \Gamma(t_1,t_2)$ forms a polytope with a finite number of vertices, which can be effectively computed and $\Gamma'(t_1,t_2)$ consists of these vertices. The infimum (minimum) is obtained at one of these vertices [BBLM17, Remark 4.5]. This allows us to always reduce to a finite set of couplings for finite-state systems.

6.2. **Decomposing the Behavioural Metrics Function.** As mentioned above, for a coalgebra $\xi \colon X \to FX$ the behavioural pseudo-metric $d : X \times X \to \mathbb{M}$ is the least fixpoint of the behavioural metrics function $\mathcal{W} = (\_^F) \circ (\xi \times \xi)$ where $(\_^F)$ is the Wasserstein lifting.

The Wasserstein lifting can be decomposed as $\_^F = \min_u \circ \tilde{F}$ where $\tilde{F} : \mathbb{M}^{X\times X} \to \mathbb{M}^{F(X\times X)}$ is a predicate lifting – which we require to be non-expansive (cf. Lemma 5.2) – and $\min_u$ is the minimum over the coupling function $u \colon F(X \times X) \to FX \times FX$ defined as $u(t) = (F\pi_1(t), F\pi_2(t))$, which means that $\min_u \colon \mathbb{M}^{F(X\times X)} \to \mathbb{M}^{FX\times FX}$ (see Table 1).

Therefore the behavioural metrics function can be expressed as

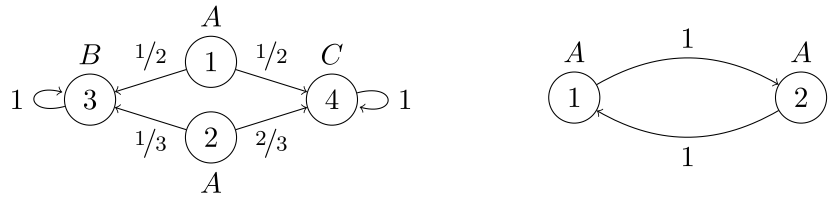


Figure 4: Two probabilistic transition systems.

$$\mathcal{W} = (\xi \times \xi)^* \circ \min_u \circ \tilde{F}$$

Explicitly, for $d \in [0,1]^{X \times X}$ and $x, y \in X$,

$$\begin{aligned}\mathcal{W}(d)(x,y) &= \min_u \circ \tilde{F}(d)(\xi(x), \xi(y)) = \min_{u(t)=(\xi(x),\xi(y))} \tilde{F}(d)(t) \\ &= \min_{t \in \Gamma(\xi(x),\xi(y))} \tilde{F}d(t) = d^F(\xi(x), \xi(y))\end{aligned}$$

Note that the fixpoint equation for behavioural metrics is sometimes equipped with a discount factor that reduces the effect of deviations in the (far) future and ensures that the fixpoint is unique by contractivity of the function. Here we focus on the undiscounted case where the fixpoint equation may have several solutions.

6.3. **A Worked-out Case Study on Behavioural Metrics.** As a case study we consider probabilistic transition systems (Markov chains) with labelled states. These are given by a finite set of states X, a function $\eta\colon X \to \mathcal{D}X$ mapping each state $x \in X$ to a probability distribution on X and a labelling function $\ell\colon X \to \Lambda$, where Λ is a fixed set of labels (for examples see Figure 4). Two such systems are depicted in Figure 4.

This is represented by a coalgebra $\xi : X \to \Lambda \times \mathcal{D}X$ for the functor $FX = \Lambda \times \mathcal{D}(X)$, where Λ is a fixed set of labels.

In particular, we are interested in computing behavioural metrics for such systems. We let $\mathbb{M} = [0,1]$ and consider the Wasserstein lifting for $\mathcal{D}$ explained earlier in Example 6.2 that lifts a distance function $d\colon X \times X \to [0,1]$ to $d^{\mathcal{D}}\colon \mathcal{D}(X) \times \mathcal{D}(X) \to [0,1]$.

For instance, the best transport plan for the system on the left-hand side of Figure 4 and the distributions $\eta(1), \eta(2)$ (where $\eta(1)(3) = 1/2$, $\eta(1)(4) = 1/2$, $\eta(2)(3) = 1/3$, $\eta(2)(4) = 2/3$) is t with $t(3,3) = 1/3$, $t(3,4) = 1/6$, $t(4,4) = 1/2$ and 0 otherwise.

For the functor $FX = \Lambda \times \mathcal{D}(X)$ we observe that couplings of $(a_1, p_1), (a_2, p_2) \in FX$ only exist if $a_1 = a_2$ and – if they do not exist – the distance is the infimum of an empty set, hence 1. If $a_1 = a_2$, couplings correspond to the usual Wasserstein couplings of probability distributions p_1, p_2.

Hence, the behavioural metric is defined as the least fixpoint of the function

$$\mathcal{B}\colon [0,1]^{X \times X} \to [0,1]^{X \times X}$$

$$\mathcal{B}(d)(x_1, x_2) = \begin{cases} 1 & \text{if } \ell(x_1) \neq \ell(x_2) \\ d^{\mathcal{D}}(\eta(x_1), \eta(x_2)) & \text{otherwise} \end{cases}$$

Based on the decomposition of $\mathcal{W}$ explained in Section 6.2, the function $\mathcal{B}$ can be written as

$$\mathcal{B} = \max_\rho \circ (c_k \otimes \mathcal{W}) = \max_\rho \circ (c_k \otimes (\eta \times \eta)^* \circ \min_u \circ \tilde{\mathcal{D}}),$$

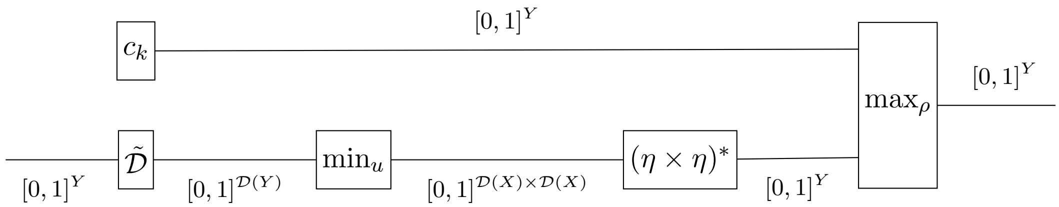


Figure 5: Decomposition of the fixpoint function $\mathcal{B}$ for computing behavioural metrics.

where we use the functions given in Table 2 (Section 3). More concretely, the types of the components and the parameters k, u, ρ are given as follows, where $Y = X \times X$:

- $c_k\colon [0,1]^\emptyset \to [0,1]^Y$ where $k(x,x') = 1$ if $\ell(x) \neq \ell(x')$ and 0 otherwise.
- $\tilde{\mathcal{D}}\colon [0,1]^Y \to [0,1]^{\mathcal{D}(Y)}$.
- $\min_u\colon [0,1]^{\mathcal{D}(Y)} \to [0,1]^{\mathcal{D}(X)\times\mathcal{D}(X)}$ where $u\colon \mathcal{D}(Y) \to \mathcal{D}(X) \times \mathcal{D}(X)$, $u(t) = (p,q)$ with $p(x) = \sum_{x'\in X} t(x,x')$, $q(x) = \sum_{x'\in X} t(x',x)$.
- $(\eta \times \eta)^*\colon [0,1]^{\mathcal{D}(X)\times\mathcal{D}(X)} \to [0,1]^Y$.
- $\max_\rho\colon [0,1]^{Y+Y} \to [0,1]^Y$ where $\rho\colon Y + Y \to Y$ is the obvious map from the coproduct (disjoint union of sets) $Y + Y$ to Y.

This decomposition is depicted diagrammatically in Figure 5.

If we consider only finite state spaces, we can, as explained in Remark 5.9 and after Definition 6.3, restrict ourselves to finite subsets of $\mathcal{D}(X)$ and $\mathcal{D}(Y)$.

By giving a transport plan as above, it is possible to provide an upper bound for the Wasserstein lifting and hence there are strategy iteration algorithms that can approach a fixpoint from above. The problem with these algorithms is that they might get stuck at a fixpoint that is not the least. Hence, it is essential to be able to determine whether a given fixpoint is indeed the smallest one (see, e.g., [BBL+21]).

Consider, for instance, the transition system in Figure 4 on the right. It contains two states $1,2$ on a cycle. In fact these two states should be indistinguishable and hence, if $d = \mu\mathcal{B}$ is the least fixpoint of $\mathcal{B}$, then $d(1,2) = d(2,1) = 0$. However, the metric a with $a(1,2) = a(2,1) = 1$ (0 otherwise) is also a fixpoint and the question is how to determine that it is not the least.

For this, we use the techniques outlined in Section 3.2. We associate $\mathcal{B}$ with an approximation $\mathcal{B}^a_\#$ on subsets of $X \times X$ such that, given $Y' \subseteq X \times X$, the set $\mathcal{B}^a_\#(Y')$ intuitively contains all pairs (x_1, x_2) such that, decreasing function a by some value δ over Y', resulting in a function b (defined as $b(x_1,x_2) = a(x_1,x_2) \ominus \delta$ if $(x_1,x_2) \in Y'$ and $b(x_1,x_2) = a(x_1,x_2)$ otherwise) and applying $\mathcal{B}$, we obtain a function $\mathcal{B}(b)$, where the same decrease takes place at (x_1,x_2) (i.e., $\mathcal{B}(b)(x_1,x_2) = \mathcal{B}(a)(x_1,x_2) \ominus \delta$). We will later discuss in detail how to compute $\mathcal{B}^a_\#$. Here, it can be seen that $\mathcal{B}^a_\#(\{(1,2)\}) = \{(2,1)\}$, since a decrease at $(1,2)$ will cause a decrease at $(2,1)$ in the next iteration. In fact the greatest fixpoint of $\mathcal{B}^a_\#$, which here is $\{(1,2),(2,1)\}$, gives us those elements that have a potential for decrease (intuitively there is "slack" or "wiggle room") and form a "vicious cycle".

By the results outlined in Section 3.2 it holds that a is the least fixpoint of $\mathcal{B}$ iff the the greatest fixpoint of $\mathcal{B}^a_\#$ is the empty set.

As discussed earlier, we can here indeed exploit the fact that the approximation is compositional, i.e., $\mathcal{B}^a_\#$ can be built out of the approximations of $\max_\rho$, c_k, $(\delta \times \delta)^*$, $\min_u$, $\tilde{\mathcal{D}}$ (see Table 2).

6.4. **Approximation of the Behavioural Metrics Function.** While we previously did not spell out how to compute the approximation of the Wasserstein lifting $\mathcal{W}$, we explain this here in detail. We rely on the decomposition of $\mathcal{W}$ as given in Section 6.2, which can be used to derive its d-approximation (for a given distance function d).

Proposition 6.4. *Let $F\colon \mathbf{Set} \to \mathbf{Set}$ be a functor and let $\tilde{F}$ be a corresponding predicate lifting (for $\mathbb{M}$-valued predicates). Assume that $\tilde{F}$ is non-expansive and finitely coupled and fix a coalgebra $\xi\colon X \to FX$, where X is finite. Let $Y = X \times X$. For $d \in \mathbb{M}^Y$ and $Y' \subseteq [Y]^d$ we have*

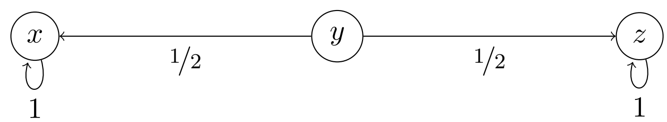
$$\mathcal{W}^d_\#(Y') = \{(x,y) \in [Y]^d \mid \ \exists t \in \tilde{F}^d_\#(Y'), u(t) = (\xi(x), \xi(y)), \tilde{F}d(t) = \mathcal{W}(d)(x,y)\}.$$

Intuitively, the definition of $\mathcal{W}^d_\#(Y')$ in Proposition 6.4 says that (x,y) is contained in this set, whenever an optimal coupling for the successors of x, y (i.e., a coupling reaching the minimum in the Wasserstein lifting) is contained in $\tilde{F}^d_\#(Y')$. Note that $\tilde{F}^d_\#$ has already been characterised earlier in Section 5.2.

We now illustrate this result by two simple and concrete examples involving coalgebras over the distribution and powerset functor.

Example 6.5. We continue with the case study from Section 6.3 on probabilistic transition systems. We omit labels, i.e. Λ is a singleton, which implied $\mathcal{B} = \mathcal{W}$.

Let $X = \{x, y, z\}$. We define a coalgebra $\xi\colon X \to \Lambda \times \mathcal{D}X$ via $\xi(x)(x) = 1$, $\xi(y)(y) = \xi(y)(z) = 1/2$ and $\xi(z)(z) = 1$. All other distances are 0.



Since all states have the same label, they are in fact probabilistically bisimilar and hence have behavioural distance 0, given by the least fixpoint of $\mathcal{W}$. Now consider the pseudo-metric $d\colon X \times X \to [0,1]$ with $d(x,y) = d(x,z) = d(y,z) = 1$ and 0 for the reflexive pairs. This is a also a fixpoint of $\mathcal{W}$ ($d = \mathcal{W}(d)$), but it clearly over-estimates the true behavioural metric.

We can detect this by computing the greatest fixpoint of $\mathcal{W}^d_\#$, which is

$$Y' = \{(x,y), (y,x), (x,z), (z,x), (y,z), (z,y)\} \neq \emptyset,$$

containing the pairs that still have slack in d and whose distances can be reduced. We explain why $Y' = \mathcal{W}^d_\#(Y')$ by focusing on the example pair (x,y) and check that $(x,y) \in \mathcal{W}^d_\#(Y')$. For this we use the definition of $\mathcal{W}^d_\#$ given in Proposition 6.4.

A valid coupling $t \in \mathcal{D}(X \times X)$ of $\xi(x), \xi(y)$ is given by $t(x,y) = t(x,z) = 1/2$. It satisfies $u(t) = (\xi(x), \xi(y))$ and is optimal since it is the only one. We obtain the Wasserstein lifting

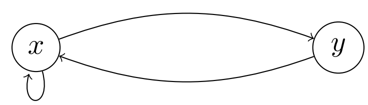
$$\mathcal{W}(d)(x,y) = d^{\mathcal{D}}(\xi(x), \xi(y)) = \min_{t' \in \Gamma(\xi(x), \xi(y))} \tilde{\mathcal{D}}d(t') = \tilde{\mathcal{D}}d(t) = 1/2 \cdot 1 + 1/2 \cdot 1 = 1.$$

It is left to show that $t \in \tilde{\mathcal{D}}^d_\#(Y')$, for which we use the characterisation in Proposition 5.6. We have $\bar{d}(x,y) = \bar{d}(x,z) = (1,1)$ where

$$\mathcal{D}\bar{d}(t) = p \in \mathcal{D}Z \text{ with } p(1,1) = 1/2 + 1/2 = 1.$$

From Lemma 5.7 we obtain $p \in \tilde{\mathcal{D}}^{\pi_1}_\#((0,1] \times \{1\})$ and by definition $t \in (\mathcal{D}\bar{d})^{-1}(p)$, i.e. $t \in \tilde{\mathcal{D}}^d_\#(Y')$. So we can conclude that $(x,y) \in \mathcal{W}^d_\#(Y')$.

