Domains and Event Structures for Fusions

Paolo Baldan
University of Padova
Andrea Corradini, Fabio Gadducci
University of Pisa

***Abstract*—Stable event structures, and their duality with prime algebraic domains (arising as partial orders of configurations), are a landmark of concurrency theory, providing a clear characterisation of causality in computations. They have been used for defining a concurrent semantics of several formalisms, from Petri nets to linear graph rewriting systems, which in turn lay at the basis of many visual frameworks. Stability however is restrictive for dealing with formalisms where a computational step can merge parts of the state, like graph rewriting systems with non-linear rules, which are needed to cover some relevant applications (such as the graphical encoding of calculi with name passing). We characterise, as a natural generalisation of prime algebraic domains, a class of domains that is well-suited to model the semantics of formalisms with fusions. We then identify a corresponding class of event structures, that we call connected event structures, via a duality result formalised as an equivalence of categories. Connected event structures are exactly the class of event structures the arise as the semantics of non-linear graph rewriting systems. Interestingly, the category of general unstable event structures coreflects into our category of domains, so that our result provides a characterisation of the partial orders of configurations of such event structures.**

***Index Terms*—Event structures, fusions, graph rewriting, process calculi.**

I. Introduction

Since a long time stable/prime event structures and their duality with prime algebraic domains have been considered one of the landmarks of concurrency theory, providing a clear characterisation of causality in software systems. They have been used to provide a concurrent semantics to a wide range of fundational formalisms, from Petri nets [1] to linear graph rewriting systems [2]–[4] and process calculi [5]–[7]. They are one of the standard tools for the formal treatment of (true, i.e., non-interleaving) concurrency. See, e.g., [8] for a reasoned survey on the use of such causal models. Recently, they have been used in the study of concurrency in weak memory models [9], [10] and for process mining and differencing [11]:

In order to endow a chosen formalism with an event structure semantics, a standard construction consists in viewing the class of computations as a partial order. An elements of the order is some sort of configuration, i.e., an execution trace up to an equivalence that identifies traces differing only for the order of independent steps (e.g., interchange law [12] in term rewriting, shift equivalence [13] in graph rewriting, permutation equivalence [14] in the lambda-calculus, ...), and the order relates two computations when the latter is an extension of the former. Events are then identified with configurations consisting of a maximal computation step (e.g.,

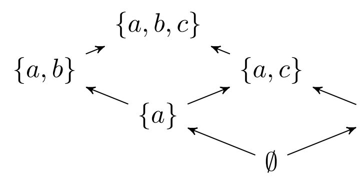
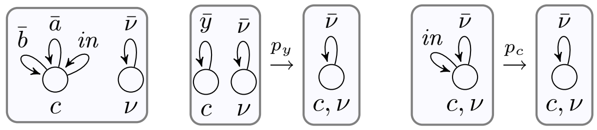


Fig. 1: The domain of configurations of the process $a.b \mid c$.

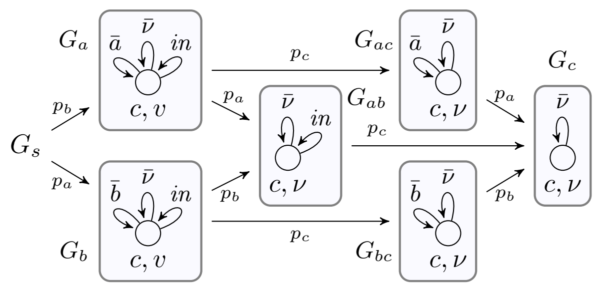
a transition of a CCS process or a transition firing for a Petri net) with all its causes. As a simple example, consider the CCS process $a.b \mid c$. The corresponding partial order is depicted in Fig. 1. The events correspond to configurations $\{a\}$ (transition a with empty set of causes), $\{a, b\}$ (transition b caused by a) and $\{c\}$ (transition a with empty set of causes). The fact that each event in a configuration has a uniquely determined set of causes, a property that for event structures is called *stability*, allows one to characterise such elements, order theoretically, as the prime elements: if they are included in a join they must be included in one of the joined elements. Each element of the partial order of configurations can be reconstructed uniquely as the join of the primes so that the partial order is prime algebraic. This duality between event structures and domains of configurations can be nicely formalised in terms of an equivalence between the category of prime event structures and that of prime algebraic domains [1], [15].

The set up described so far fails when moving to formalisms where a computational step can merge parts of the state, as it happens whenever we consider nominal calculi where as a result of name passing the received name is identified with a local one at the receiver [16], [17] or in the modelling of bonding in biological/chemical processes [18]. Whenever we think of the state of the system as some kind of graph with the dynamics described by graph rewriting, this means that rules are non-linear (precisely, in the so-called double pushout approach [19], left-linear but possibly not right-linear). In general terms, the point is that in the presence of fusions the same event can be enabled by different minimal sets of events, thus preventing the identification of a notion of causality.

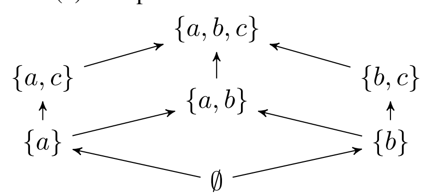
As an example consider the graph rewriting system in Fig. 2. Figure 2a reports the start graph G_s and the rewriting rules p_a, p_b, and p_c. Observe that rules p_y, where y can be either a or b, delete edge $\bar{y}$ and merge nodes c and ν. The possible rewrites are reported in Fig. 2b. For instance, applying p_a to G_s we get the graph G_b. Now, p_b can still be applied to G_b



(a) The start graph G_s and the rules p_y ($y \in \{a, b\}$) and p_c



(b) The possible rewrites



(c) The domain of configurations

Fig. 2: A graph rewriting system with fusions.

matching its left-hand side non-injectively, thus getting graph G_{ab}. Similarly, we can apply first p_b and then G_a, obtaining again G_{ab}. Observe that at least one between p_a and p_b must be applied to enable p_c, since the latter rule requires nodes c and ν to be merged. The graph rewriting system is a (simplified) representation of the π-calculus process $(\nu c)(\bar{a}(c) \mid \bar{b}(c) \mid c())$. Rules p_y, for $y \in \{a, b\}$, represent the execution of $\bar{y}(c)$ that outputs on channel y the restricted name c. The first rule that is executed extrudes name c, while the second is just a standard output. Only after the extrusion the name c is available outside the scope and the input prefix $c()$ can be consumed. Observe that in a situation where all the three rules p_a, p_b, and p_c are applied, since p_a and p_b are independent, it is not possible to define a proper notion of causality. We only know that at least one between p_a and p_b must be applied before p_c. The corresponding domain of configurations, reported in Fig. 2c, is naturally derived from the possible rewrites in Fig. 2b.

The impossibility of modelling these situations with stable event structures is well-known (see, e.g., [15] for a general discussion, [2] for graph rewriting systems or [16] for the π-calculus). One has to drop the stability requirement and replace causality by an enabling relation $\vdash$. More precisely, in the specific case we would have $\emptyset \vdash a$, $\emptyset \vdash b$, $\{a\} \vdash c$, $\{b\} \vdash c$.

The questions that we try to answer is: what can be retained of the satisfactory duality between events structures and domains, when dealing with formalisms with fusions? Which are the properties of the domain of computations that arise in this setting? What are the event structure counterparts?

The domain of configurations of the example suggests that in this context an event is still a computation that cannot be decomposed as the join of other computations hence, in order theoretical terms, it is an irreducible. However, due to unstability, irreducibles are not primes: two different irreducibles can be different minimal histories of the same event, in a way that an irreducible can be included in a computation that is the join of two computations without being included in any of the two. For instance, in the example above, $\{a, c\}$ is an irreducible, corresponding to the execution of c enabled by a, and it is included in $\{a\} \sqcup \{b, c\} = \{a, b, c\}$, although neither $\{a, c\} \subseteq \{a\}$ nor $\{a, c\} \subseteq \{b, c\}$. Uniqueness of decomposition of an element in terms of irreducibles also fails, e.g., $\{a, b, c\} = \{a\} \sqcup \{b\} \sqcup \{a, c\} = \{a\} \sqcup \{b\} \sqcup \{b, c\}$: the irreducibles $\{a, c\}$ and $\{b, c\}$ can be used interchangeably.