Example 6.6. We now consider an example in the non-deterministic setting. Let $X = \{x,y\}$ and a coalgebra $\xi\colon X \to \mathcal{P}X$ be given by $\xi(x) = \{x,y\}$, $\xi(y) = \{x\}$.



Since all states have successors, they are in fact bisimilar and hence have behavioural distance 0, given by the least fixpoint of $\mathcal{W}$. Now consider the pseudo-metric $d\colon X \times X \to [0,1]$ with $d(x,y) = d(y,x) = 1/2$ and 0 for the reflexive pairs. This is a also a fixpoint of $\mathcal{W}$ ($d = \mathcal{W}(d)$), but it clearly over-estimates the true behavioural metric.

We can detect this by computing the greatest fixpoint of $\mathcal{W}^d_\#$, which is

$$Y' = \{(x,y),(y,x)\} \neq \emptyset,$$

containing the pairs that still have slack in d and whose distances can be reduced. We explain why $Y' = \mathcal{W}^d_\#(Y')$ by focusing on the example pair (x,y) and check that $(x,y) \in \mathcal{W}^d_\#(Y')$. For this we use the definition of $\mathcal{W}^d_\#$ given in Proposition 6.4.

A valid (and optimal) coupling $t \in \mathcal{P}(X \times X)$ of $\xi(x), \xi(y)$ is given by $t = \{(x,x),(y,x)\}$. It satisfies $u(t) = (\xi(x), \xi(y))$. We obtain the Wasserstein lifting

$$\mathcal{W}(d)(x,y) = d^{\mathcal{P}}(\xi(x),\xi(y)) = \min_{t \in \Gamma(\xi(x),\xi(y))} \tilde{\mathcal{P}}d(t) = \tilde{\mathcal{P}}d(t) = \max\{0, 1/2\} = 1/2.$$

It is left to show that $t \in \tilde{\mathcal{P}}^d_\#(Y')$, for which we use the characterisation in Proposition 5.6. We have $\bar{d}(x,x) = \bar{d}(y,y) = (0,0)$, $\bar{d}(x,y) = \bar{d}(y,x) = (1/2, 1)$ and

$$\mathcal{P}\bar{d}(t) = S = \{(0,0),(1/2,1)\}$$

From Lemma 5.8 we obtain $S \in \tilde{\mathcal{P}}^{\pi_1}_\#((\mathbb{M}\backslash\{0\}) \times \{1\})$ and $t \in (\mathcal{P}\bar{d})^{-1}(S)$, i.e. $t \in \tilde{\mathcal{P}}^d_\#(Y')$. So we can conclude that $(x,y) \in \mathcal{W}^d_\#(Y')$.

7. GS-Monoidality

We will now show that the categories $\mathbb{C}_f$ and $\mathbb{A}_f$ can be turned into gs-monoidal categories, making # a gs-monoidal functor. This will give us a method to assemble functions and their approximations compositionally and this will form the basis for the tool. We first define gs-monoidal categories in detail (cf. [GH97, Definition 7] and [FGTC23]).

Definition 7.1 (gs-monoidal categories)**.** A *strict gs-monoidal category* is a strict symmetric monoidal category, where $\otimes$ denotes the tensor and e its unit and symmetries are given by $\rho_{a,b}\colon a \otimes b \to b \otimes a$. For every object a there exist morphisms $\nabla_a\colon a \to a \times a$ (*duplicator*) and $!_a\colon a \to e$ (*discharger*) satisfying the axioms given below. (See also their visualisations as string diagrams in Figure 6.)

functoriality of tensor	*naturality*
$(g \otimes g') \circ (f \otimes f') = (g \circ f) \otimes (g' \circ f')$	$(f' \otimes f) \circ \rho_{a,a'} = \rho_{b,b'} \circ (f \otimes f')$
$id_{a \otimes b} = id_a \otimes id_b$	
monoidality	*symmetry*
$(f \otimes g) \otimes h = f \otimes (g \otimes h)$	$\rho_{e,e} = id_e \qquad \rho_{b,a} \circ \rho_{a,b} = id_{a \otimes b}$
$f \otimes id_e = f = id_e \otimes f$	$\rho_{a \otimes b,c} = (\rho_{a,c} \otimes id_b \circ (id_a \otimes \rho_{b,c})$
gs-monoidality	
$!_e = \nabla_e = id_e$	
coherence axioms	*monoidality axioms*
$(id_a \otimes \nabla_a) \circ \nabla_a = (\nabla_a \otimes id_a) \circ \nabla_a$	$!_{a \otimes b} = !_a \otimes !_b$
$id_a = (id_a \otimes !_a) \circ \nabla_a$	$(id_a \otimes \rho_{a,b} \otimes id_b) \circ (\nabla_a \otimes \nabla_b) = \nabla_{a \otimes b}$
$\rho_{a,a} \circ \nabla_a = \nabla_a$	(or, equiv. $\nabla_a \otimes \nabla_b = (id_a \otimes \rho_{b,a} \otimes id_b) \circ \nabla_{a \otimes b}$

A functor $F : \mathbb{C} \to \mathbb{D}$, where $\mathbb{C}$ and $\mathbb{D}$ are gs-monoidal categories, is *gs-monoidal* if the following holds:

monoidality	*symmetry*	*gs-monoidality*
$F(e) = e'$	$F(\rho_{a,b}) = \rho'_{F(a),F(b)}$	$F(!_a) = !'_{F(a)}$
$F(a \otimes b) = F(a) \otimes' F(b)$		$F(\nabla_a) = \nabla'_{F(a)}$

where the primed operators are from category $\mathbb{D}$, the others from $\mathbb{C}$.

Note that the visualisations of the axioms in Figure 6 match the images in Figure 2 and Figure 5. However, instead of labelling the wires with the types of objects as in the previous figures, we here label the wires with a fixed object.

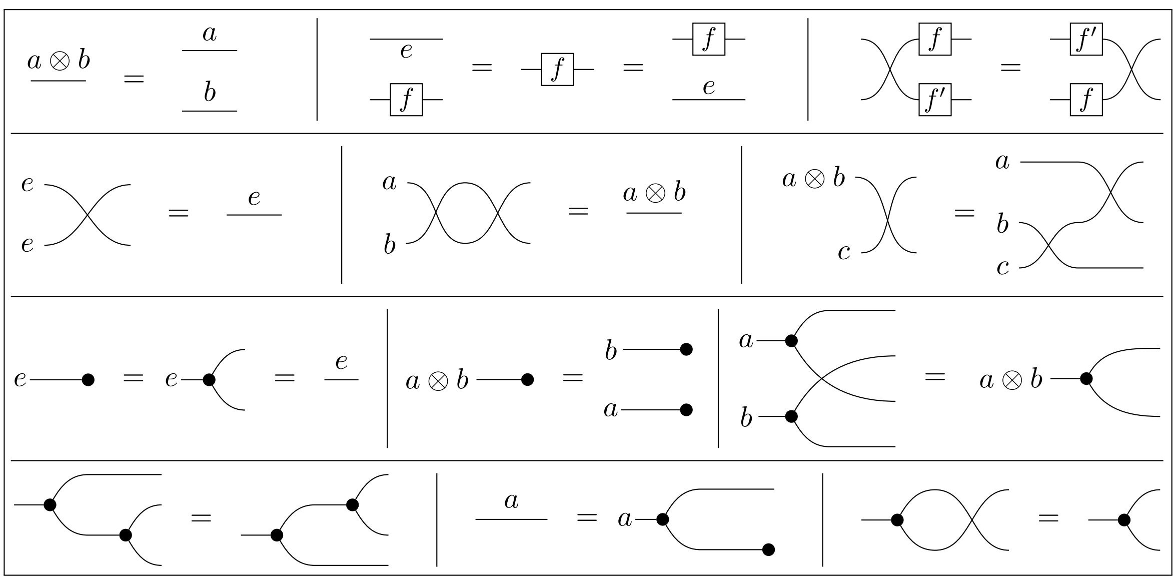


Figure 6: String diagrams for the axioms of gs-monoidal categories.

In fact, in order to obtain strict gs-monoidal categories with disjoint union as tensor, we will work with the skeleton categories where every finite set Y is represented by an isomorphic copy $\{1, \ldots, |Y|\}$. This enables us to make disjoint union strict, i.e., associativity holds on the nose and not just up to isomorphism. In particular for finite sets Y, Z, we define disjoint union as $Y + Z = \{1, \ldots, |Y|, |Y| + 1, \ldots, |Y| + |Z|\}$.

Theorem 7.2 ($\mathbb{C}_f$ is gs-monoidal)**.** *The category $\mathbb{C}_f$ with the following operators is gs-monoidal:*

(1) *The tensor $\otimes$ on objects $a \in \mathbb{M}^Y$ and $b \in \mathbb{M}^Z$ is defined as*

$$a \otimes b = a + b \in \mathbb{M}^{Y+Z}$$

where for $k \in Y + Z$ we have $(a+b)(k) = a(k)$ if $k \leq |Y|$ and $(a+b)(k) = b(k - |Y|)$ if $|Y| < k \leq |Y| + |Z|$.

On arrows $f : a \dashrightarrow b$ and $g : a' \dashrightarrow b'$ (with $a' \in \mathbb{M}^{Y'}$, $b' \in \mathbb{M}^{Z'}$) tensor is given by

$$f \otimes g : \mathbb{M}^{Y+Y'} \to \mathbb{M}^{Z+Z'}, \quad (f \otimes g)(u) = f(\overleftarrow{u}_Y) + g(\vec{u}_Y)$$

for $u \in \mathbb{M}^{Y+Y'}$ where $\overleftarrow{u}_Y \in \mathbb{M}^Y$ and $\vec{u}_Y \in \mathbb{M}^{Y'}$ are defined as

$$\overleftarrow{u}_Y(k) = u(k) \text{ for } 1 \leq k \leq |Y| \text{ and } \vec{u}_Y(k) = u(|Y| + k) \text{ for } 1 \leq k \leq |Y'|.$$

(2) *The symmetry $\rho_{a,b} : a \otimes b \dashrightarrow b \otimes a$ for $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ is defined for $u \in \mathbb{M}^{Y+Z}$ as*

$$\rho_{a,b}(u) = \vec{u}_Y + \overleftarrow{u}_Y.$$

(3) *The unit e is the unique mapping $e : \emptyset \to \mathbb{M}$.*

(4) *The duplicator $\nabla_a : a \dashrightarrow a \otimes a$ for $a \in \mathbb{M}^Y$ is defined for $u \in \mathbb{M}^Y$ as*

$$\nabla_a(u) = u + u.$$

(5) *The discharger $!_a : a \dashrightarrow e$ for $a \in \mathbb{M}^Y$ is defined for $u \in \mathbb{M}^Y$ as $!_a(u) = e$.*

We now turn to the category of approximations $\mathbb{A}_f$. Note that here functions have as parameters sets of the form $U \subseteq [Y]^a \subseteq Y$. Hence, (the cardinality of) Y cannot be determined directly from U and we need extra care with the tensor.

Theorem 7.3 ($\mathbb{A}_f$ is gs-monoidal)**.** *The category $\mathbb{A}_f$ with the following operators is gs-monoidal:*

(1) *The tensor $\otimes$ on objects $a \in \mathbb{M}^Y$ and $b \in \mathbb{M}^Z$ is again defined as $a \otimes b = a + b$.*

On arrows $f : a \dashrightarrow b$ and $g : a' \dashrightarrow b'$ (where $a' \in \mathbb{M}^{Y'}$, $b' \in \mathbb{M}^{Z'}$ and $f : \mathcal{P}([Y]^a) \to \mathcal{P}([Z]^{b'})$, $g : \mathcal{P}([Y']^{a'}) \to \mathcal{P}([Z']^{b'})$ are the underlying functions), the tensor is given by

$$f \otimes g : \mathcal{P}([Y + Y']^{a+a'}) \to \mathcal{P}([Z + Z']^{b+b'}), \quad (f \otimes g)(U) = f(\overleftarrow{U}_Y) \cup_Z g(\vec{U}_Y)$$

where

$$\overleftarrow{U}_Y = U \cap \{1, \ldots, |Y|\} \text{ and } \vec{U}_Y = \{k \mid |Y| + k \in U\}.$$

Furthermore:

$$U \cup_Y V = U \cup \{|Y| + k \mid k \in V\} \quad (\text{where } U \subseteq Y)$$

(2) *The symmetry $\rho_{a,b} : a \otimes b \dashrightarrow b \otimes a$ for $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ is defined for $U \subseteq [Y + Z]^{a+b}$ as*

$$\rho_{a,b}(U) = \vec{U}_Y \cup_Z \overleftarrow{U}_Y \subseteq [Z + Y]^{b+a}$$

(3) *The unit e is again the unique mapping $e : \emptyset \to \mathbb{M}$.*

(4) *The duplicator $\nabla_a : a \dashrightarrow a \otimes a$ for $a \in \mathbb{M}^Y$ is defined for $U \subseteq [Y]^a$ as*

$$\nabla_a(U) = U \cup_Y U \subseteq [Y + Y]^{a+a}.$$

(5) *The discharger $!_a : a \dashrightarrow e$ for $a \in \mathbb{M}^Y$ is defined for $U \subseteq [Y]^a$ as $!_a(U) = \emptyset$.*

Finally, the approximation # is indeed gs-monoidal, i.e., it preserves all the additional structure (tensor, symmetry, unit, duplicator and discharger).

Theorem 7.4 (# is gs-monoidal)**.** $\#\colon \mathbb{C}_f \to \mathbb{A}_f$ *is a gs-monoidal functor.*

8. UDEfix: A Tool for Fixpoints Checks

8.1. **Overview.** We present a tool, called UDEfix, which exploits gs-monoidality as discussed before and allows the user to construct functions $f\colon \mathbb{M}^Y \to \mathbb{M}^Y$ (with Y finite) as a circuit. As basic components, UDEfix can handle all functions presented in Section 4 (Table 2) and addition/subtraction by a fixed constant w, denoted add_w/sub_w (both are non-expansive functions). Since the approximation functor # is gs-monoidal on the finitary subcategories, this circuit can then be transformed automatically, in a compositional way, into the corresponding approximation $f_\#^a$, for some given $a \in \mathbb{M}^Y$. By computing the greatest fixpoint of $f_\#^a$ and checking for emptiness, UDEfix can check whether $a = \mu f$. In addition, it is possible to check whether a given post-fixpoint a is below the least fixpoint μf (recall that in this case the check is sound but not complete). The dual checks (for greatest fixpoint and pre-fixpoints) are implemented as well.