Building on the previous observation, we introduce an equivalence on irreducibles identifying those that can be used interchangeably in the decompositions of an element (intuitively, different minimal histories of the same event). Based on this we give a weaker notion of primality (i.e., up to interchangeability) such that the class of domains suited for modelling the semantics of formalisms with fusions are defined as the class of weak prime algebraic domains.

Given a weak prime algebraic domain, a corresponding event structure can be obtained by taking as events the set of irreducibles, quotiented under the (transitive closure of the) interchangeability relation. The resulting class of event structures is a (mild) restriction of the general unstable event structures in [15] that we call connected event structures. Categorically, we get an equivalence between the category of weak prime algebraic domains and the one of connected event structures, generalising the equivalence between prime algebraic domains and prime event structures.

We also show that, in the same way as prime algebraic domains/prime event structures are exactly what is needed for Petri nets/linear graph rewriting systems, weak prime algebraic domains/connected event structures are exactly what is needed for non-linear graph rewriting systems: each rewriting system maps to a connected event structure and viceversa each connected event structure arises as the semantics of some rewriting system. This supports the adequateness of weak prime algebraic domains and connected event structure as semantics structures for formalims with fusions.

Interestingly, we can also show that the category of general unstable event structures [15] coreflects into our category of weak prime algebraic domains. Therefore our notion of weak prime algebraic domain can be seen as a novel characterisation of the partial order of configurations of such event structures that is alternative to those based on intervals in [20], [21]. Our characterisation is a natural generalisation of the one for prime event structures, with irreducibles (instead of primes) having a tight connection with events. The correspondence is established at a categorical level, as a coreflection of categories, something that, to the best of our knowledge, had not been done before in the literature.

The rest of the paper is structured as follows. In Section II we recall the basics of (prime) event structures and their correspondence with prime algebraic domains. In Section III we introduce weak prime algebraic domains, connected event structures and establish a duality result. In Section IV we show the intimate connection between weak prime algebraic domains or equivalently connected event structures, and non-linear graph rewriting systems. Finally, in Section V we wrap up the main contributions of the paper and we sketch further advances and possible connections with related works.

II. Background: domains and event structures

This section recalls the notion of event structures, as introduced in [15], and their duality with partial orders.

A. Event structures

In the paper, we focus on event structures with binary conflict. This choice plays a role only in the relation with graph rewriting (Section IV), while the duality results in Section III can be easily rephrased for event strutures with non-binary conflicts expressed by means of a consistency predicate (see Appendix A). Given a set X we denote by $\mathbf{2}^X$ and $\mathbf{2}^X_{\mathit{fin}}$ the powerset and the set of finite subsets of X, respectively. For $m, n \in \mathbb{N}$, we denote by $[m, n]$ the set $\{m, m+1, \ldots, n\}$.

Definition 1 (event structure) An *event structure* (*es* for short) is a tuple $\langle E, \vdash, \# \rangle$ such that

- E is a set of events;
- $\vdash \subseteq \mathbf{2}^E_{\mathit{fin}} \times E$ is the *enabling* relation satisfying $X \vdash e$ and $X \subseteq Y$ implies $Y \vdash e$;
- $\# \subseteq E \times E$ is the conflict relation.

An $X \subseteq E$ is consistent if there is no $e, e' \in X$ with $(e \# e')$.

An es $\langle E, \vdash, \# \rangle$ is often denoted simply by E. Computations are captured by the notion of configuration.

Definition 2 (configuration, live es) A *configuration* of an es E is a finite consistent $C \subseteq E$ which is *secured*, i.e., C can be totally ordered as $\langle e_1, \ldots, e_n \rangle$ in a way that $\{e_1, \ldots, e_{k-1}\} \vdash e_k$ for all $k \in [1, n]$. The set of configurations of an es E is denoted by $\mathit{Conf}(E)$. An es is *live* if conflict is saturated, i.e., for all $e, e' \in E$, if there is no $C \in \mathit{Conf}(E)$ such that $\{e, e'\} \subseteq C$ then $e \# e'$ and moreover $\neg(e \# e)$ for all $e \in E$.

Observe that when a configuration C is secured via a total ordering $\langle e_1, \ldots, e_n \rangle$, in particular, $\emptyset \vdash e_1$. In this setting, two events are concurrent when they are consistent and enabled by the same configuration.

Remark 1 In the rest of the paper, we restrict to live es, where conflict is saturated (this corresponds to inheritance of conflict in prime event structures) and each event is executable. Hence the qualification live is omitted.

Since the enabling predicate is over finite sets of events, we can consider minimal sets of events enabling a given one.

Definition 3 (minimal enabling) Given an es $\langle E, \vdash, \# \rangle$ define $C \vdash_0 e$ when $C \in \mathit{Conf}(E)$, $C \vdash e$ and for any other configuration $C' \subseteq C$, if $C' \vdash e$ then $C' = C$.

The classes of stable and prime es play an important role in the paper.

Definition 4 (stable and prime es) An es $\langle E, \vdash, \# \rangle$ is *stable* if $X \vdash e$, $Y \vdash e$, and $X \cup Y \cup \{e\}$ consistent imply $X \cap Y \vdash e$. It is *prime* if $X \vdash e$ and $Y \vdash e$ imply $X \cap Y \vdash e$.

For stable es, given a configuration C and an event $e \in C$, there is a unique minimal configuration $C' \subseteq C$ such that $C' \vdash_0 e$. The set C' can be seen as the set of causes of the event e in the configuration C. This gives a well-defined notion of causality that is local to each configuration. In a prime es, for any event e there is a unique minimal enabling $C \vdash_0 e$, thus providing a global notion of causality. In general, in possibly unstable es, due to the presence of consistent *or-enablings*, there might be distinct minimal enablings in the same configuration.

Example 1 A simple example of unstable es is the following: the set of events is $\{a, b, c\}$, the conflict relation $\#$ is the empty one and the minimal enablings are $\emptyset \vdash_0 a$, $\emptyset \vdash_0 b$, $\{a\} \vdash_0 c$, and $\{b\} \vdash_0 c$. Thus, event c has two minimal enablings and these are consistent, hence $\{a, b\} \vdash c$. This is the event structure discussed in the introduction, whose domain of configurations is reported in Fig. 2c.

The class of es can be turned into a category.

Definition 5 (category of es) A morphism of es $f : E_1 \to E_2$ is a partial function $f : E_1 \to E_2$ such that for all $e_1, e_1' \in E_1$ with $f(e_1)$, $f(e_1')$ defined

- if $f(e_1) \# f(e_1')$ then $e_1 \# e_1'$
- if $f(e_1) = f(e_1')$ and $e_1 \neq e_1'$ then $e_1 \# e_1'$;
- if $X_1 \vdash_1 e_1$, for $X_1 \subseteq E$, then $f(X_1) \vdash_2 f(e_1)$.

We denote by ES the category of es and their morphisms and by sES and pES the full subcategories of stable and prime es.

B. Domains

A preordered or partially ordered set $\langle D, \sqsubseteq \rangle$ is often denoted simply as D, omitting the (pre)order relation. Given an element $x \in D$, we write $\downarrow x$ to denote the set $\{y \in D \mid y \sqsubseteq x\}$. We denote by $\preceq$ the immediate predecessor relation, i.e., $x \preceq y$ when $x \sqsubseteq y$ and for all z, if $x \sqsubseteq z \sqsubseteq y$ then $z \in \{x, y\}$. A subset $X \subseteq D$ is *consistent* if it has an upper bound $d \in D$ (i.e., $x \sqsubseteq d$ for all $x \in X$). It is *pairwise consistent* if each two elements subset Y of X is consistent. A subset $X \subseteq D$ is *directed* if $X \neq \emptyset$ and each pair of elements in X has an upper bound in X. It is an *ideal* if it is directed and downward closed. Given D, its ideal completion, denoted $\mathsf{Idl}(D)$, is the set of ideals of D, ordered by subset inclusion. The *least upper bound* and the *greatest lower bound* of a subset $X \subseteq D$ (if they exist) are denoted by $\bigsqcup X$ and $\bigsqcap X$, respectively.

Definition 6 (domains) A partial order D is *coherent* if for any pairwise consistent $X \subseteq D$ the least upper bound $\bigsqcup X$ exists. An element $d \in D$ is *compact* if for any directed $X \subseteq D$, $d \sqsubseteq \bigsqcup X$ implies $d \sqsubseteq x$ for some $x \in X$. The set of compact elements of D is denoted by $\mathsf{K}(D)$. A coherent partial order D is *algebraic* for any $x \in D$ we have

$x = \bigsqcup(\downarrow x \cap \mathsf{K}(D))$. We say that D is *finitary* if for any element $a \in \mathsf{K}(D)$ the set $\downarrow a$ is finite. We refer to algebraic finitary coherent partially ordered sets as *domains*.

For a domain D we can think of its elements as "pieces of information" expressing the states of evolution of a process. Compact elements represent states that are reached after a finite number of steps. Thus algebraicity essentially says that any infinite computation can be approximated with arbitrary precision by the finite ones. More formally, when D is algebraic it is determined by $\mathsf{K}(D)$, i.e., $D \simeq \mathsf{Idl}(\mathsf{K}(D))$.

For an es, the configurations ordered by subset inclusion form a domain. When the es is stable, if an event with its minimal history is in the join of different configurations, then it belongs, with the same history, to one of such configurations. In order-theoretic terms, minimal histories are prime elements, representing the building blocks of computations.

Definition 7 (primes and prime algebraicity) Let D be a domain. A *prime* is an element $p \in \mathsf{K}(D)$ such that, for any pairwise consistent $X \subseteq \mathsf{K}(D)$, if $p \sqsubseteq \bigsqcup X$ then $p \sqsubseteq x$ for some $x \in X$. The set of prime elements of D is denoted by $pr(D)$. The domain D is *prime algebraic* (or simply *prime*) if for all $x \in D$ we have $x = \bigsqcup(\downarrow x \cap pr(D))$.

Prime domains are the domain theoretical counterpart of stable and prime es. For a stable es $\langle E, \#, \vdash \rangle$, the ideal completion of $\langle \mathit{Conf}(E), \subseteq \rangle$ is a prime domain, denoted $\mathcal{D}_S(E)$. Viceversa, given a prime domain D, the triple $\langle pr(D), \#, \vdash \rangle$, where $p \# p'$ if $\{p, p'\}$ is not consistent and $X \vdash p$ when $(\downarrow p \cap pr(D)) \setminus \{p\} \subseteq X$, is a prime es, denoted $\mathcal{E}_S(D)$.

This correspondence can be elegantly formulated at categorical level. We recall the notion of domain morphism [15].

Definition 8 (category of prime domains) Let D_1, D_2 be prime domains. A morphism $f : D_1 \to D_2$ is a total function such that

1) $d_1 \preceq d_1'$ implies $f(d_1) \preceq f(d_1')$;
2) for $X_1 \subseteq D_1$ consistent, $f(\bigsqcup X_1) = \bigsqcup f(X_1)$;
3) for $X_1 \subseteq D_1$ consistent, $f(\bigsqcap X_1) = \bigsqcap f(X_1)$;

We denote by pDom the category of prime domains and their morphisms.

Theorem 1 (duality) *There are functors $\mathcal{D}_S : \mathsf{sES} \to \mathsf{pDom}$ and $\mathcal{E}_S : \mathsf{pDom} \to \mathsf{sES}$ establishing a coreflection. It restricts to an equivalence of categories between pDom and pES.*

III. Weak prime domains and connected es

In this section we characterise a class of domains, and the corresponding brand of es, that are suited for expressing the semantics of computational formalisms with fusions.

A. Weak prime algebraic domains

We show that domains arising in the presence of fusions are characterised by resorting to a weakened notion of prime element. We start recalling the notion of irreducible element.

Definition 9 (irreducibles) Let D a be a domain. An *irreducible* of D is an element $i \in \mathsf{K}(D)$ such that, for any pairwise consistent $X \subseteq \mathsf{K}(D)$, if $i = \bigsqcup X$ then $i \in X$. The set of irreducibles of D is denoted by $ir(D)$ and, for $d \in D$, we define $ir(d) = \downarrow d \cap ir(D)$.

Irreducibles in domains have a simple characterisation.

Lemma 1 (unique predecessor for irreducibles) *Let D be a domain and $i \in D$. Then $i \in ir(D)$ iff it has a unique immediate predecessor, denoted $p(i)$.*

Proof: Assume that $i \in D$ has a unique immediate predecessor $d \prec i$, and let $X \subseteq \mathsf{K}(D)$ be consistent and such that $i = \bigsqcup X$. Hence for any $x \in X$ we have $x \sqsubseteq i$. Assume by contradiction that $i \notin X$. This means that all elements $x \in X$ must be below the immediate predecessor $x \sqsubseteq d$, and therefore $i = \bigsqcup X \sqsubseteq d \prec i$, which is a contradiction. Hence it must be $i \in X$, which means that i is irreducible.

Vice versa, let i be irreducible and let $d_1, d_2 \prec i$ be immediate predecessors. Since D is a domain and $\{d_1, d_2\}$ is consistent, we can take $d = d_1 \sqcup d_2$ and we know $d_1 \sqsubseteq d \sqsubseteq i$. Since i is irreducible it cannot be $d = i$, therefore $d = d_1$ and thus $d_1 = d_2$. This means that i has a unique immediate predecessor. ■

We next observe that any domain is actually irreducible algebraic, namely it can be generated by the irreducibles.

Proposition 1 (domains are irreducible algebraic) *Let D be a domain. Then for any $d \in D$ it holds $d = \bigsqcup ir(d)$.*

Proof: We first prove that for any compact element $d \in \mathsf{K}(D)$ it holds that $d = \bigsqcup(\downarrow d \cap ir(D))$. The thesis then immediately follows from algebraicity. Since D is a domain, $\downarrow d$ is finite, hence we can proceed by induction on $|\downarrow d|$. When $|\downarrow d| = 1$, we have that $d = \bot$, hence $\downarrow d \cap ir(D) = \emptyset$ and indeed $\bot = \bigsqcup \emptyset$. When $|\downarrow d| = k > 1$ consider the immediate predecessors of d and denote them $d_1, \ldots, d_n \prec d$. Since D is a domain and $\{d_1, \ldots, d_n\}$ is consistent, there exists $\bigsqcup\{d_1, \ldots, d_n\} = d'$ and $d_i \sqsubseteq d' \sqsubseteq d$. There are two cases

- $d' = d_i$, for all $i \in [1, n]$, i.e., d has a unique immediate predecessor, hence it is an irreducible and thus clearly $d = \bigsqcup(\downarrow d \cap ir(D))$ or
- $d = d' = \bigsqcup\{d_1, \ldots, d_n\}$. Since, in turn, by inductive hypothesis $d_i = \bigsqcup(\downarrow d_i \cap ir(D))$ and $\downarrow d \cap ir(D) = \bigcup_{i=1}^{n}(\downarrow d_i \cap ir(D))$, we immediately get the thesis.

■

Now note that any prime is an irreducible. If D is a prime domain then also the converse holds. i.e., the irreducibles coincide with the primes.