The tool is shipped with pre-defined functions implementing examples concerning case studies on termination probability, bisimilarity, simple stochastic games, energy games, behavioural metrics and Rabin automata.

UDEfix is a Windows-Tool created in Python, which can be obtained from `https://github.com/TimoMatt/UDEfix`.

Building the desired function $f\colon \mathbb{M}^Y \to \mathbb{M}^Y$ requires three steps:

- Choosing the MV-algebra $\mathbb{M}$ of interest.
- Creating the required basic functions by specifying their parameters.
- Assembling f from these basic functions.

8.2. **Tool Areas.** Concretely, the GUI of UDEfix is separated into three areas: the Content area, Building area and Basic-Functions area. Under File-Settings the user can set the MV-algebra. Currently the MV-chains $[0, k]$ (algebra 1) and $\{0, \dots, k\}$ (algebra 2) for arbitrary k are supported (see Example 3.3)

Basic-Functions Area: The Basic-Functions area contains the basic functions, encompassing those listed in Table 2 (Section 3), as well as addition and subtraction by a constant w. Via drag-and-drop (or right-click) these basic functions can be added to the Building area to create a Function box. Each such box requires three (in the case of $\tilde{\mathcal{D}}$ two) Contents: The Input set, the Output set and an additional required parameter (see Table 3). These Contents can be created in the Content area.

Basic Function	c_k	u^*	$\min_{\mathcal{R}} / \max_{\mathcal{R}}$	add_w	sub_w
Req. Parameter	$k \in \mathbb{M}^Z$	$u\colon Z \to Y$	$R \subseteq Y \times Z$	$w \in \mathbb{M}^Y$	$w \in \mathbb{M}^Y$

Table 3: Additional parameters for the basic functions from Table 2.

Additionally the Basic-Functions area offers functionalities for composing functions via disjoint union (more concretely, this is handled by the auxiliary Higher-Order Function) and the Testing functionality for fixpoint checks which we will discuss in the next paragraph.

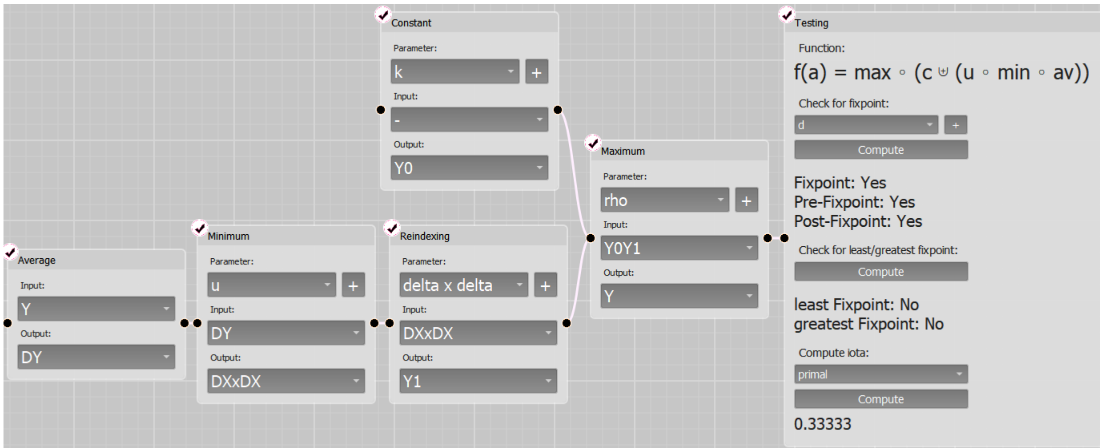


Figure 7: Assembling the function $\mathcal{B}$ from Section 6.3.

Building Area: The user can connect the created Function boxes to construct the function f of interest. Composing functions is as simple as connecting two Function boxes in the correct order by mouse-click. Disjoint union is achieved by connecting two boxes to the same box. Note that Input and Output sets of connected Function boxes need to match. As an example, in Figure 7 we show how the function $\mathcal{B}$, discussed in Section 6.3, for computing the behavioural distance of a labeled Markov chain can be assembled. The construction exactly follows the structure of the diagram in Figure 5. Here, the parameters are instantiated for the labeled Markov chain displayed in Figure 4 (left-hand side).

The special box Testing is always required to appear at the end. Here, the user can enter some mapping $a\colon Y \to \mathbb{M}$, test if a is a fixpoint of the function f of interest and then verify if $a = \mu f$. As explained before, this is realised by computing the greatest fixpoint of the approximation $\nu f_{\#}^{a}$. In case this is not empty and thus $a \neq \mu f$, the tool can produce a suitable value which can be used for decreasing a, needed for iterating to the least fixpoint from above (respectively increasing a for iterating to the greatest fixpoint from below). There is also support for comparison with pre- and post-fixpoints.

Example 8.1. As an example, consider the left-hand system in Figure 4 and consider the function $\mathcal{B}$ from Section 6.3 whose least fixpoint corresponds to the behavioural distance for a labeled Markov chain.

We now define a fixpoint of $\mathcal{B}$, which is not the least, namely $d\colon Y \to [0,1]$ with $d(3,3) = 0$, $d(1,1) = 1/2$, $d(1,2) = d(2,1) = d(2,2) = 2/3$ and 1 for all other pairs. Note that d is not a pseudo-metric, but it is a fixpoint of $\mathcal{B}$. For understanding why d is a fixpoint, consider, for instance, the pair $(1,2)$. Due to the labels, an optimal coupling for the successors of $1,2$ assigns $(3,3) \mapsto 1/3$, $(4,4) \mapsto 1/2$, $(3,4) \mapsto 1/6$. Hence, by the fixpoint equation, we have

$$d(1,2) = 1/3 \cdot d(3,3) + 1/2 \cdot d(4,4) + 1/6 \cdot d(3,4) = 1/3 \cdot 0 + 1/2 \cdot 1 + 1/6 \cdot 1 = 1/2 + 1/6 = 2/3.$$

A similar argument can be made for the remaining pairs.

By clicking Compute in the Testing-box, UDEfix displays that d is a fixpoint and tells us that d is in fact not the least and not the greatest fixpoint. It also computes the greatest fixpoints of the approximations step by step (via Kleene iteration) and displays the results to the user. In this case $\nu f_{\#}^{a} = \{(4,4)\}$, indicating that the distance $d(4,4) = 1$ over-estimates

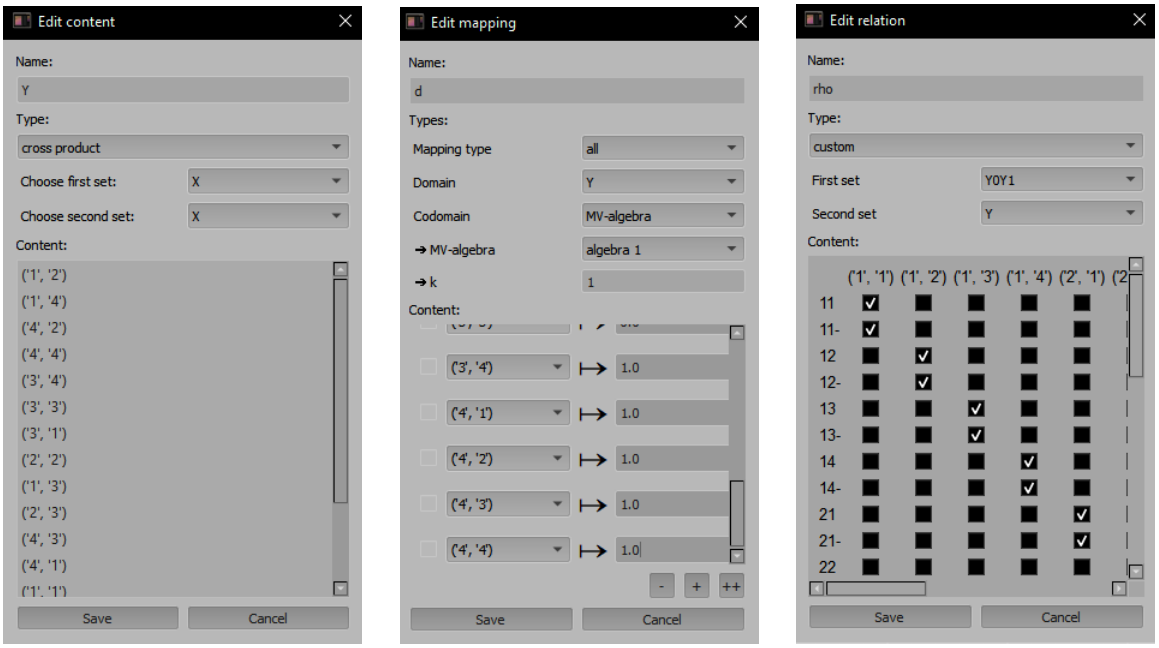


Figure 8: Contents: Set Y, Mapping d, Relation ρ.

the true value and can be decreased. The fact that also other pairs over-estimate their value (but to a lesser degree), will be detected in later steps in the iteration to the least fixpoint from above.

Content Area: Here the user can create sets, mappings and relations which are used to specify the basic functions. The user can create a variety of different types of sets, such as $X = \{1, 2, 3, 4\}$, which is a basic set of numbers, and the set $D = \{p_1, p_2, p_3, p_4\}$ which is a set of mappings representing probability distributions. These objects are called Content.

Once Input and Output sets are created we can specify the required parameters (cf. Table 3) for a function. Here, the created sets can be chosen as domain and co-domain. Relations can be handled in a similar fashion: Given the two sets one wants to relate, creating a relation can be easily achieved by checking some boxes. Some useful in-built relations like "is-element-of"-relation and projections to the i-th component are pre-defined.

By clicking on the icon "+" in a Function box, a new function with the chosen Input and Output sets is created. The additional parameters (cf. Table 3) have domains and co-domains which need to be created by the user or are provided by the chosen MV-algebra.

The Testing function a (i.e., the candidate (pre/post-)fixpoint) is a mapping as well and can be created as all other functions.

In Figure 8 we give examples of how contents can be created: we show the creation of a set ($Y = X \times X$), a distance function (d) and a relation (ρ).

8.3. **Tutorial.** We now provide a small tutorial intended to clarify the use of the tool. It deals with a simple example: a function whose least fixpoint provides the termination probability of a Markov chain. Specifically, we continue the example from Section 2 and Example 4.8, and we consider the Markov chain (S, T, η) in Figure 3.

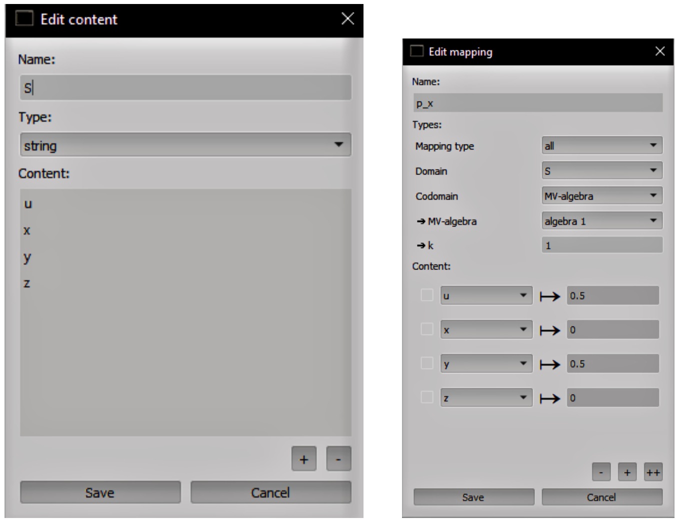


Figure 9: Creation of set S and probability distribution p_x for the example.

Remember that the function $\mathcal{T}$, whose least fixpoint is the termination probability, can be decomposed as follows:

$$\mathcal{T} = (\eta^* \circ \tilde{\mathcal{D}}) \otimes c_k$$

In order to start, one first chooses the correct MV-algebra under Settings and creates the sets $S, T, S \setminus T$ (Input and Output sets). The creation of set S is exemplified in Figure 9 (left-hand side). The tool supports several types of sets and operators on sets (such as complement, which makes it easy to create $S \backslash T$. Next, we create the set D of probability distributions, consisting of the mappings p_x, p_y, p_z. In Figure 9 (right-hand side) we show the creation of p_x.

Now we create the basic function boxes and connect them in the correct way (see Figure 10). The additional parameters according to Table 3 – in this case the map c_k and the reindexing η^* based on the successor map – can be created by clicking the icon "+" in the corresponding box (see Figure 11) (left-hand side for k and middle for η).

We can also assemble several test functions (e.g., possible candidate fixpoints), among them the greatest fixpoint $a_1 = \nu\mathcal{T}$ (see Figure 11, right-hand side).

When testing a_1 we obtain the results depicted in Figure 12. In fact $\nu\mathcal{T}_{\#}^{a_1} = \{y, z\} \neq \emptyset$, which tells us that a_1 is not the least fixpoint.

Similarly, one can test $a_2 = \mu\mathcal{T}$, the least fixpoint, obtaining $\nu\mathcal{T}_{\#}^{a_2} = \emptyset$. This allows the user to deduce that a_2 is indeed the least fixpoint. As mentioned before, one can also test whether a pre-fixpoint is below the greatest fixpoint or a post-fixpoint above the least fixpoint, although such tests are sound but not complete.

9. Conclusion, Related and Future Work

We have shown that a framework originally introduced in [BEKP21, BEKP23a] for analysing fixpoint of non-expansive functions over MV-algebras can be naturally cast into a gs-monoidal setting. When considering the finitary categories, both the non-expansive functions of interest and their approximations live in two gs-monoidal categories and are related by a gs-monoidal functor $\#$. We also developed a general theory for constructing approximations

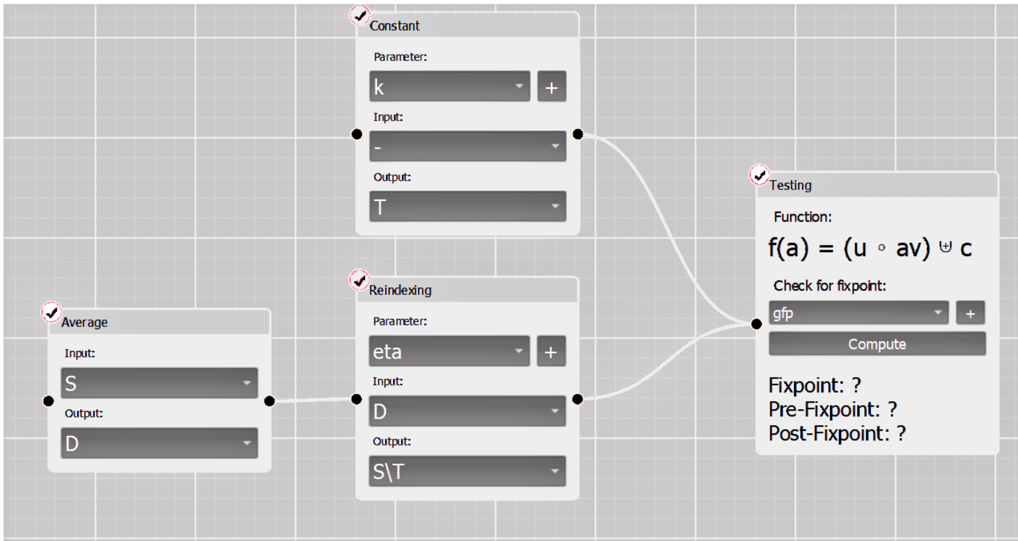


Figure 10: Assembling the function $\mathcal{T}$.

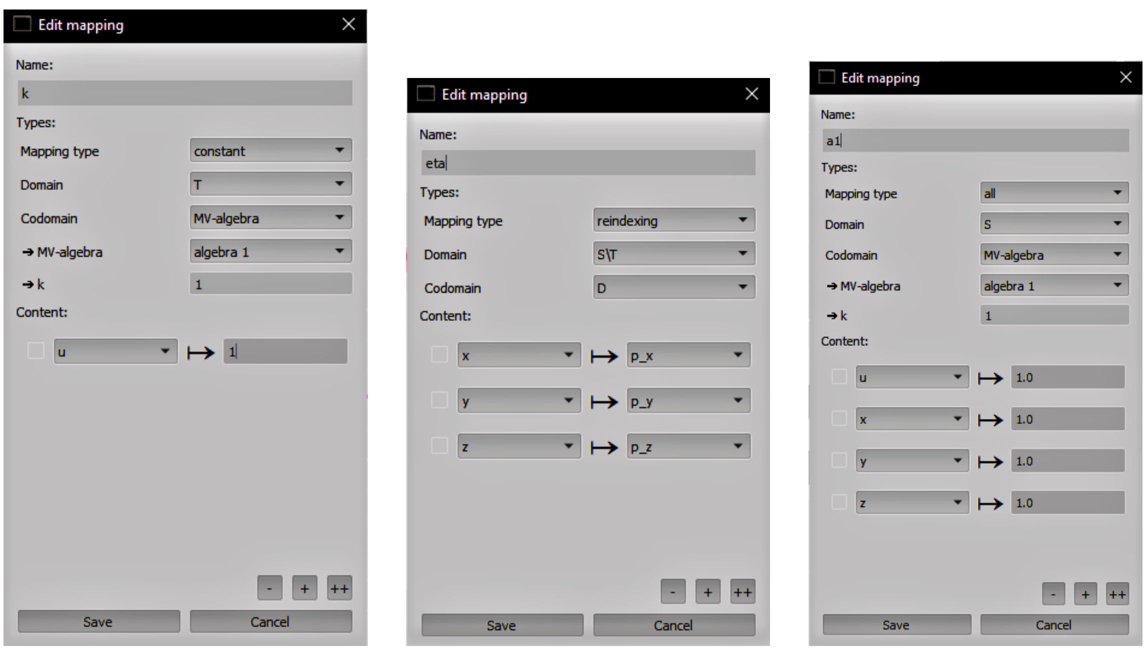


Figure 11: Creation of the parameters k and η^* and the greatest fixpoint $a_1 = \mu\mathcal{T}$.

of predicate liftings, which find natural application in the definition of behavioural metrics over coalgebras.

Graph compositionality has been studied from several angles and has always been an important part of the theory of graph rewriting. For instance, one way to explain the double-pushout approach [EPS73] is to view the graph to be rewritten as a composition of a left-hand side and a context, then compose the context with the right-hand side to obtain the result of the rewriting step.

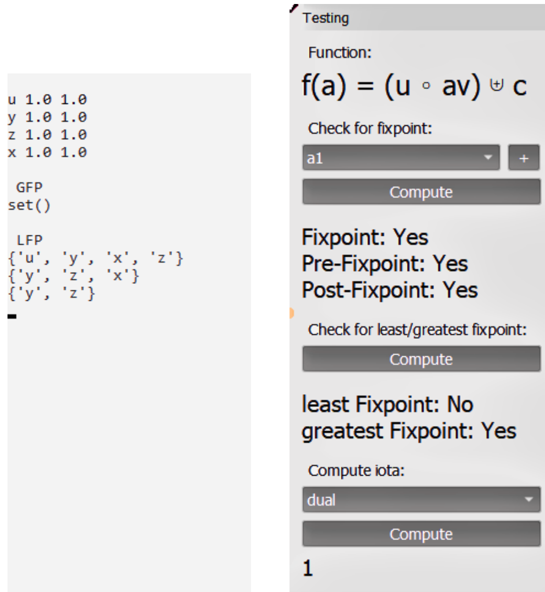


Figure 12: Checking the candidate fixpoint a_1.

Several algebras have been proposed for a compositional view on graphs, see for instance [BC87, Kön02, BK04]. For the compositional modelling of graphs and graph-like structures it has in particular proven useful to use the notion of monoidal categories [Mac71], i.e., categories equipped with a tensor product. There are several variants of such categories, such as gs-monoidal categories, that have been shown to be suitable for specifying term graph rewriting (see e.g. [Gad96, GH97]).

In our work, the compositionality properties arising from the gs-monoidal view of the theory are at the basis of the development of the prototypical tool UDEfix. The tool allows one to build the concrete function of interest out of some basic components. Then the approximation of the function can be obtained compositionally from the approximations of the components and one can check whether some fixpoint is the least or greatest fixpoint of the function of interest. Additionally, one can use the tool to show that some pre-fixpoint is above the greatest fixpoint or some post-fixpoint is below the least fixpoint.

Related work: This paper is based on fixpoint theory, coalgebras, as well as on the theory of monoidal categories. Monoidal categories [Mac71] are categories equipped with a tensor. It has long been realised that monoidal categories can have additional structure such as braiding or symmetries. Here we base our work on so called gs-monoidal categories [CG99, GH97], called s-monoidal in [Gad96]. These are symmetric monoidal categories, equipped with a discharger and a duplicator. Note that "gs" originally stood for "graph substitution" as such categories were first used for modelling term graph rewriting.

We view gs-monoidal categories as a means to compositionally build monotone non-expansive functions on complete lattices, for which we are interested in the (least or greatest) fixpoint. Such fixpoints are ubiquitous in computer science, here we are in particular interested in applications in concurrency theory and games, such as bisimilarity [San11], behavioural metrics [DGJP04, vB17, CvBW12, BBKK18] and simple stochastic games [Con90].

In recent work we have considered strategy iteration procedures inspired by games for solving fixpoint equations [BEKP23b].

Fixpoint equations also arise in the context of coalgebra [Rut00], a general framework for investigating behavioural equivalences for systems that are parameterised – via a functor – over their branching type (labelled, non-deterministic, probabilistic, etc.). Here in particular we are concerned with coalgebraic behavioural metrics [BBKK18], based on a generalisation of the Wasserstein or Kantorovich lifting [Vil09]. Such liftings require the notion of predicate liftings, well-known in coalgebraic modal logics [Sch08], lifted to a quantitative setting [BKP18].

Future work: One important question is still open: we defined an approximation $\#$, relating the concrete category $\mathbb{C}$ of functions of type $\mathbb{M}^Y \to \mathbb{M}^Z$ – where Y, Z might be infinite – to their approximations, living in $\mathbb{A}$. It is unclear whether $\#$ is a lax or even proper functor, i.e., whether it (laxly) preserves composition. For finite sets functoriality derives from a non-trivial result in [BEKP23a] and it is unclear whether it can be extended to the infinite case. If so, this would be a valuable step to extend the theory to infinite sets.

In this paper we illustrated the approximation for predicate liftings via the powerset and the distribution functor. It would be worthwhile to study more functors and hence broaden the applicability to other types of transition systems.

Concerning UDEfix, we plan to extend the tool to compute fixpoints, either via Kleene iteration or strategy iteration (strategy iteration from above and below), as detailed in [BEKP23b]. Furthermore for convenience it would be useful to have support for generating fixpoint functions directly from a given coalgebra respectively transition system.

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Appendix A. Proofs and Additional Material for §4 (A Categorical View of the Approximation Framework)

Lemma 4.2 (well-definedness)**.** *The categories $\mathbb{C}$ and $\mathbb{A}$ are well-defined and the $\#^\delta$ are lax functors, i.e., identities are preserved and $\#^\delta(f) \circ \#^\delta(g) \subseteq \#^\delta(f \circ g)$ for composable arrows f, g in $\mathbb{C}$.*

Proof.

(1) $\mathbb{C}$ *is a well-defined category*: Given arrows $f\colon a \dashrightarrow b$ and $g\colon b \dashrightarrow c$ then $g \circ f$ is non-expansive (since non-expansiveness is preserved by composition) and $(g\circ f)(a) = g(b) = c$, thus $g \circ f\colon a \dashrightarrow c$. Associativity holds and the identities are the units of composition as for standard function composition.

(2) $\mathbb{A}$ *is a well-defined category*: Given arrows $f\colon a \dashrightarrow b$ and $g\colon b \dashrightarrow c$ then $g \circ f$ is monotone (since monotonicity is preserved by composition) and hence $g \circ f\colon a \dashrightarrow c$.

Again associativity and the fact that the identities are units is immediate.

(3) $\#^\delta\colon \mathbb{C} \to \mathbb{A}$ *is a lax functor*: we first check that identities are preserved. Let $U \subseteq [Y]^a$, then

$$\begin{aligned}
\#^\delta(id_a)(U) &= (id_a)^{a,\delta}_\#(U)\\
&= \{y \in [Y]^{id_a(a)} \mid id_a(a)(y) \ominus id_a(a \ominus \delta_U)(y) \sqsupseteq \delta\}\\
&= \{y \in [Y]^a \mid a(y) \ominus (a \ominus \delta_U)(y) \sqsupseteq \delta\}\\
&= U = id_a(U) = id_{\#^\delta(a)}(U).
\end{aligned}$$

where in the second last line we use the fact that $U \subseteq [Y]^a$.

Let $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$, $c \in \mathbb{M}^V$, $f\colon a \dashrightarrow b$, $g\colon b \dashrightarrow c$ be arrows in $\mathbb{C}$ and $Y' \subseteq [Y]^a$. Then

$$\begin{aligned}
(\#^\delta(g) \circ \#^\delta(f))(Y') &= g^{b,\delta}_\#(f^{a,\delta}_\#(Y'))\\
&= (\gamma^{c,\delta} \circ g \circ \alpha^{b,\delta} \circ \gamma^{b,\delta} \circ f \circ \alpha^{a,\delta})(Y')\\
&\subseteq (\gamma^{g(f(a)),\delta} \circ g \circ f \circ \alpha^{a,\delta})(Y')\\
&= (g \circ f)^{a,\delta}_\#(Y')\\
&= \#^\delta(g \circ f)(Y')
\end{aligned}$$

The inequality holds since for $c \in \mathbb{M}^Z$:

$$\alpha^{b,\delta}(\gamma^{b,\delta}(c)) = \alpha^{b,\delta}(\{y \in Y \mid b(y) \ominus c(y) \sqsupseteq \delta\}) = b \ominus \delta_{\{y\in Y \mid b(y)\ominus c(y)\sqsupseteq\delta\}} \sqsupseteq c.$$

Then the inequality follows from the antitonicity of $\gamma^{c,\delta}$. (Remember that we are working with a contra-variant Galois connection.) □

Lemma 4.4. *Given $a \in \mathbb{M}^Y$, $g\colon Z \to Y$ and $0 \sqsubset \delta \in \mathbb{M}$, then we have*

(1) $\alpha^{a\circ g,\delta} \circ g^{-1} = g^* \circ \alpha^{a,\delta}$

(2) $\gamma^{a\circ g,\delta} \circ g^* = g^{-1} \circ \gamma^{a,\delta}$

This implies that for two $\mathbb{C}$-arrows $f\colon a \dashrightarrow b$, $h\colon b \dashrightarrow c$, it holds that $\#(h\circ f) = \#(h)\circ\#(f)$ whenever f or h is a reindexing, i.e., is contained in $\mathbb{C}^*$.

Proof.

(1) Let $Y' \subseteq [Y]^a$. Then

$$g^*(\alpha^{a,\delta}(Y')) = g^*(a \ominus \delta_{Y'}) = (a \ominus \delta_{Y'}) \circ g = a \circ g \ominus \delta_{Y'} \circ g$$

$$= a \circ g \ominus \delta_{g^{-1}(Y')} = \alpha^{a \circ g, \delta}(g^{-1}(Y'))$$

where we use that $(\delta_{Y'} \circ g)(z) = \delta$ if $g(z) \in Y'$, equivalent to $z \in g^{-1}(Y')$, and 0 otherwise. Hence $\delta_{Y'} \circ g = \delta_{g^{-1}(Y')}$.

(2) Let $b \in \mathbb{M}^Y$ with $a \ominus \delta \sqsubseteq b \sqsubseteq a$. Then

$$\begin{aligned}\gamma^{a \circ g, \delta} \circ g^*(b) &= \{z \in Z \mid a(g(z)) \ominus b(g(z)) \sqsupseteq \delta\} = \{z \in Z \mid g(z) \in \gamma^{a,\delta}(b)\} \\ &= g^{-1}(\gamma^{a,\delta}(b))\end{aligned}$$

It is left to show that $\#(h \circ f) = \#(h) \circ \#(f)$ whenever f or h is a reindexing. Recall that on reindexings it holds that $\#(g^*) = g^{-1}$.