Proposition 2 (irreducibles vs. primes) *Let D be a domain. Then D is a prime domain iff $pr(D) = ir(D)$.*

Proof: Let D be a prime domains. We observed that $pr(D) \subseteq ir(D)$ holds in general domains. For the converse inclusion, let $i \in ir(D)$. By prime algebraicity $i = \bigsqcup \downarrow i \cap pr(D)$. Since i is irreducible, there exists $p \in \downarrow i \cap pr(D)$ such that $i = p$, hence i is a prime.

Vice versa, if D is a domain, by Proposition 1 we know that D is irreducible algebraic. Hence, if $pr(D) = ir(D)$, we immediately conclude that D is prime. ■

Quite intuitively, in the domain of configurations of an es the irreducibles are minimal histories of events. For instance, in the domain depicted in Fig. 2c the irreducibles are $\{a\}$, $\{b\}$, $\{a,c\}$, and $\{b,c\}$. For stable es, the domain is prime and thus, as observed above, irreducibles coincide with primes. This fails in unstable es, as we can see in our running example: while $\{a\}$ and $\{b\}$ are primes, the two minimal histories of c, namely $\{a,c\}$ and $\{b,c\}$, are not. In fact, $\{a,c\} \subseteq \{a\} \sqcup \{b,c\}$, but neither $\{a,c\} \subseteq \{a\}$ nor $\{a,c\} \subseteq \{b,c\}$.

The key observation is that in general an event corresponds to a class of irreducibles, like $\{a,c\}$ and $\{b,c\}$ in our example. Additionally, two irreducibles corresponding to the same event can be used, to a certain extent, interchangeably for building the same configuration. For instance, $\{a,b,c\} = \{a,b\} \cup \{a,c\} = \{a,b\} \cup \{b,c\}$. We next formalise this intuition, i.e., we interpret irreducibles in a domain as minimal histories of some event and we identify classes of irreducibles corresponding to the same event.

We start by observing that in a prime domain any element admits a unique decomposition in terms of irreducibles.

Lemma 2 (unique decomposition) *Let D be a prime domain. If $X, X' \subseteq ir(D)$ are downward closed sets of irreducibles such that $\bigsqcup X = \bigsqcup X'$ then $X = X'$.*

Proof: Let $X, X' \subseteq ir(D)$ be downward closed sets of irreducibles such that $\bigsqcup X = \bigsqcup X'$. Take any $i' \in X'$. Then $i' \sqsubseteq x = \bigsqcup X$. Since the domain is prime algebraic, and thus i' is prime, there must exist $i \in X$ such that $i' \sqsubseteq i$ and thus $i' \in X$. Therefore $X' \subseteq X$. By symmetry also the converse inclusion holds, whence equality. ■

The result above no longer holds in domains arising in the presence of fusions. For instance, in the domain in Fig. 2c, $X = \{\{a\},\{a,c\},\{b\}\}$, $X' = \{\{a\},\{b\},\{b,c\}\}$ and $X'' = \{\{a\},\{b\},\{b,c\},\{a,c\}\}$ are all decompositions for $\{a,b,c\}$. The idea is to identify irreducibles that can be used interchangeably in a decomposition.

Definition 10 (interchangeability) Let D be a domain and $i, i' \in ir(D)$. We write $i \leftrightarrow i'$ if for all $X \subseteq ir(D)$ such that $X \cup \{i\}$ and $X \cup \{i'\}$ are downward closed and consistent we have $\bigsqcup(X \cup \{i\}) = \bigsqcup(X \cup \{i'\})$ and for some such X it holds $\bigsqcup X \neq \bigsqcup(X \cup \{i\})$.

In words, $i \leftrightarrow i'$ means that i and i' produce the same effect when added to a decomposition that already includes their predecessors and there is at least one situation in which the addition of i and i' produces some effect. Hence, intuitively, i and i' correspond to the execution of the same event.

Clearly, interchangeable irreducibles need to be consistent.

Lemma 3 *Let D be a domain and $i, i' \in ir(D)$ such that $i \leftrightarrow i'$. Then i and i' are consistent. And if $i \sqsubseteq i'$ then $i = i'$.*

Proof: The fist part is obvious, since, by definition, there must exist $X \subseteq ir(D)$ such that $X \cup \{i\}$ and $X \cup \{i'\}$ downward closed, and $\bigsqcup(X \cup \{i\}) = \bigsqcup(X \cup \{i'\})$. Therefore $i, i' \sqsubseteq \bigsqcup(X \cup \{i\}) = \bigsqcup(X \cup \{i'\})$ and thus i, i' are consistent.

For the second part, let $i \sqsubseteq i'$. If $i \neq i'$ and we let $X = ir(p(i'))$, it turns out that $X \cup \{i\} = X$ and $X \cup \{i'\}$ are consistent and downward closed. Moreover $\bigsqcup X \cup \{i\} = \bigsqcup XX = p(i') \neq \bigsqcup X \cup \{i'\} = i'$, contradicting $i \leftrightarrow i'$. ■

We now give some equivalent characterisations of interchangeability.

Lemma 4 (characterising $\leftrightarrow$) *Let D be a domain and $i, i' \in ir(D)$. Then the following are equivalent*

1) *$i \leftrightarrow i'$;*
2) *i, i' consistent and for all $d \in \mathsf{K}(D)$ such that $p(i), p(i') \sqsubseteq d$, $d \sqcup i = d \sqcup i'$ and for some such d it holds $d \neq d \sqcup i$;*
3) *i, i' consistent and $i \sqcup p(i') = p(i) \sqcup i' \neq p(i) \sqcup p(i')$.*

Proof:

($1 \to 2$) Assume that $i \leftrightarrow i'$ and let $d \in \mathsf{K}(D)$ such that $p(i), p(i') \sqsubseteq d$. If we let $X = ir(d)$ we have that $ir(i) \setminus \{i\} \subseteq X$ and similarly $ir(i') \setminus \{i'\} \subseteq X$. Therefore $X \cup \{i\}$ and $X \cup \{i'\}$ are downward closed and consistent. Hence $d \sqcup i = \bigsqcup X \sqcup i = \bigsqcup(X \cup \{i\}) = \bigsqcup(X \cup \{i'\}) = \bigsqcup X \sqcup i' = d \sqcup i'$.

Moreover, if we consider the set $X \subseteq ir(D)$ required by the definition of interchangeability, such that $\bigsqcup X \neq \bigsqcup(X \cup \{i\}) = \bigsqcup(X \cup \{i'\})$ and define $d = \bigsqcup X$, we obtain $d \neq \bigsqcup(X \cup \{i\}) = \bigsqcup X \sqcup i = d \sqcup i$, as desired.

($2 \to 3$) Assume (2). Let $p = p(i) \sqcup p(i')$. Clearly, $p(i), p(i') \sqsubseteq p$. Therefore $i \sqcup p(i') = i \sqcup p(i) \sqcup p(i') = i \sqcup p = p \sqcup i' = p(i) \sqcup p(i') \sqcup i' = p(i) \sqcup i'$.

Moreover, $p(i) \sqcup p(i') \neq p(i) \sqcup i'$, otherwise for any $d \in \mathsf{K}(D)$ such that $p(i), p(i') \sqsubseteq d$, we would have $d \sqsubseteq d \sqcup i = d \sqcup p(i') \sqcup i = d \sqcup p(i) \sqcup p(i')$, contradicting the second part of condition (2).

($3 \to 1$) Assume (3). Let $X \subseteq ir(D)$ be such that $X \cup \{i\}$ and $X \cup \{i'\}$ are downward closed and consistent sets of irreducibles. This implies that $ir(p(i)) \subseteq X$ and similarly $ir(p(i')) \subseteq X$. Hence, if we let $P = (\downarrow p(i) \cup \downarrow p(i')) \cap ir(D)$

$$P \subseteq X \text{ and } \bigsqcup P = p(i) \sqcup p(i')$$

Therefore

$$\begin{aligned}
&\bigsqcup(X \cup \{i\}) = \\
&\quad = (\bigsqcup X \setminus P) \sqcup \bigsqcup P \sqcup i = \\
&\quad = (\bigsqcup X \setminus P) \sqcup p(i) \sqcup p(i') \sqcup i = \\
&\quad = (\bigsqcup X \setminus P) \sqcup i \sqcup p(i') = \\
&\quad = (\bigsqcup X \setminus P) \sqcup p(i) \sqcup i' = \\
&\quad = (\bigsqcup X \setminus P) \sqcup p(i) \sqcup p(i') \sqcup i' = \\
&\quad = (\bigsqcup X \setminus P) \sqcup \bigsqcup P \sqcup i' = \\
&\quad = \bigsqcup(X \cup \{i'\})
\end{aligned}$$

Moreover, if we take $X = ir(p(i)) \cup ir(p(i'))$ then $X \cup \{i\}$ and $X \cup \{i'\}$ are downward closed and consistent. Moreover, $\bigsqcup X = p(i) \sqcup p(i') \neq p(i) \sqcup i' = \sqcup(X \cup \{i'\})$, as desired. ■

The interchangeability relation is reflexive and symmetric, but not transitive: in the domain of Fig. 3, $i \leftrightarrow i'$ and $i' \leftrightarrow i''$

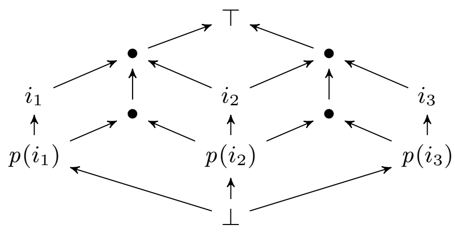


Fig. 3: Interchangeability need not be transitive.

but not $i \leftrightarrow i''$. The same holds in the domain obtained by removing the top element. We see later that on weak prime algebraic domains $\leftrightarrow$ is transitive on consistent irreducibles (see Lemma 10).

We now introduce weak primes: they weaken the property of prime elements, requiring that it holds up to interchangeability.

Definition 11 (weak prime) Let D be a domain. A *weak prime* of D is an element $i \in ir(D)$ such that for any consistent $X \subseteq D$, if $i \sqsubseteq \bigsqcup X$ then there exist $i' \in ir(D)$ with $i \leftrightarrow i'$ and $d \in X$ such that $i' \sqsubseteq d$. We denote by $wpr(D)$ the set of weak primes of D.

Clearly, any prime is a weak prime since interchangeability is reflexive. Moreover, in prime domains interchangeability is the identity and also the converse holds.

Lemma 5 (weak primes in prime domains) *Let D be a prime domain. Then $\leftrightarrow$ is the identity and $wpr(D) = pr(D)$.*

Proof: Let $i, i' \in ir(D)$ be such that $i \leftrightarrow i'$.

If i and i' are comparable, i.e., $i \sqsubseteq i'$ or $i' \sqsubseteq i$, by Lemma 3, we deduce $i = i'$ and we are done.

Otherwise, let $X = (ir(i) \setminus \{i\}) \cup (ir(i') \setminus \{i'\})$. Note that $X \cup \{i\}$ and $X \cup \{i'\}$ are consistent, since by Lemma 3, i and i' are so. Moreover $X \cup \{i\}$ and $X \cup \{i'\}$ are downward closed, and thus, since $i \leftrightarrow i'$, we deduce $\bigsqcup(X \cup \{i\}) = \bigsqcup(X \cup \{i'\})$. Since D is prime, by Lemma 2, this implies that $X \cup \{i\} = X \cup \{i'\}$. Since i and i' are uncomparable, $i, i' \notin X$ and thus we conclude $i = i'$. ∎

We argue that the domain of configurations arising in the presence of fusions can be characterised domain-theoretically by asking that all irreducibles are weak primes, i.e., that the domain is algebraic with respect to weak primes.

Definition 12 (weak prime algebraic domains) Let D be a domain. A domain D is a *weak prime algebraic domain* (or simply *weak prime domain*) if for any $d \in D$ it holds $d = \bigsqcup(\downarrow d \cap wpr(D))$.

In the same way as prime domains are domains where all irreducibles are primes (see Proposition 2), we can provide a characterisation of weak prime domains in terms of coincidence between irreducibles and weak primes.

Proposition 3 (weak prime domains, again) *Let D be a domain. It is a weak prime domain iff all irreducibles are weak primes.*

A domain is often built as the ideal completion of its compact elements. We next provide a characterisation of domains and weak prime domains based on the generators.

Lemma 6 (weak prime domains from generators) *Let $(P, \sqsubseteq)$ be a finitary partial order such that for all $d, d', d'' \in P$, if $\{d, d', d''\}$ is pairwise consistent then $d \sqcup d'$ exists and is consistent with d''. Then $\mathsf{Idl}(P)$ is a domain with $\mathsf{K}(\mathsf{Idl}(P)) = \{\downarrow d \mid d \in P\} \simeq P$.*

Additionally, let $\leftrightarrow$ be transitive on consistent irreducibles and for all $i \in ir(P)$, $d, d' \in P$ consistent, if $i \sqsubseteq d \sqcup d'$ then there is $i' \in ir(P)$, $i \leftrightarrow i'$ such that $i' \sqsubseteq d$ or $i' \sqsubseteq d'$. Then $\mathsf{Idl}(P)$ is a weak prime domain.

Proof: Let $(P, \sqsubseteq)$ be a finitary partial order such that for all $d, d', d' \in P'$, if $\{d, d', d''\}$ is pairwise consistent then $d \sqcup d'$ exists and is consistent with d''.

The fact that $\mathsf{Idl}(P)$ is a complete partial order with $\mathsf{K}(\mathsf{Idl}(P)) = \{\downarrow d \mid d \in P\} \simeq P$ is a standard result. Moreover, let $X \subseteq \mathsf{Idl}(P)$ pairwise consistent. Consider $A = \bigcup\{I \mid I \in X\}$. Observe that for any finite $Y \subseteq A$ there exists $\bigsqcup Y$ in P. In fact, let $Y = \{y_1, \ldots, y_n\}$. This means that there are $I_1, \ldots, I_n$ such that $y_i \in I_i$ for each $i \in [1, n]$. Since X is pairwise consistent in $\mathsf{Idl}(P)$, we deduce that Y is pairwise consistent in P. Since y_1, y_2 are consistent, and both are consistent with $y_3, \ldots, y_n$, by (2) there exists $y_1 \sqcup y_2$ and it is consistent with $y_3, \ldots, y_n$, i.e., $\{y_1 \sqcup y_2, y_3, \ldots, y_n\}$ is again pairwise consistent. Iterating the reasoning we get the existence of $y_1 \sqcup y_2 \sqcup \ldots \sqcup y_n = \bigsqcup Y$, as desired. Now, if we define $I' = \{\bigsqcup Y \mid Y \subseteq_{fin} A\}$, then I' is an ideal and $I' = \bigsqcup X$.