Let $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$, $c \in \mathbb{M}^W$ and assume first that f is a reindexing, i.e., $f = g^*$ for some $g\colon Z \to Y$. Let $Y' \subseteq [Y]^a$, then

$$\begin{aligned}\#(h \circ f) = (h \circ f)^a_\# &= \bigcup_{\delta \sqsupset 0} (\gamma^{h(f(a)),\delta} \circ h \circ f \circ \alpha^{a,\delta})(Y') \\ &= \bigcup_{\delta \sqsupset 0} (\gamma^{h(f(a)),\delta} \circ h \circ g^* \circ \alpha^{a,\delta})(Y') \\ &= \bigcup_{\delta \sqsupset 0} (\gamma^{h(f(a)),\delta} \circ h \circ \alpha^{a \circ g,\delta})(g^{-1}(Y')) && (1) \\ &= \bigcup_{\delta \sqsupset 0} (\gamma^{h(f(a)),\delta} \circ h \circ \alpha^{f(a),\delta})(\#(g^*)(Y')) \\ &= (\#(h) \circ \#(f))(Y')\end{aligned}$$

Now we assume that h is a reindexing, i.e., $h = g^*$ for some $g\colon W \to Z$. Let again $Y' \subseteq [Y]^a$, then:

$$\begin{aligned}\#(h \circ f) = (h \circ f)^a_\# &= \bigcup_{\delta \sqsupset 0} (\gamma^{h(f(a)),\delta} \circ h \circ f \circ \alpha^{a,\delta})(Y') \\ &= \bigcup_{\delta \sqsupset 0} (\gamma^{f(a) \circ g,\delta} \circ g^* \circ f \circ \alpha^{a,\delta})(Y') \\ &= \bigcup_{\delta \sqsupset 0} g^{-1}((\gamma^{f(a),\delta} \circ f \circ \alpha^{a,\delta})(Y')) && (2) \\ &= g^{-1}(\bigcup_{\delta \sqsupset 0} (\gamma^{f(a),\delta} \circ f \circ \alpha^{a,\delta})(Y')) && \text{[preimage preserves union]} \\ &= \#(g^*)(\bigcup_{\delta \sqsupset 0} (\gamma^{f(a),\delta} \circ f \circ \alpha^{a,\delta})(Y')) \\ &= (\#(h) \circ \#(f))(Y') && \square\end{aligned}$$

Lemma 4.7 (approximation functor for finitary categories)**.** *The approximation map $\#\colon \mathbb{C} \to \mathbb{A}$ restricts to $\#\colon \mathbb{C}_f \to \mathbb{A}_f$, which is a (proper) functor.*

Proof. Clearly the restriction to categories based on finite sets is well-defined.

We show that $\#$ is a (proper) functor. Let $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$, $c \in \mathbb{M}^V$, $f\colon a \dashrightarrow b$, $g\colon b \dashrightarrow c$ and $Y' \subseteq [Y]^a$. Then

$$\#(g \circ f) = (g \circ f)^a_\# = g^{f(a)}_\# \circ f^a_\# = g^b_\# \circ f^a_\# = \#(g) \circ \#(f),$$

The second inequality above is a consequence of the compositionality result in [BEKP23a, Proposition D.3]. This requires finiteness of the sets Y, Z, V.

The rest follows from Lemma 4.2. $\square$

Appendix B. Proofs and Additional Material for §5 (Predicate Liftings)

Lemma B.1 (complete MV-algebras are quantales)**.** *Let* $\mathbb{M}$ *be a complete MV-algebra. Then* $(\mathbb{M}, \oplus, \sqsupseteq)$ *is a unital commutative quantale, i.e., a quantale with neutral element for* $\oplus$*.*

Proof. We know $\mathbb{M}$ is a complete lattice. Binary meets are given by

$$x \sqcap y = \overline{\overline{x \oplus \overline{y}} \oplus \overline{y}}. \tag{B.1}$$

Moreover $\oplus$ is associative and commutative, with 0 as neutral element.

It remains to show that $\oplus$ distributes with respect to $\sqcap$ (note that $\sqcap$ is the join for the reverse order), i.e., that for all $X \subseteq \mathbb{M}$ and $a \in \mathbb{M}$, it holds

$$a \oplus \bigsqcap X = \bigsqcap \{a \oplus x \mid x \in X\}$$

Clearly, since $\bigsqcap X \leq x$ for all $x \in X$ and $\oplus$ is monotone, we have $a \oplus \bigsqcap X \sqsubseteq \bigsqcap\{a \oplus x \mid x \in X\}$. In order to show that $a \oplus \bigsqcap X$ is the greatest lower bound, let z be another lower bound for $\{a \oplus x \mid x \in X\}$, i.e., $z \sqsubseteq a \oplus x$ for all $x \in X$. Then observe that for $x \in X$, using (B.1), we get

$$x \sqsupseteq x \sqcap \overline{a} = \overline{\overline{(x \oplus a)} \oplus a} \sqsupseteq \overline{\overline{z} \oplus a} = z \ominus a$$

Therefore $\bigsqcap X \sqsupseteq z \ominus a$ and thus

$$a \oplus \bigsqcap X \sqsupseteq a \oplus (z \ominus a) \sqsupseteq z$$

as desired. $\square$

Lemma 5.2 (non-expansive predicate lifting)**.** *Let* $ev \colon F\mathbb{M} \to \mathbb{M}$ *be an evaluation map and assume that its corresponding lifting* $\tilde{F} \colon \mathbb{M}^Y \to \mathbb{M}^{FY}$ *is well-behaved. Then* $\tilde{F}$ *is non-expansive iff for all* $\delta \in \mathbb{M}$ *it holds that* $\tilde{F}\delta_Y \sqsubseteq \delta_{FY}$*.*

Proof. The proof is inspired by [WS22, Lemma 3.9] and uses the fact that a monotone function $f \colon \mathbb{M}^Y \to \mathbb{M}^Z$ is non-expansive iff $f(a \oplus \delta) \sqsubseteq f(a) \oplus \delta$ for all a, δ.

($\Rightarrow$) Fix a set Y and assume that $\tilde{F} \colon \mathbb{M}^Y \to \mathbb{M}^{FY}$ is non-expansive. Then

$$\tilde{F}(\delta) = \tilde{F}(0 \oplus \delta) \sqsubseteq \tilde{F}(0) \oplus \delta \sqsubseteq 0 \oplus \delta = \delta$$

($\Leftarrow$) Now assume that $\tilde{F}(\delta) \sqsubseteq \delta$. Then, using the lemma referenced above,

$$\tilde{F}(a \oplus \delta) \sqsubseteq \tilde{F}(a) \oplus \tilde{F}(\delta) \sqsubseteq \tilde{F}(a) \oplus \delta$$

Above we write δ for both δ_Y, δ_{FY} and both deductions rely on the fact that $\tilde{F}$ is well-behaved. $\square$

Proposition 5.5. *Let* $\tilde{F}$ *be a (non-expansive) predicate lifting. There is a natural transformation* $\beta \colon \# \Rightarrow \#\tilde{F}$ *between functors* $\#, \#\tilde{F} \colon \mathbb{C}^* \to \mathbb{A}$*, whose components, for* $a \in \mathbb{M}^Y$*, are* $\beta_a \colon a \dashrightarrow \tilde{F}(a)$ *in* $\mathbb{A}$*, defined by* $\beta_a(U) = \tilde{F}^a_\#(U)$ *for* $U \subseteq [Y]^a$*.*

That is, the following diagrams commute for every $g \colon Z \to Y$ *(the diagram on the left indicates the formal arrows, while the one on the right reports the underlying functions).*

$$\begin{tikzcd} \#(a) \arrow[r, dashed, "\#(g^*)"] \arrow[d, dashed, "\beta_a"'] & \#(a\circ g) \arrow[d, dashed, "\beta_{a\circ g}"] \\ \#(\tilde{F}a) \arrow[r, dashed, "\#(\tilde{F}(g^*))"] & \#(\tilde{F}(a\circ g)) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}([Y]^a) \arrow[r, "g^{-1}"] \arrow[d, "\tilde{F}^a_\#"'] & \mathcal{P}([Z]^{a\circ g}) \arrow[d, "\tilde{F}^{a\circ g}_\#"] \\ \mathcal{P}([FY]^{\tilde{F}(a)}) \arrow[r, "(Fg)^{-1}"] & \mathcal{P}([FZ]^{\tilde{F}(a\circ g)}) \end{tikzcd}$$

Proof. We first define a natural transformation $\eta\colon \mathrm{Id}_{\mathbb{C}^*} \Rightarrow \tilde{F}$ (between the identity functor and $\tilde{F}$) with components $\eta_a\colon a \dashrightarrow \tilde{F}(a)$ (for $a \in \mathbb{M}^Y$) by defining $\eta_a(b) = \tilde{F}(b)$ for $b \in \mathbb{M}^Y$. The η_a are non-expansive by assumption. In addition, η is natural due to the definition of a predicate lifting, i.e., $(Fg)^* \circ \tilde{F} = \tilde{F} \circ g^*$ for $g\colon Z \to Y$.

Now we apply $\#$ and use the fact that $\#$ is functorial, even for the full category $\mathbb{C}$, whenever one of the two arrows to which $\#$ is applied is a reindexing (see Lemma 4.4). Furthermore we observe that $\beta = \#(\eta)$. This immediately gives commutativity of the diagram on the left. (The diagram on the right just displays the underlying functions.) $\square$

Proposition 5.6 (Approximations for predicate liftings)**.** *Let $\tilde{F}$ be a predicate lifting. We fix $Y' \subseteq Y$ and let $\chi_{Y'}\colon Y \to \{0,1\}$ be its characteristic function. Furthermore let $a\colon Y \to \mathbb{M}$ be a predicate. Let $\pi_1\colon \mathbb{M} \times \{0,1\} \to \mathbb{M}$, $\pi_2\colon \mathbb{M} \times \{0,1\} \to \{0,1\}$ be the projections and define $\bar{a}\colon Y \to \mathbb{M} \times \{0,1\}$ via $\bar{a}(y) = (a(y), \chi_{Y'}(y))$ as the mediating morphism into the product (see diagram below).*

$$\begin{tikzcd} \mathbb{M} & Y \arrow[l, "a"'] \arrow[r, "\chi_{Y'}"] \arrow[d, "\bar{a}"] & \{0,1\} \\ & \mathbb{M} \times \{0,1\} \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

Then

$$\tilde{F}^a_\#(Y') = (F\bar{a})^{-1}(\tilde{F}^{\pi_1}_\#((\mathbb{M}\backslash\{0\}) \times \{1\})).$$

Proof. Let $a \in \mathbb{M}^Y$ and $Y' \subseteq [Y]^a$. Note that $\bar{a}^{-1}((\mathbb{M}\backslash\{0\}) \times \{1\}) = Y'$ and $a = \pi_1 \circ \bar{a}$, thus by Proposition 5.5:

$$\begin{aligned} \tilde{F}^a_\#(Y') &= \tilde{F}^{\pi_1\circ\bar{a}}_\#(\bar{a}^{-1}((\mathbb{M}\backslash\{0\}) \times \{1\})) \\ &= (F\bar{a})^{-1}(\tilde{F}^{\pi_1}_\#((\mathbb{M}\backslash\{0\}) \times \{1\})) \end{aligned}$$ $\square$

Lemma 5.7. *Consider the lifting of the distribution functor presented in Example 5.3 and let $\mathbb{M} = [0,1]$. Then we have*

$$\tilde{\mathcal{D}}^{\pi_1}_\#((0,1] \times \{1\}) = \{p \in \mathcal{D}Z \mid supp(p) \in (0,1] \times \{1\}\}.$$

Proof. Let $\delta > 0$. We define

$$\tilde{\pi}_1^\delta := \alpha^{\pi_1,\delta}((0,1] \times \{1\})$$

where $\tilde{\pi}_1^\delta(x,0) = x$, $\tilde{\pi}_1^\delta(x,1) = x \ominus \delta$ for $x \in [0,1]$. Note that $[\mathcal{D}Z]^{\tilde{\mathcal{D}}\pi_1} = \{p \in DZ \mid \exists (x,b) \in supp(p)$ with $x \geq 0\}$. Now

$$\begin{aligned} \tilde{\mathcal{D}}^{\pi_1,\delta}_\#((0,1] \times \{1\}) &= \{p \in [\mathcal{D}Z]^{\tilde{\mathcal{D}}\pi_1} \mid \tilde{\mathcal{D}}\pi_1(p) \ominus \tilde{\mathcal{D}}(\tilde{\pi}_1^\delta)(p) \geq \delta\} \\ &= \{p \in [\mathcal{D}Z]^{\tilde{\mathcal{D}}\pi_1} \mid \big(\sum_{x\in[0,1]} x \cdot p(x,0) \oplus \sum_{x\in[0,1]} x \cdot p(x,1)\big) \end{aligned}$$

$$\ominus \big(\sum_{x\in[0,1]} x \cdot p(x,0) \oplus \sum_{x\in[0,1]} (x \ominus \delta)\cdot p(x,1)\big) \geq \delta\}$$
$$= \{p \in [\mathcal{D}Z]^{\tilde{\mathcal{D}}\pi_1} \mid \sum_{x\in[0,\delta)} x \cdot p(x,1) + \sum_{x\in[\delta,1]} \delta \cdot p(x,1) \geq \delta\}$$
$$= \{p \in [\mathcal{D}Z]^{\tilde{\mathcal{D}}\pi_1} \mid \mathit{supp}(p) \in [\delta,1]\times\{1\}\}.$$

Where the second last equality uses the fact that $x \ominus (x \ominus \delta) = \delta$ if $x \geq \delta$ and x otherwise.