For the second part, we need to show that under the hypotheses, if $I \in ir(\mathsf{Idl}(P))$ and $X \subseteq \mathsf{Idl}(P)$ pairwise consistent and $I \subseteq \bigsqcup X$ then there exists $I' \leftrightarrow I$ and $A \in X$ such that $I' \subseteq A$. It is immediate to see that $ir(\mathsf{Idl}(P)) = \{\downarrow i \mid i \in ir(P)\}$. Thus let $I = ir(i)$ for some $i \in ir(P)$. The fact that $I \subseteq \bigsqcup X = \bigsqcup\{\downarrow d \mid d \in \bigcup X\}$ means that $i \sqsubseteq \bigsqcup\{d_1, \ldots, d_n\}$ for some finite subset $\{d_1, \ldots, d_n\} \subseteq \bigcup X$. Since $i \sqsubseteq d_1 \sqcup \bigsqcup\{d_2, \ldots, d_n\}$, by the hypothesis there is $i_1 \leftrightarrow i$ such that $i_1 \sqsubseteq d_1$ or $i_1 \sqsubseteq \bigsqcup\{d_2, \ldots, d_n\} = d_2 \sqcup \bigsqcup\{d_3, \ldots, d_n\}$. In the second case, again by the hypothesis, there is $i_2 \leftrightarrow i_1$, hence by transitivity $i_2 \leftrightarrow i$ such that $i_2 \sqsubseteq d_2$ or $i_1 \sqsubseteq \bigsqcup\{d_3, \ldots, d_n\}$. Thus we finally get the existence of some $i' \leftrightarrow i$ and $j \in [1, n]$ such that $i' \sqsubseteq d_j$. Recalling that $d_j \in \bigcup X$, there is $I \in X$ such that $d_j \in I$, hence $\downarrow i_j \subseteq \downarrow d_j \subseteq I$. Noting that $i \leftrightarrow i_j$ implies that also in $\mathsf{Idl}(P)$ the irreducibles $\downarrow i$ and $\downarrow i_j$ are interchangeable, i.e., $\downarrow i \leftrightarrow \downarrow i_j$, we conclude. ∎

We finally introduce a category of weak prime domains by defining a notion of morphism.

Definition 13 (category of weak prime domains) A weak prime domain morphism $f : D_1 \to D_2$ is a total function such that

1) $d_1 \preceq d'_1$ implies $f(d_1) \preceq f(d'_1)$;
2) for $X_1 \subseteq D_1$ consistent, $f(\bigsqcup X_1) = \bigsqcup f(X_1)$;

3) for $d_1, d'_1 \in D_1$ consistent, if $d_1 \sqcap d'_1 \preceq d_1$ then $f(d_1 \sqcap d'_1) = f(d_1) \sqcap f(d'_1)$;

We denote by wDom the category of weak prime domains and their morphisms.

Compared with the notion of morphism for prime domains in Definition 8, taken from [15], note that we still require the preservation of $\preceq$ and $\sqcup$ of consistent sets (conditions (1) and (2)). However, the third condition, i.e., preservation of $\sqcap$, is weakened to preservation in some cases. General preservation of meets is indeed not expected in the presence of fusions. Consider e.g. the running example in Example 1 and another es $E' = \{c\}$ with $\emptyset \vdash c$ and the morphism $f : E \to E'$ that forgets a and b. Then $f(\{a, c\}) \sqcap f(\{b, c\}) = \{c\} \sqcap \{c\} = \{c\} \neq f(\{a, c\} \sqcap \{b, c\}) = f(\emptyset) = \emptyset$. Intuitively, the condition $d_1 \sqcap d'_1 \prec d_1$ means that d'_1 includes the computation modelled by d_1 apart from a final step, hence $d_1 \sqcap d'_1$ coincides with d_1 when such final step is removed. Since domain morphisms preserve immediate precedence (i.e., single steps), also $f(d_1)$ differs from $f(d'_1)$ for the execution of a final step and the meet $f(d_1) \sqcap f(d'_1)$ is $f(d_1)$ without such step, and thus it coincides with $f(d_1 \sqcap d'_1)$.

In general we only have

$$f(\bigsqcap X_1) \sqsubseteq \bigsqcap f(X_1)$$

In fact, for all $x_1 \in X_1$, we have $\bigsqcap X_1 \sqsubseteq x_1$, hence $f(\bigsqcap X_1) \sqsubseteq f(x_1)$ and thus $f(\bigsqcap X_1) \sqsubseteq \bigsqcap f(X_1)$. Still, when restricted to prime domains, our notion of morphism can be shown to boil down to the original one, i.e., the full subcategory of wDom having prime domains as objects is pDom.

Lemma 7 (meet preservation for prime domains) *Let D_1, D_2 be prime domains and $f : D_1 \to D_2$ a weak prime domain morphism. Then $f(\bigsqcap X_1) = \bigsqcap f(X_1)$.*

Proof: We first show that for $d_1, d'_1 \in \mathsf{K}(D_1)$, it holds that $f(d_1 \sqcap d'_1) = f(d_1) \sqcap f(d'_1)$. We proceed by induction on $k = |\downarrow d_1 \cap pr(D)|$.

When $k = 0$ we have $d_1 = \bot$. Since f preserves joins, we have that $f(\bot) = f(\bigsqcup \emptyset) = \bigsqcup f(\emptyset) = \bigsqcup \emptyset = \bot$. Hence

$$f(d_1 \sqcap d'_1) = f(\bot \sqcap d'_1) = f(\bot) = \bot = \bot \sqcap f(d'_1) = f(\bot) \sqcap f(d'_1) = f(d_1) \sqcap f(d'_1).$$

Suppose now $k > 0$. We distinguish two subcases. If d_1 is not prime then, recalling that in a prime domain, primes and irreducibles coincide, d_1 is not irreducible and thus $d_1 = e_1 \sqcup f_1$ with $d_1 \neq e_1, f_1 \neq \bot$. It is immediate to see that $|\downarrow e_1 \cap pr(D)| < k$ and $|\downarrow f_1 \cap pr(D)| < k$. Moreover, since any prime algebraic domain is distributive we have $d_1 \sqcap d'_1 = (e_1 \sqcup f_1) \sqcap d'_1 = (e_1 \sqcap d'_1) \sqcup (f_1 \sqcap d'_1)$. Summing up

$$\begin{aligned}
&f(d_1 \sqcap d'_1) = \\
&\quad f((e_1 \sqcap d'_1) \sqcup (f_1 \sqcap d'_1)) = && [\text{Preservation of } \sqcup] \\
&\quad f(e_1 \sqcap d'_1) \sqcup f(f_1 \sqcap d'_1) = && [\text{Inductive hypothesis}] \\
&\quad (f(e_1) \sqcap f(d'_1)) \sqcup (f(f_1) \sqcap f(d'_1)) = && [\text{Distributivity}] \\
&\quad (f(e_1) \sqcup f(f_1)) \sqcap f(d'_1) = && [\text{Preservation of } \sqcup] \\
&\quad f(e_1 \sqcup f_1) \sqcap f(d'_1) = \\
&\quad f(d_1) \sqcap f(d'_1)
\end{aligned}$$

If instead d_1 is prime then note that if $d_1 \sqsubseteq d'_1$ the thesis is immediate: by monotonicity $f(d_1) \sqsubseteq f(d'_1)$. Thus $f(d_1 \sqcap d'_1) = f(d_1) = f(d_1) \sqcap f(d'_1)$ as desired. Therefore, let us assume that $d_1 \not\sqsubseteq d'_1$. In this case $d_1 \sqcap d'_1 = p(d_1) \sqcap d'_1$, since the set of lower bounds of $\{d_1, d'_1\}$ and of $\{p(d_1), d'_1\}$ is the same. Observe that

$$p(d_1) = d_1 \sqcap (p(d_1) \sqcup d'_1) \tag{1}$$

In fact, by distributivity, $d_1 \sqcap (p(d_1) \sqcup d'_1) = (d_1 \sqcap p(d_1)) \sqcup (d_1 \sqcap d'_1) = p(d_1) \sqcup (p(d_1) \sqcap d'_1) = p(d_1)$

Therefore

$$\begin{aligned}
&f(d_1 \sqcap d'_1) = \\
&\quad f(p(d_1) \sqcap d'_1) = && [\text{Inductive hypothesis}] \\
&\quad f(p(d_1)) \sqcap f(d'_1) = && [\text{Using (1)}] \\
&\quad f(d_1 \sqcap (p(d_1) \sqcup d'_1)) \sqcap f(d'_1) = && [\text{By Definition 13(3)}] \\
&\quad f(d_1) \sqcap f(p(d_1) \sqcup d'_1)) \sqcap f(d'_1) = && [\text{Preservation of } \sqcup] \\
&\quad f(d_1) \sqcap f(d'_1)
\end{aligned}$$

as desired. This extends to the meet of finite sets of compacts, by associativity of $\sqcap$, and to infinite sets of compacts by observing that, given an infinite set X, by finitariness we can identify a finite subset $F \subseteq X$ such that $\bigsqcap X = \bigsqcap F$. ■

Proposition 4 *The category of prime domains* pDom *is the full subcategory of* wDom *having prime domains as objects.*

B. Connected es

We show that, given an es, its set of configurations, ordered by subset inclusion, is a weak prime domain. More precisely, since configurations are finite sets, they represent the compact elements of the domain, which is thus obtained by ideal completion. Moreover, the correspondence can be lifted to a functor. We also identify a subclass of es that we call connected es and that are the exact counterpart of weak prime domains (in the same way as prime es correspond to prime algebraic domains).

Definition 14 (poset of configurations of an es) Let E be an es. We denote by $\mathcal{D}(E)$ the ideal completion

$\mathsf{Idl}(\langle \mathit{Conf}(E), \subseteq \rangle)$. Given an es morphism $f : E_1 \to E_2$, its image $\mathcal{D}(f) : \mathcal{D}(E_1) \to \mathcal{D}(E_2)$ is defined as $\mathcal{D}(f)(C_1) = \{f(e_1) : e_1 \in C_1\}$.

We first need some technical facts, collected in the following lemma. Recall that in the setting of unstable es we can have distinct consistent minimal enablings for an event. When $C \vdash_0 e$, $C' \vdash_0 e$, and $C \cup C' \cup \{e\}$ is consistent, we write $C \overset{e}{\frown} C'$. We denote by $\overset{e}{\frown}^*$ the transitive closure of the relation $\overset{e}{\frown}$.

Lemma 8 *Let $\langle E, \vdash, \mathit{Con}\rangle$ be an es. Then*

1) *$\mathcal{D}(E)$ is a domain, $\mathsf{K}(\mathcal{D}(E)) = \mathit{Conf}(E)$, join is union and $C \preceq C'$ iff $C = C' \cup \{e\}$ for some $e \in E$;*
2) *$C \in \mathit{Conf}(E)$ is irreducible iff $C = C' \cup \{e\}$ and $C' \vdash_0 e$; it is denoted as $I = \langle C', e\rangle$;*
3) *for $C \in \mathit{Conf}(E)$, we have $ir(C) = \{\langle C', e'\rangle \mid e' \in C \ \wedge\ C' \subseteq C \ \wedge\ C' \vdash_0 e'\}$;*
4) *for $\langle C_1, e_1\rangle, \langle C_2, e_2\rangle \in ir(\mathcal{D}(E))$, we have $\langle C_1, e_1\rangle \leftrightarrow \langle C_2, e_2\rangle$ iff $e = e_1 = e_2$ and $C_1 \overset{e}{\frown} C_2$.*

Proof:

1) Observe that the poset of configurations $\langle \mathit{Conf}(E), \sqsubseteq \rangle$ is finitary since configurations are finite and given $C, C', C'' \in \mathit{Conf}(E)$, pairwise consistent, the join $C \sqcup C'$ is the union $C \sqcup C'$, which is consistent with C''. The fact that $\mathcal{D}(E)$ is a domain thus follows by Lemma 6.
Concerning immediate precedence, let $C, C' \in \mathit{Conf}(E)$. If $C' = C \cup \{e\}$ then clearly $C \prec C'$, since the order is subset inclusion. Vice versa, if $C \prec C'$ by definition $C \subseteq C'$ and it must be $|C' \setminus C| = 1$. In fact, let $C = \{e_1, \dots, e_m\}$ with $\langle e_1, \dots, e_m\rangle$ a secured execution and let $C' \setminus C = \{e_{m+1}, \dots, e_n\}$. Since enabling is monotone, there is a secured execution of C' of the kind $\langle e_1, \dots, e_m, e_{m+1}, \dots, e_n\rangle$. Now, if it were $|C' \setminus C| > 1$ the set $C'' = \{e_1, \dots, e_m, e_{m+1}\}$ would be a configuration, with $C \subset C'' \subset C'$, contradicting $C \prec C'$.
2) Let $C \in \mathit{Conf}(E)$ be a configuration and assume that $C = C' \cup \{e\}$ such that $C' \vdash_0 e$. If $C = C_1 \cup C_2$ for $C_1, C_2 \in \mathit{Conf}(E)$, then e must occur either in C_1 or in C_2. If $e \in C_1$, since C_1 is secured, there exists $C_1' \subseteq C_1 \setminus \{e\}$ such that $C_1' \vdash e$. Hence, by monotonicity of enabling, $C_1 \setminus \{e\} \vdash e$. Since $C' \vdash_0 e$ and $C_1 \setminus \{e\} \subseteq C'$ it follows that $C_1 \setminus \{e\} = C'$ and thus $C_1 = C'$.
Vice versa, let $C \in \mathit{Conf}(E)$ be irreducible. Consider a secured execution $\langle e_1, \dots, e_n\rangle$ of configuration C. Note that for any $k \in [1, n-1]$ it must be $\{e_1, \dots, e_{k-1}\} \not\vdash e_n$. Otherwise, if it were $\{e_1, \dots, e_{k-1}\} \vdash e_n$ for some $k \in [1, n-1]$, we would have that $C' = \{e_1, \dots, e_k, e_n\}$ and $C'' = \{e_1, \dots, e_{n-1}\}$ are two proper subconfigurations of C such that $C = C' \cup C''$, violating the fact that C is irreducible. But this means exactly that $\{e_1, \dots, e_{n-1}\} \vdash_0 e_n$, as desired.
3) Immediate.
4) Let $I_j = \langle C_j, e_j\rangle \in ir(\mathcal{D}(E))$ for $j \in \{1, 2\}$ be irreducibles. Assume $I_1 \leftrightarrow I_2$. By Lemma 4(3), observing that $p(I_j) = C_j$, we must have $I_1 \cup C_2 = C_1 \cup I_2$, namely $C_1 \cup \{e_1\} \cup C_2 = C_1 \cup C_2 \cup \{e_2\}$, from which we conclude that it must be $e_1 = e_2$, i.e., as desired $I_j = \langle C_j, e\rangle$, where $e = e_1 = e_2$ for $j \in \{1, 2\}$. Additionally, I_1 and I_2 are consistent by Lemma 3, meaning that $C_1 \overset{e}{\frown} C_2$.
For the converse, consider two irreducibles $I_1 = \langle C_1, e\rangle$ and $I_2 = \langle C_2, e\rangle$, such that $C_1 \overset{e}{\frown} C_2$. Hence $C_1 \vdash_0 e$, $C_2 \vdash_0 e$ and $C = C_1 \cup C_2 \cup \{e\}$ is consistent. Since $I_1, I_2 \subseteq C$, they are consistent in $\mathcal{D}(E)$. Moreover, $p(I_1) = C_1$, $p(I_2) = C_2$ and $I_1 \cup C_2 = I_2 \cup C_1 = C \neq C_1 \cup C_2$. Hence by Lemma 4(3), we have $I_1 \leftrightarrow I_2$, as desired.

■

Concerning point 1, observe that the meet in the domain of configurations is $C \sqcap C' = \bigcup\{C'' \in \mathit{Conf}(E) \mid C'' \subseteq C \wedge C'' \subseteq C'\}$, which is usually smaller than the intersection. For instance, in Fig. 2, $\{a, c\} \sqcap \{b, c\} = \emptyset \neq \{c\}$. Point 2 says that irreducibles are configurations of the form $C \cup \{e\}$ that admits a secured execution in which the event e appears as the last one and cannot be switched with any other. In other words, irreducibles are minimal histories of events. Point 3 characterises the irreducibles in a configuration. According to point 4, two irreducibles are interchangeable when they are different histories for the same event.