Now, we obtain

$$\tilde{\mathcal{D}}^{\pi_1}_{\#}((0,1]\times\{1\}) = \bigcup_{\delta \sqsupset 0} \tilde{\mathcal{D}}^{\pi_1,\delta}_{\#}((0,1]\times\{1\}) = \{p \in \mathcal{D}Z \mid \mathit{supp}(p) \in (0,1]\times\{1\}\}. \qquad \square$$

Lemma 5.8. *Consider the lifting of the finite powerset functor from Example 5.4 with arbitrary* $\mathbb{M}$. *Then we have*

$$(\tilde{\mathcal{P}}_f)^{\pi_1}_{\#}((\mathbb{M}\backslash\{0\})\times\{1\}) = \{S \in [\mathcal{P}_f Z]^{\tilde{\mathcal{P}}_f\pi_1} \mid \exists (s,1)\in S\ \forall (s',0)\in S:\ s \sqsupset s'\}.$$

Proof. Let $\delta \sqsupset 0$ and define $\tilde{\pi}_1^{\delta}$ as in the proof of Lemma 5.7. Then

$$(\tilde{\mathcal{P}}_f)^{\pi_1,\delta}_{\#}((\mathbb{M}\backslash\{0\})\times\{1\}) = \{S \in [\mathcal{P}_f Z]^{\tilde{\mathcal{P}}_f\pi_1} \mid \tilde{\mathcal{P}}_f\pi_1(S) \ominus \tilde{\mathcal{P}}_f(\tilde{\pi}_1^{\delta})(S) \sqsupseteq \delta\}$$
$$= \{S \in [\mathcal{P}_f Z]^{\tilde{\mathcal{P}}_f\pi_1} \mid \max_{(s,b)\in S} s \ominus \big(\max_{(s,b)\in S} s \ominus b\cdot\delta\big) \sqsupseteq \delta\}$$
$$= \{S \in [\mathcal{P}_f Z]^{\tilde{\mathcal{P}}_f\pi_1} \mid \exists (s,1)\in S\ \forall (s',0)\in S:\ s\ominus\delta \sqsupseteq s'\}$$

For the last step we note that this condition ensures that the second maximum equates to $\max_{(s,b)\in S} s \ominus \delta$ which is required for the inequality to hold. Now, we obtain

$$(\tilde{\mathcal{P}}_f)^{\pi_1}_{\#}((\mathbb{M}\backslash\{0\})\times\{1\}) = \bigcup_{\delta\sqsupset 0} \tilde{F}^{\pi_1,\delta}_{\#}((\mathbb{M}\backslash\{0\})\times\{1\})$$
$$= \{S \in [\mathcal{P}_f Z]^{\tilde{\mathcal{P}}_f\pi_1} \mid \exists (s,1)\in S\ \forall (s',0)\in S:\ s \sqsupset s'\}. \qquad \square$$

Appendix C. Proofs and Additional Material for §6 (Wasserstein Lifting and Behavioural Metrics)

Proposition 6.4. *Let* $F\colon \mathbf{Set} \to \mathbf{Set}$ *be a functor and let* $\tilde{F}$ *be a corresponding predicate lifting (for* $\mathbb{M}$*-valued predicates). Assume that* $\tilde{F}$ *is non-expansive and finitely coupled and fix a coalgebra* $\xi\colon X \to FX$, *where* X *is finite. Let* $Y = X \times X$. *For* $d \in \mathbb{M}^Y$ *and* $Y' \subseteq [Y]^d$ *we have*

$$\mathcal{W}^d_{\#}(Y') = \{(x,y)\in[Y]^d \mid\ \exists t \in \tilde{F}^d_{\#}(Y'), u(t) = (\xi(x),\xi(y)), \tilde{F}d(t) = \mathcal{W}(d)(x,y)\}.$$

Proof. We first remark that since X is finite and $\tilde{F}$ is finitely coupled it is sufficient to restrict to finite subsets of $F(X\times X)$ and $FX \times FX$ (cf. Remark 5.9). In other words $\mathcal{W}$ can be obtained as composition of functions living in $\mathbb{C}_f$, hence $\#$ is a proper functor and approximations can obtained compositionally. We exploit this fact in the following.

More concretely, we restrict u to $u\colon V \to W$, where $V \subseteq F(X\times X)$, $W \subseteq FX \times FX$. We require that W contains all pairs $(\xi(x),\xi(y))$ for $x,y\in X$ and $V = \bigcup_{(t_1,t_2)\in W} \Gamma'(t_1,t_2)$. Hence both V, W are finite.

The function $\tilde{F}$ is restricted accordingly to a map $\mathbb{M}^Y \to \mathbb{M}^V$ as explained in Remark 5.9.

For $d \in \mathbb{M}^Y$ and $Y' \subseteq [Y]^d$ we have, using the fact that $\mathcal{W} = (\xi \times \xi)^* \circ \min_u \circ \tilde{F}$, compositionality and the approximations listed in Table 2:

$$\begin{aligned}
\mathcal{W}^d_\#(Y') &= \{(x,y) \in [Y]^d \mid (\xi(x), \xi(y)) \in (\min_u)^{\tilde{F}(d)}_\#(\tilde{F}^d_\#(Y') \cap V)\} \\
&= \{(x,y) \in [Y]^d \mid \arg \min_{t \in u^{-1}(\xi(x),\xi(y))} \tilde{F}(d)(t) \cap \tilde{F}^d_\#(Y') \cap V \neq \emptyset\} \\
&= \{(x,y) \in [Y]^d \mid \exists t \in \tilde{F}^d_\#(Y') \cap V, u(t) = (\xi(x), \xi(y)), \\
&\qquad\qquad \tilde{F}d(t) = \min\nolimits_{t' \in V} \tilde{F}d(t')\} \\
&= \{(x,y) \in [Y]^d \mid \exists t \in \tilde{F}^d_\#(Y') \cap V, u(t) = (\xi(x), \xi(y)), \\
&\qquad\qquad \tilde{F}d(t) = \min\nolimits_{t' \in \Gamma'(\xi(x),\xi(y))} \tilde{F}d(t')\} \\
&= \{(x,y) \in [Y]^d \mid \exists t \in \tilde{F}^d_\#(Y'), u(t) = (\xi(x), \xi(y)), \tilde{F}d(t) = \min\nolimits_{t' \in \Gamma'(\xi(x),\xi(y))} \tilde{F}d(t')\} \\
&= \{(x,y) \in [Y]^d \mid \exists t \in \tilde{F}^d_\#(Y'), u(t) = (\xi(x), \xi(y)), \tilde{F}d(t) = \mathcal{W}(d)(x,y)\}
\end{aligned}$$

The first equality is based on Remark 5.9 and uses the fact that the approximation for the restricted $\tilde{F}$ maps Y' to $\tilde{F}^d_\#(Y') \cap V$.

The second-last inequality also needs explanation, in particular, we have to show that the set on the second-last line is included in the one on the previous line, although we omitted the intersection with V.

Hence let $(x,y) \in [Y]^d$ such that there exists $s \in \tilde{F}^d_\#(Y')$, $u(s) = (\xi(x), \xi(y))$ and $\tilde{F}d(s) = \min_{t' \in \Gamma'(\xi(x),\xi(y))} \tilde{F}d(t')$. We have to show that there exists a t with the same properties that is also included in V.

The fact that $s \in \tilde{F}^d_\#(Y')$ implies that $\tilde{F}(d)(s) \ominus \tilde{F}(d \ominus \delta_{Y'})(s) \sqsupseteq \delta$ for small enough δ, using the fact that $\tilde{F}^d_\# = \gamma^{\tilde{F}(d),\delta} \circ \tilde{F} \circ \alpha^{d,\delta}$ (for an appropriate value δ).

Since the minimum of the Wasserstein lifting is always reached in $\Gamma'(\xi(x), \xi(y))$, independently of the argument, there exists $t \in \Gamma'(\xi(x), \xi(y)) \subseteq V$ (hence $u(t) = (\xi(x), \xi(y))$), such that

$$\tilde{F}(d \ominus \delta_{Y'})(t) = \min\nolimits_{t' \in \Gamma(\xi(x),\xi(y))} \tilde{F}(d \ominus \delta_{Y'})(t').$$

This implies that $\tilde{F}(d \ominus \delta_{Y'})(t) \sqsubseteq \tilde{F}(d \ominus \delta_{Y'})(s)$ (since $s \in \Gamma(\xi(x), \xi(y))$). From the assumption $\tilde{F}d(s) = \min_{t' \in \Gamma'(\xi(x),\xi(y))} \tilde{F}d(t')$ we obtain $\tilde{F}d(s) \sqsubseteq \tilde{F}d(t)$. Hence, using the fact that $\ominus$ is monotone in the first and antitone in the second argument, we have:

$$\delta \sqsubseteq \tilde{F}(d)(s) \ominus \tilde{F}(d \ominus \delta_{Y'})(s) \sqsubseteq \tilde{F}(d)(t) \ominus \tilde{F}(d \ominus \delta_{Y'})(t) \sqsubseteq \delta.$$

The last inequality follows from non-expansiveness. Hence

$$\tilde{F}(d)(s) \ominus \tilde{F}(d \ominus \delta_{Y'})(s) = \tilde{F}(d)(t) \ominus \tilde{F}(d \ominus \delta_{Y'})(t) = \delta,$$

which in particular implies that $t \in \tilde{F}^d_\#(Y')$.

In order to conclude we have to show that $\tilde{F}d(t) = \min_{t' \in \Gamma'(\xi(x),\xi(y))} \tilde{F}d(t')$. We first observe that in an MV-chain $\mathbb{M}$, whenever $x \sqsubseteq y$ (for $x, y \in \mathbb{M}$) we can infer that $(y \ominus x) \oplus x = y$ (this follows for instance from Lemma 2.4(6) in [BEKP23a] and duality). The inequality $\tilde{F}(d \ominus \delta_{Y'}) \sqsubseteq \tilde{F}(d)$ holds by monotonicity and we can conclude that

$$\begin{aligned}
\tilde{F}d(t) &= (\tilde{F}(d)(t) \ominus \tilde{F}(d \ominus \delta_{Y'})(t)) \oplus \tilde{F}(d \ominus \delta_{Y'})(t)) \\
&= (\tilde{F}(d)(s) \ominus \tilde{F}(d \ominus \delta_{Y'})(s)) \oplus \tilde{F}(d \ominus \delta_{Y'})(t)) \\
&\sqsubseteq (\tilde{F}(d)(s) \ominus \tilde{F}(d \ominus \delta_{Y'})(s)) \oplus \tilde{F}(d \ominus \delta_{Y'})(s)) = \tilde{F}(d)(s).
\end{aligned}$$

The other inequality $\tilde{F}d(s) \sqsubseteq \tilde{F}d(t)$ holds anyway and hence $\tilde{F}d(t) = \tilde{F}d(s)$. This finally implies, as desired, that

$$\tilde{F}d(t) = \tilde{F}d(s) = \min_{t' \in \Gamma'(\xi(x),\xi(y))} \tilde{F}d(t'). \qquad \square$$

Appendix D. Proofs and Additional Material for §7 (GS-Monoidality)

Theorem 7.2 ($\mathbb{C}_f$ is gs-monoidal)**.** *The category $\mathbb{C}_f$ with the following operators is gs-monoidal:*

(1) *The tensor $\otimes$ on objects $a \in \mathbb{M}^Y$ and $b \in \mathbb{M}^Z$ is defined as*

$$a \otimes b = a + b \in \mathbb{M}^{Y+Z}$$

where for $k \in Y + Z$ we have $(a+b)(k) = a(k)$ if $k \le |Y|$ and $(a+b)(k) = b(k - |Y|)$ if $|Y| < k \le |Y| + |Z|$.

On arrows $f\colon a \dashrightarrow b$ and $g\colon a' \dashrightarrow b'$ (with $a' \in \mathbb{M}^{Y'}$, $b' \in \mathbb{M}^{Z'}$) tensor is given by

$$f \otimes g\colon \mathbb{M}^{Y+Y'} \to \mathbb{M}^{Z+Z'}, \quad (f \otimes g)(u) = f(\overleftarrow{u}_Y) + g(\overrightarrow{u}_Y)$$

for $u \in \mathbb{M}^{Y+Y'}$ where $\overleftarrow{u}_Y \in \mathbb{M}^Y$ and $\overrightarrow{u}_Y \in \mathbb{M}^{Y'}$ are defined as

$$\overleftarrow{u}_Y(k) = u(k) \text{ for } 1 \le k \le |Y| \text{ and } \overrightarrow{u}_Y(k) = u(|Y| + k) \text{ for } 1 \le k \le |Y'|.$$

(2) *The symmetry $\rho_{a,b}\colon a \otimes b \dashrightarrow b \otimes a$ for $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ is defined for $u \in \mathbb{M}^{Y+Z}$ as*

$$\rho_{a,b}(u) = \overrightarrow{u}_Y + \overleftarrow{u}_Y.$$

(3) *The unit e is the unique mapping $e\colon \emptyset \to \mathbb{M}$.*

(4) *The duplicator $\nabla_a\colon a \dashrightarrow a \otimes a$ for $a \in \mathbb{M}^Y$ is defined for $u \in \mathbb{M}^Y$ as*

$$\nabla_a(u) = u + u.$$

(5) *The discharger $!_a\colon a \dashrightarrow e$ for $a \in \mathbb{M}^Y$ is defined for $u \in \mathbb{M}^Y$ as $!_a(u) = e$.*

Proof. In the following let $a \in \mathbb{M}^Y$, $a' \in \mathbb{M}^{Y'}$, $b \in \mathbb{M}^Z$, $b' \in \mathbb{M}^{Z'}$, $c \in \mathbb{M}^W$, $c' \in \mathbb{M}^{W'}$ be objects in $\mathbb{C}_f$.

We know that $\mathbb{C}_f$ is a well-defined category from Lemma 4.2. We also note that disjoint unions of non-expansive functions are non-expansive. Moreover, given $f\colon a \dashrightarrow b$ and $g\colon a' \dashrightarrow b'$, that

$$\begin{aligned}
&(f \otimes g)(a \otimes a') = (f \otimes g)(a + a') \\
&= f(\overleftarrow{(a+a')}_Y) + g(\overrightarrow{(a+a')}_Y) = f(a) + g(a') \\
&= b + b' = b \otimes b'.
\end{aligned}$$

Thus $f \otimes g$ is a well-defined arrow $a \otimes a' \dashrightarrow b \otimes b'$.

We next verify all the axioms of gs-monoidal categories given in Definition 7.1. In general the calculations are straightforward, but they are provided here for completeness.

In the sequel we will often use the fact that $\overleftarrow{u}_Y + \overrightarrow{u}_Y = u$ whenever Y is a subset of the domain of u.

(1) functoriality of tensor:

- $id_{a\otimes b} = id_a \otimes id_b$
  Let $u \in \mathbb{M}^{Y+Z}$. Then
$$(id_a \otimes id_b)(u) = id_a(\overleftarrow{u}_Y) + id_b(\vec{u}_Y) = \overleftarrow{u}_Y + \vec{u}_Y = u = id_{a\otimes b}(u)$$
- $(g \otimes g') \circ (f \otimes f') = (g \circ f) \otimes (g' \circ f')$
  This is required to hold when both sides are defined. Hence let $f\colon a \dashrightarrow b$, $g\colon b \dashrightarrow c$, $f'\colon a' \dashrightarrow b'$, $g'\colon b' \dashrightarrow c'$ and $u \in \mathbb{M}^{Y+Y'}$. We obtain:
$$\begin{aligned}&(g \otimes g') \circ (f \otimes f')(u) = (g \otimes g')(f(\overleftarrow{u}_Y) + f'(\vec{u}_Y))\\ &= g(f(\overleftarrow{u}_Y)) + g'(f'(\vec{u}_Y)) = ((g \circ f) \otimes (g' \circ f'))(\overleftarrow{u}_Y + \vec{u}_Y)\\ &= ((g \circ f) \otimes (g' \circ f'))(u)\end{aligned}$$

(2) monoidality:
- $f \otimes id_e = f = id_e \otimes f$
  Let $f\colon a \dashrightarrow b$ and $u \in \mathbb{M}^Y$. It holds that:
$$\begin{aligned}&(f \otimes id_e)(u) = (f \otimes id_e)(u + e) = f(u) + id_e(e)\\ &= f(u) + e = f(u) = e + f(u)\\ &= id_e(e) + f(u) = (id_e \otimes f)(e + u) = (id_e \otimes f)(u)\end{aligned}$$
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$
  Let $f\colon a \dashrightarrow a'$, $g\colon b \dashrightarrow b'$ and $h\colon c \dashrightarrow c'$ and $u \in \mathbb{M}^{Y+Z+W}$, then
$$\begin{aligned}&((f \otimes g) \otimes h)(u) = (f \otimes g)(\overleftarrow{u}_{Y+Z}) + h(\vec{u}_{Y+Z})\\ &= (f(\overleftarrow{(\overleftarrow{u}_{Y+Z})}_Y) + g(\overrightarrow{(\overleftarrow{u}_{Y+Z})}_Y)) + h(\vec{u}_{Y+Z})\\ &= f(\overleftarrow{u}_Y) + (g(\overleftarrow{(\vec{u}_Y)}_Z) + h(\overrightarrow{(\vec{u}_Y)}_Z))\\ &= f(\overleftarrow{u}_Y) + (g \otimes h)(\vec{u}_Y) = (f \otimes (g \otimes h))(u)\end{aligned}$$
  where we use the fact that $\overrightarrow{(\overleftarrow{u}_{Y+Z})}_Y = \overleftarrow{(\vec{u}_Y)}_Z$.

(3) naturality:
- $(f' \otimes f) \circ \rho_{a,a'} = \rho_{b,b'} \circ (f \otimes f')$
  Let $f\colon a \dashrightarrow b$ and $f'\colon a' \dashrightarrow b'$. Then for $u \in \mathbb{M}^{Y+Y'}$
$$\begin{aligned}&(\rho_{b,b'} \circ (f \otimes f'))(u) = \rho_{b,b'}(f(\overleftarrow{u}_Y) + f'(\vec{u}_Y))\\ &= f'(\vec{u}_Y) + f(\overleftarrow{u}_Y) = (f' \otimes f)(\vec{u}_Y + \overleftarrow{u}_Y) = ((f' \otimes f) \circ \rho_{a,a'})(u)\end{aligned}$$

(4) symmetry:
- $\rho_{e,e} = id_e$
  We note that e is the unique function from $\emptyset$ to $\mathbb{M}$ and furthermore $e \otimes e = e + e = e$. Then
$$\rho_{e,e}(e) = \rho_{e,e}(e + e) = e + e = e = id_e(e)$$
- $\rho_{b,a} \circ \rho_{a,b} = id_{a\otimes b}$
  Let $u \in \mathbb{M}^{Y+Z}$, then:
$$(\rho_{b,a} \circ \rho_{a,b})(u) = \rho_{b,a}(\vec{u}_Y + \overleftarrow{u}_Y) = \overleftarrow{u}_Y + \vec{u}_Y = u = id_{a\otimes b}(u)$$

- $\rho_{a\otimes b,c} = (\rho_{a,c} \otimes id_b) \circ (id_a \otimes \rho_{b,c})$
  Let $u \in \mathbb{M}^{Y+Z+W}$, then:

$$\begin{aligned}
&((\rho_{a,c} \otimes id_b) \circ (id_a \otimes \rho_{b,c}))(u)\\
&= (\rho_{a,c} \otimes id_b)(id_a(\overleftarrow{u}_Y) + \rho_{b,c}(\vec{u}_Y))\\
&= (\rho_{a,c} \otimes id_b)(\overleftarrow{u}_Y + \overrightarrow{(\vec{u}_Y)}_Z + \overleftarrow{(\vec{u}_Y)}_Z)\\
&= \rho_{a,c}(\overleftarrow{u}_Y + \vec{u}_{Y+Z}) + id_b(\overleftarrow{(\vec{u}_Y)}_Z) = \vec{u}_{Y+Z} + \overleftarrow{u}_Y + \overrightarrow{(\overleftarrow{u}_{Y+Z})}_Y\\
&= \vec{u}_{Y+Z} + \overleftarrow{u}_{Y+Z} = \rho_{a\otimes b,c}(u)
\end{aligned}$$

  where we use the fact that $\overrightarrow{(\vec{u}_Y)}_Z = \vec{u}_{Y+Z}$ and $\overleftarrow{(\vec{u}_Y)}_Z = \overrightarrow{(\overleftarrow{u}_{Y+Z})}_Y$.

(5) gs-monoidality:

- $!_e = \nabla_e = id_e$
  Since e is the unique function of type $\emptyset \to \mathbb{M}$ and $e + e = e$, we obtain:

$$!_e(e) = e = id_e(e) = e = e + e = \nabla_e(e)$$

- coherence axioms:
  For $u \in \mathbb{M}^Y$, we note that $\overleftarrow{(u+u)}_Y = \overrightarrow{(u+u)}_Y = u$.
  - $(id_a \otimes \nabla_a) \circ \nabla_a = (\nabla_a \otimes id_a) \circ \nabla_a$
    Let $u \in \mathbb{M}^Y$, then:

$$\begin{aligned}
&((id_a \otimes \nabla_a) \circ \nabla_a)(u) = (id_a \otimes \nabla_a)(u+u)\\
&= id_a(u) + \nabla_a(u) = u + u + u = \nabla_a(u) + id_a(u)\\
&= (\nabla_a \otimes id_a)(u+u) = (\nabla_a \otimes id_a)(\nabla_a(u))
\end{aligned}$$

  - $id_a = (id_a \otimes !_a) \circ \nabla_a$
    Let $u \in \mathbb{M}^Y$, then:

$$\begin{aligned}
&((id_a \otimes !_a) \circ \nabla_a)(u) = (id_a \otimes !_a)(u+u)\\
&= id_a(u) + !_a(u) = id_a(u) + e = id_a(u)
\end{aligned}$$

  - $\rho_{a,a} \circ \nabla_a = \nabla_a$
    Let $u \in \mathbb{M}^Y$, then:

$$(\rho_{a,a} \circ \nabla_a)(u) = \rho_{a,a}(u+u) = u + u = \nabla_a(u)$$

- monoidality axioms:
  - $!_{a\otimes b} = !_a \otimes !_b$
    Let $u \in \mathbb{M}^{Y+Z}$, then:

$$!_{a\otimes b}(u) = e = e + e = !_a(\overleftarrow{u}_Y) + !_b(\vec{u}_Y) = (!_a \otimes !_b)(u)$$

  - $\nabla_a \otimes \nabla_b = (id_a \otimes \rho_{b,a} \otimes id_b) \circ \nabla_{a\otimes b}$
    Let $u \in \mathbb{M}^{Y+Z}$, then:

$$\begin{aligned}
&(id_a \otimes \rho_{b,a} \otimes id_b)(\nabla_{a\otimes b}(u)) = (id_a \otimes \rho_{b,a} \otimes id_b)(u+u)\\
&= (id_a \otimes \rho_{b,a} \otimes id_b)(\overleftarrow{u}_Y + \vec{u}_Y + \overleftarrow{u}_Y + \vec{u}_Y)\\
&= \overleftarrow{u}_Y + \overleftarrow{u}_Y + \vec{u}_Y + \vec{u}_Y = \nabla_a(\overleftarrow{u}_Y) + \nabla_b(\vec{u}_Y)\\
&= (\nabla_a \otimes \nabla_b)(\overleftarrow{u}_Y + \vec{u}_Y) = (\nabla_a \otimes \nabla_b)(u)
\end{aligned}$$

□

Theorem 7.3 ($\mathbb{A}_f$ is gs-monoidal)**.** *The category $\mathbb{A}_f$ with the following operators is gs-monoidal:*

(1) *The tensor $\otimes$ on objects $a \in \mathbb{M}^Y$ and $b \in \mathbb{M}^Z$ is again defined as $a \otimes b = a + b$.*
On arrows $f\colon a \dashrightarrow b$ and $g\colon a' \dashrightarrow b'$ (where $a' \in \mathbb{M}^{Y'}$, $b' \in \mathbb{M}^{Z'}$ and $f\colon \mathcal{P}([Y]^a) \to \mathcal{P}([Z]^{b'})$, $g\colon \mathcal{P}([Y']^{a'}) \to \mathcal{P}([Z']^{b'})$ are the underlying functions), the tensor is given by
$$f \otimes g\colon \mathcal{P}([Y+Y']^{a+a'}) \to \mathcal{P}([Z+Z']^{b+b'}), \quad (f \otimes g)(U) = f(\overleftarrow{U}_Y) \cup_Z g(\vec{U}_Y)$$
where
$$\overleftarrow{U}_Y = U \cap \{1, \dots, |Y|\} \text{ and } \vec{U}_Y = \{k \mid |Y| + k \in U\}.$$
Furthermore:
$$U \cup_Y V = U \cup \{|Y| + k \mid k \in V\} \quad (\text{where } U \subseteq Y)$$
(2) *The symmetry $\rho_{a,b}\colon a \otimes b \dashrightarrow b \otimes a$ for $a \in \mathbb{M}^Y$, $b \in \mathbb{M}^Z$ is defined for $U \subseteq [Y+Z]^{a+b}$ as*
$$\rho_{a,b}(U) = \vec{U}_Y \cup_Z \overleftarrow{U}_Y \subseteq [Z+Y]^{b+a}$$
(3) *The unit e is again the unique mapping $e\colon \emptyset \to \mathbb{M}$.*
(4) *The duplicator $\nabla_a\colon a \dashrightarrow a \otimes a$ for $a \in \mathbb{M}^Y$ is defined for $U \subseteq [Y]^a$ as*
$$\nabla_a(U) = U \cup_Y U \subseteq [Y+Y]^{a+a}.$$
(5) *The discharger $!_a\colon a \dashrightarrow e$ for $a \in \mathbb{M}^Y$ is defined for $U \subseteq [Y]^a$ as $!_a(U) = \emptyset$.*

Proof. Let $a \in \mathbb{M}^Y$, $a' \in \mathbb{M}^{Y'}$, $b \in \mathbb{M}^Z$, $b' \in \mathbb{M}^{Z'}$, $c \in \mathbb{M}^W$, $c' \in \mathbb{M}^{W'}$ be objects in $\mathbb{A}_f$.

We know that $\mathbb{A}_f$ is a well-defined category from Lemma 4.2. We note that, disjoint unions of monotone functions are monotone, making the tensor well-defined.

We now verify the axioms of gs-monoidal categories (see Definition 7.1). The calculations are mostly straightforward.

In the following we will often use the fact that $\overleftarrow{U}_Y \cup_Y \vec{U}_Y = U$ whenever $U \in \mathcal{P}([Z]^b)$ and $Y \subseteq Z$.

(1) functoriality of tensor:
- $id_{a\otimes b} = id_a \otimes id_b$
  Let $U \subseteq [Y+Z]^{a+b}$, then:
$$\begin{aligned}(id_a \otimes id_b)(U) &= (id_a \otimes id_b)(\overleftarrow{U}_Y \cup_Y \vec{U}_Y)\\ &= id_a(\overleftarrow{U}_Y) \cup_Y id_b(\vec{U}_Y) = \overleftarrow{U}_Y \cup_Y \vec{U}_Y = U = id_{a\otimes b}(U)\end{aligned}$$
- $(g \otimes g') \circ (f \otimes f') = (g \circ f) \otimes (g' \circ f')$
  Let $f\colon a \dashrightarrow b$, $g\colon b \dashrightarrow c$, $f'\colon a' \dashrightarrow b'$, $g'\colon b' \dashrightarrow c'$ and $u \in \mathbb{M}^{Y+Y'}$. We obtain:
$$\begin{aligned}((g \otimes g') \circ (f \otimes f'))(U) &= (g \otimes g')(f(\overleftarrow{U}_Y) \cup_Z f'(\vec{U}_Y))\\ &= g(f(\overleftarrow{U}_Y)) \cup_W g'(f'(\vec{U}_Y)) = ((g \circ f) \otimes (g' \circ f'))(\overleftarrow{U}_Y \cup_Y \vec{U}_Y)\\ &= ((g \circ f) \otimes (g' \circ f'))(U)\end{aligned}$$

(2) monoidality:
- $f \otimes id_e = f = id_e \otimes f$
  Let $f\colon a \dashrightarrow b$ and $U \subseteq [Y]^a$. It holds that:
$$(f \otimes id_e)(U) = f(\overleftarrow{U}_Y) \cup_Z id_e(\vec{U}_Y) = f(U) \cup_Z id_e(\emptyset) = f(U) \cup_Z \emptyset$$

$$\begin{aligned}
&= f(U) = \emptyset \cup_\emptyset f(U) = id_e(\emptyset) \cup_\emptyset f(U) = id_e(\overleftarrow{U}_\emptyset) \cup_\emptyset f(\overrightarrow{U}_\emptyset) \\
&= (id_e \otimes f)(U)
\end{aligned}$$

where we use $\overleftarrow{U}_Y = U$ and $\overrightarrow{U}_Y = \emptyset$, since $U \subseteq Y$, as well as $\overleftarrow{U}_\emptyset = \emptyset$ and $\overrightarrow{U}_\emptyset = U$.

- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$
  Let $f\colon a \dashrightarrow a'$, $g \in\colon b \dashrightarrow b'$ and $h\colon c \dashrightarrow c'$ and $U \subseteq [Y + Z + W]^{a+b+c}$. Then:

$$\begin{aligned}
&((f \otimes g) \otimes h)(U) = (f \otimes g)(\overleftarrow{U}_{Y+Z}) \cup_{Y'+Z'} h(\overrightarrow{U}_{Y+Z}) \\
&= \big(f(\overleftarrow{(\overleftarrow{U}_{Y+Z})_Y}) \cup_{Y'} g(\overrightarrow{(\overleftarrow{U}_{Y+Z})_Y})\big) \cup_{Y'+Z'} h(\overrightarrow{U}_{Y+Z}) \\
&= \big(f(\overleftarrow{U}_Y) \cup_{Y'} g(\overleftarrow{(\overrightarrow{U}_Y)_Z})\big) \cup_{Y'+Z'} h(\overrightarrow{U}_{Y+Z}) \\
&= f(\overleftarrow{U}_Y) \cup_{Y'} \big(g(\overleftarrow{(\overrightarrow{U}_Y)_Z}) \cup_{Z'} h(\overrightarrow{(\overrightarrow{U}_Y)_Z})\big) \\
&= f(\overleftarrow{U}_Y) \cup_{Y'} (g \otimes h)(\overrightarrow{U}_Y) = (f \otimes (g \otimes h))(U)
\end{aligned}$$

  where we use $\overleftarrow{(\overleftarrow{U}_{Y+Z})_Y} = \overleftarrow{U}_Y$, $\overrightarrow{(\overleftarrow{U}_{Y+Z})_Y} = \overleftarrow{(\overrightarrow{U}_Y)_Z}$ and $\overrightarrow{U}_{Y+Z} = \overrightarrow{(\overrightarrow{U}_Y)_Z}$.

(3) naturality:

- $(f' \otimes f) \circ \rho_{a,a'} = \rho_{b,b'} \circ (f \otimes f')$
  Let $f\colon a \dashrightarrow b$ and $f'\colon a' \dashrightarrow b'$. Then for $U \subseteq [Y + Y']^{a+a'}$ it holds that:

$$\begin{aligned}
&(\rho_{b,b'} \circ (f \otimes f'))(U) \\
&= \rho_{b,b'}(f(\overleftarrow{U}_Y) \cup_Z f'(\overrightarrow{U}_Y)) \\
&= \overrightarrow{(f(\overleftarrow{U}_Y) \cup_Z f'(\overrightarrow{U}_Y))_Z} \cup_{Z'} \overleftarrow{(f(\overleftarrow{U}_Y) \cup_Z f'(\overrightarrow{U}_Y))_Z} \\
&= f'(\overrightarrow{U}_Y) \cup_{Z'} f(\overleftarrow{U}_Y) \\
&= f'(\overleftarrow{(\overrightarrow{U}_Y \cup_{Y'} \overleftarrow{U}_Y)_{Y'}}) \cup_{Z'} f(\overrightarrow{(\overrightarrow{U}_Y \cup_{Y'} \overleftarrow{U}_Y)_{Y'}}) \\
&= (f' \otimes f)(\overrightarrow{U}_Y \cup_{Y'} \overleftarrow{U}_Y) = (f' \otimes f)(\rho_{a,a'}(U))
\end{aligned}$$

  where we use $\overleftarrow{(U \cup_Y V)_Y} = U$ and $\overrightarrow{(U \cup_Y V)_Y} = V$.

(4) symmetry:

- $\rho_{e,e} = id_e$
  Note that the only possibly argument is $\emptyset$ and hence:

$$\rho_{e,e}(\emptyset) = \overrightarrow{\emptyset}_\emptyset \cup_\emptyset \overleftarrow{\emptyset}_\emptyset = \emptyset \cup_\emptyset \emptyset = \emptyset = id_e(\emptyset)$$

- $\rho_{b,a} \circ \rho_{a,b} = id_{a \otimes b}$
  Let $U \subseteq [Y + Z]^{a+b}$, then:

$$\begin{aligned}
&(\rho_{b,a} \circ \rho_{a,b})(U) = \rho_{b,a}(\overrightarrow{U}_Y \cup_Z \overleftarrow{U}_Y) \\
&= \overrightarrow{(\overrightarrow{U}_Y \cup_Z \overleftarrow{U}_Y)_Z} \cup_Y \overleftarrow{(\overrightarrow{U}_Y \cup_Z \overleftarrow{U}_Y)_Z} = \overleftarrow{U}_Y \cup_Y \overrightarrow{U}_Y = U \\
&= id_{a \otimes b}(U)
\end{aligned}$$

- $\rho_{a \otimes b, c} = (\rho_{a,c} \otimes id_b) \circ (id_a \otimes \rho_{b,c})$
  Let $U \subseteq [Y + Z + W]^{a+b+c}$, then:

$$((\rho_{a,c} \otimes id_b) \circ (id_a \otimes \rho_{b,c}))(U)$$

$$
\begin{aligned}
&= (\rho_{a,c} \otimes id_b)(id_a(\overleftarrow{U}_Y) \cup_Y \rho_{b,c}(\vec{U}_Y)) \\
&= (\rho_{a,c} \otimes id_b)(\overleftarrow{U}_Y \cup_Y (\overrightarrow{(\vec{U}_Y)}_Z \cup_W \overleftarrow{(\vec{U}_Y)}_Z)) \\
&= (\rho_{a,c} \otimes id_b)(\overleftarrow{U}_Y \cup_Y (\vec{U}_{Y+Z} \cup_W \overleftarrow{(\vec{U}_Y)}_Z)) \\
&= (\rho_{a,c} \otimes id_b)((\overleftarrow{U}_Y \cup_Y \vec{U}_{Y+Z}) \cup_{Y+W} \overleftarrow{(\vec{U}_Y)}_Z)) \\
&= \rho_{a,c}(\overleftarrow{U}_Y \cup_Y \vec{U}_{Y+Z}) \cup_{W+Y} id_b(\overleftarrow{(\vec{U}_Y)}_Z) \\
&= (\vec{U}_{Y+Z} \cup_W \overleftarrow{U}_Y) \cup_{W+Y} \overleftarrow{(\vec{U}_Y)}_Z \\
&= \vec{U}_{Y+Z} \cup_W (\overleftarrow{U}_Y \cup_Y \overleftarrow{(\vec{U}_Y)}_Z) \\
&= \vec{U}_{Y+Z} \cup_W (\overleftarrow{(\overleftarrow{U}_{Y+Z})}_Y \cup_Y \overrightarrow{(\overleftarrow{U}_{Y+Z})}_Y) \\
&= \vec{U}_{Y+Z} \cup_W \overleftarrow{U}_{Y+Z} = \rho_{a\otimes b,c}(U)
\end{aligned}
$$

where we use $\overrightarrow{(\vec{U}_Y)}_Z = \vec{U}_{Y+Z}$, $\overleftarrow{U}_Y = \overleftarrow{(\overleftarrow{U}_{Y+Z})}_Y$ and $\overleftarrow{(\vec{U}_Y)}_Z = \overrightarrow{(\overleftarrow{U}_{Y+Z})}_Y$.

(5) gs-monoidality:

- $!_e = \nabla_e = id_e$

  In this case $\emptyset$ is the only possible argument and we have:

  $$!_e(\emptyset) = \emptyset = id_e(\emptyset) = \emptyset = \emptyset \cup_\emptyset \emptyset = \nabla_e(\emptyset)$$

- coherence axioms:

  For $U \subseteq [Y]^a$, we note that $\overleftarrow{(U \cup_Y U)}_Y = \overrightarrow{(U \cup_Y U)}_Y = U$.

  – $(id_a \otimes \nabla_a) \circ \nabla_a = (\nabla_a \otimes id_a) \circ \nabla_a$

  Let $U \subseteq [Y]^a$, then:

  $$
  \begin{aligned}
  &((id_a \otimes \nabla_a) \circ \nabla_a)(U) = (id_a \otimes \nabla_a)(U \cup_Y U) \\
  &= id_a(U) \cup_Y \nabla_a(U) = U \cup_Y (U \cup_Y U) \\
  &= (U \cup_Y U) \cup_{Y+Y} U = \nabla_a(U) \cup_{Y+Y} id_a(U) \\
  &= (\nabla_a \otimes id_a)(U \cup_Y U) = (\nabla_a \otimes id_a)(\nabla_a(U))
  \end{aligned}
  $$

  – $id_a = (id_a \otimes !_a) \circ \nabla_a$

  Let $U \subseteq [Y]^a$, then:

  $$
  \begin{aligned}
  &((id_a \otimes !_a) \circ \nabla_a)(U) = (id_a \otimes !_a)(U \cup_Y U) \\
  &= id_a(U) \cup_Y !_a(U) = id_a(U) \cup_Y \emptyset \\
  &= id_a(U)
  \end{aligned}
  $$

  – $\rho_{a,a} \circ \nabla_a = \nabla_a$

  Let $U \subseteq [Y]^a$, then:

  $$(\rho_{a,a} \circ \nabla_a)(U) = \rho_{a,a}(U \cup_Y U) = U \cup_Y U = \nabla_a(U)$$

- monoidality axioms:

  – $!_{a\otimes b} = !_a \otimes !_b$

  Let $U \subseteq [Y+Z]^{a+b}$, then:

  $$!_{a\otimes b}(U) = \emptyset = \emptyset \cup_\emptyset \emptyset = !_a(\overleftarrow{U}_Y) \cup_\emptyset !_b(\vec{U}_Y)$$

$$= (!_a \otimes !_b)(\overleftarrow{U}_Y \cup_Y \overrightarrow{U}_Y) = (!_a \otimes !_b)(U)$$

– $\nabla_a \otimes \nabla_b = (id_a \otimes \rho_{b,a} \otimes id_b) \circ \nabla_{a \otimes b}$

Let $U \subseteq [Y+Z]^{a+b}$, then:

$$\begin{aligned}
&(id_a \otimes \rho_{b,a} \otimes id_b)(\nabla_{a\otimes b}(U)) \\
&= (id_a \otimes (\rho_{b,a} \otimes id_b))(U \cup_{Y+Z} U) \\
&= id_a(\overleftarrow{(U \cup_{Y+Z} U)}_Y) \cup_Y (\rho_{b,a} \otimes id_b)(\overrightarrow{(U \cup_{Y+Z} U)}_Y) \\
&= id_a(\overleftarrow{U}_Y) \cup_Y (\rho_{b,a} \otimes id_b)(\overrightarrow{(U \cup_{Y+Z} U)}_Y) \\
&= \overleftarrow{U}_Y \cup_Y (\rho_{b,a} \otimes id_b)(\overrightarrow{(U \cup_{Y+Z} U)}_Y) \\
&= \overleftarrow{U}_Y \cup_Y (\rho_{b,a} \otimes id_b)((\overrightarrow{U}_Y \cup_Z \overleftarrow{U}_Y) \cup_{Z+Y} \overrightarrow{U}_Y) \\
&= \overleftarrow{U}_Y \cup_Y ((\overleftarrow{U}_Y \cup_Y \overrightarrow{U}_Y) \cup_{Y+Z} \overrightarrow{U}_Y) \\
&= (\overleftarrow{U}_Y \cup_Y \overleftarrow{U}_Y) \cup_{Y+Y} (\overrightarrow{U}_Y \cup_Z \overrightarrow{U}_Y) \\
&= \nabla_a(\overleftarrow{U}_Y) \cup_{Y+Y} \nabla_b(\overrightarrow{U}_Y) \\
&= (\nabla_a \otimes \nabla_b)(\overleftarrow{U}_Y \cup_Y \overrightarrow{U}_Y) = (\nabla_a \otimes \nabla_b)(U)
\end{aligned}$$

where we use the fact that

$$\overleftarrow{(U \cup_{Y+Z} U)}_Y = \overleftarrow{U}_Y \text{ and } \overrightarrow{(U \cup_{Y+Z} U)}_Y = (\overrightarrow{U}_Y \cup_Z \overleftarrow{U}_Y) \cup_{Z+Y} \overrightarrow{U}_Y.$$

□

Theorem 7.4 (# is gs-monoidal)**.** *$\#\colon \mathbb{C}_f \to \mathbb{A}_f$ is a gs-monoidal functor.*

Proof. We write $e', \otimes', !', \nabla', \rho'$ for the corresponding operators in category $\mathbb{A}_f$. Note that by definition $e = e'$ and $\otimes$, $\otimes'$ agree on objects.

First, categories $\mathbb{C}_f$ and $\mathbb{A}_f$ are gs-monoidal by Theorem 7.2 and 7.3.

Furthermore we verify that:

(1) monoidality:
- $\#(e) = e'$
  We have $\#(e) = e = e'$
- $\#(a \otimes b) = \#(a) \otimes' \#(b)$
  We have:
  $$\#(a \otimes b) = a \otimes b = \#(a) \otimes' \#(b)$$

(2) symmetry:
- $\#(\rho_{a,b}) = \rho'_{\#(a),\#(b)}$
  Let $U \subseteq [Y+Z]^{a+b}$, then for sufficiently small $\delta \sqsupset 0$ (note that such δ exists due to finiteness):
  $$\begin{aligned}
  &\#(\rho_{a,b})(U) \\
  &= (\rho_{a,b})_{\#}^{a+b,\delta}(U) \\
  &= \{w \in [Z+Y]^{b+a} \mid \rho_{a,b}(a+b)(w) \ominus \rho_{a,b}((a+b) \ominus \delta_U)(w) \sqsupseteq \delta\} \\
  &= \{w \in [Z+Y]^{b+a} \mid (b+a)(w) \ominus ((b+a) \ominus \delta_{\rho'_{a,b}(U)})(w) \sqsupseteq \delta\} \\
  &= \rho'_{a,b}(U) = \rho'_{\#(a),\#(b)}(U)
  \end{aligned}$$

since $\rho_{a,b}$ distributes over componentwise subtraction and $\rho_{a,b}(\delta_U) = \delta_{\rho'_{a,b}(U)}$. The second-last equality holds since for all w in the set we have $(b+a)(w) \sqsupset 0$.

(3) gs-monoidality:

- $\#(!_a) = !'_{\#(a)}$
  Let $U \subseteq [Y]^a$, then for some δ:
  $$\#(!_a)(U) = (!_a)^{a,\delta}_{\#}(U) = \emptyset = !'_a(U) = !'_{\#(a)}(U)$$
  since the codomain of $(!_a)^{a,\delta}_{\#}(U)$ is $\mathcal{P}(\emptyset)$ and hence the only possible value for $(!_a)^{a,\delta}_{\#}(U)$ is $\emptyset$.
- $\#(\nabla_a) = \nabla'_{\#(a)}$
  Let $U \subseteq [Y]^a$, then for sufficiently small $\delta \sqsupset 0$:
  $$\begin{aligned}
  &\#(\nabla_a)(U)\\
  &= (\nabla_a)^{a,\delta}_{\#}(U)\\
  &= \{w \in [Y+Y]^{a+a} \mid \nabla_a(a)(w) \ominus \nabla_a(a \ominus \delta_U)(w) \sqsupseteq \delta\}\\
  &= \{w \in [Y+Y]^{a+a} \mid (a+a)(w) \ominus ((a+a) \ominus \delta_{\nabla'_a(U)})(w) \sqsupseteq \delta\}\\
  &= \nabla'_a(U) = \nabla'_{\#(a)}(U)
  \end{aligned}$$
  since ∇_a distributes over componentwise subtraction and $\nabla_a(\delta_U) = \delta_{\nabla'_a(U)}$. The second-last equality holds since for all w in the set we have $(a+a)(w) \sqsupset 0$. □