Proposition 5 *Let E be an es. Then $\mathcal{D}(E)$ is a weak prime domain. Given a morphism $f : E_1 \to E_2$, its image $\mathcal{D}(f) : \mathcal{D}(E_1) \to \mathcal{D}(E_2)$ is a weak prime domain morphism.*

Proof: By Lemma 8 we know that $\mathcal{D}(E)$ is a domain.

In order to show that $\mathcal{D}(E)$ is a weak prime domain, we exploit the characterisation in Proposition 3, i.e., we prove that all irreducibles are weak primes. Consider an irreducible, which, by Lemma 8(2), is of the shape $I = \langle C, e\rangle$ and suppose that $I \subseteq \bigsqcup X$ for some $X \subseteq \mathsf{K}(\mathcal{D}(E))$. This means that, in particular, $e \in \bigsqcup X$ and thus there is $C' \in X$ such that $e \in C'$. In turn, we can consider a minimal enabling of e in C', i.e., a minimal $C'' \subseteq C'$ such that $C'' \vdash_0 e$, and we have that $I'' = \langle C'', e\rangle$ is an irreducible $I'' \subseteq C'$. Since I and I'' are consistent, as they are both included in $\bigsqcup X$, by Lemma 8(4), $I \leftrightarrow I''$.

We next prove that given an es morphism $f : E_1 \to E_2$, its image $\mathcal{D}(f) : \mathcal{D}(E_1) \to \mathcal{D}(E_2)$ is a weak prime domain morphism.

- $C_1 \preceq C_1'$ implies $\mathcal{D}(f)(C_1) \preceq \mathcal{D}(f)(C_1')$

  Since $\mathcal{D}(f)(C_i) = \{f(d_i) \mid d_i \in C_i\}$ and by Lemma 8(1) $C_1 \preceq C_1'$ iff $C_1' = C_1 \cup \{e_1\}$ for some event e_1, the result follows immediately.

- for $X_1 \subseteq \mathcal{D}(E_1)$ consistent, $\mathcal{D}(f)(\bigsqcup X_1) = \bigsqcup \mathcal{D}(f)(X_1)$

Since $\mathcal{D}(f)$ takes the image as set and $\bigsqcup$ on consistent sets is union, the result follows.

- for $C_1, C_1' \in \mathcal{D}(E_1)$ consistent such that $C_1 \sqcap C_1' \prec C_1$ it holds $f(C_1 \sqcap C_1') = f(C_1) \sqcap f(C_1')$

  The condition $C_1 \sqcap C_1' \prec C_1$ means that $C_1 \sqcap C_1' = C_1 \setminus \{e_1\}$ for some $e_1 \in E_1$. Clearly $e_1 \notin C_1'$, otherwise also $C_1 \subseteq C_1'$ and thus $C_1 \cap C_1' = C_1$. Therefore in this case, the meet coincides with intersection, $C_1 \sqcap C_1' = C_1 \cap C_1' = C_1 \setminus \{e_1\}$. Since for the events in $C_1 \cup C_1'$, by definition of event structure morphism, f is injective, we have that $f(C_1) \cap f(C_1') = f(C_1 \cap C_1')$. As a general fact, $f(C_1) \sqcap f(C_1') \subseteq f(C_1) \cap f(C_1')$. Therefore, putting things together, we conclude

  $$f(C_1) \sqcap f(C_1') \subseteq f(C_1) \cap f(C_1') = f(C_1 \cap C_1') = f(C_1 \sqcap C_1')$$

  The converse inequality holds in any domain (as observed after Definition 13) and thus the result follows. ■

A special role is played by the subclass of *connected* es.

Definition 15 (connected es) A *connected es* is an es E such that whenever $C \vdash_0 e$ and $C' \vdash_0 e$ then $C \overset{e}{\frown}^{*} C'$. We denote by cES the full subcategory of ES having connected es as objects.

In words, different minimal enablings for the same event must be pairwise connected by a chain of consistency. For instance, the es in Example 1 is a connected es. Only event c has two minimal histories $\{a\} \vdash_0 c$ and $\{b\} \vdash_0 c$ and obviously $\{a\} \overset{c}{\frown} \{b\}$. Clearly, prime es are also connected es. More precisely, we have the following.

Proposition 6 (connectedness, stability, primality) *Let E be an es. Then E is prime iff E is stable and connected.*

Proof: The fact that a prime es is stable and connected follows immediately from the definitions. Conversely, let E be stable and connected. We show that E is prime, i.e., each $e \in E$ has a unique minimal enabling. Let $C, C' \in Conf(E)$ be minimal enablings for e, i.e., $C \vdash_0 e$ and $C \vdash_0 e$. Since E is connected $C \overset{e}{\frown}^{*} C'$. Let $C \overset{e}{\frown} C_1 \overset{e}{\frown} \ldots \overset{e}{\frown} C_n \overset{e}{\frown} C'$. Then stability we get that $C = C_1 = \ldots = C_n = C'$. ■

The defining property of connected es allows one to recognise that two minimal histories are relative to the same event by only looking at the partially ordered structure and thus, as we will see, from the domain of configurations of a connected es we can recover an es isomorphic to the original one and vice versa (see Theorem 2). In general, this is not possible. For instance, consider the es E' with events $E' = \{a, b, c\}$, and where $a \# b$ and minimal enablings are again $\emptyset \vdash_0 a$, $\emptyset \vdash_0 b$, $\{a\} \vdash_0 c$, and $\{b\} \vdash_0 c$. Namely, event c has two minimal enablings, but differently from what happens in the running example, these are not consistent, hence $\{a, b\} \not\vdash c$. The domain of configurations is depicted on the left of Fig. 4. Intuitively, it is not possible to recognise that $\{a, c\}$ and $\{b, c\}$

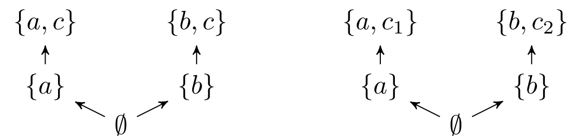


Fig. 4: Non-connected es do not uniquely determine the domain of configurations.

are different histories of the same event. In fact, note that we would get an isomorphic domain of configuration for the es E'' with events $E'' = \{a, b, c_1, c_2\}$ such that $a \# b$ and minimal enablings are again $\emptyset \vdash_0 a$, $\emptyset \vdash_0 b$, $\{a\} \vdash_0 c_1$, and $\{b\} \vdash_0 c_2$.

C. From domains to es

We show how to extract an es from a weak prime domain. As expected, events are equivalence classes of irreducibles, where the equivalence is (the transitive closure of) interchangeability.

In order to properly relate domains to the corresponding es we need to prove some properties of irreducibles and of the interchangeability relation in weak prime domains.

We already observed that the interchangeability relation in general is not transitive. We show that in weak prime domains it is transitive on consistent irreducibles. We start with a simple technical lemma.

Lemma 9 *Let D be a domain and $i, i', i'' \in ir(D)$. If $i \leftrightarrow i'$, $i \leftrightarrow i''$, and $i' \sqsubseteq i''$ then $i' = i''$.*

Proof: Assume $i \leftrightarrow i'$, $i \leftrightarrow i''$, and $i' \sqsubseteq i''$ and suppose by absurdum that $i' \neq i''$, hence $i' \sqsubseteq p(i'')$. Then we have

$$\begin{aligned} i &\sqsubseteq i \sqcup p(i') \\ &= p(i) \sqcup i' && \text{[By } i \leftrightarrow i' \text{ and Lemma 4(3)]} \\ &\sqsubseteq p(i) \sqcup p(i'') && \text{[Since } i' \sqsubseteq p(i'')\text{]} \end{aligned}$$

Therefore $p(i) \sqcup p(i'') \sqsubseteq i \sqcup p(i'') \sqsubseteq p(i) \sqcup p(i'') \sqcup p(i'') = p(i) \sqcup p(i'')$. Hence we deduce $i \sqcup p(i'') = p(i) \sqcup p(i'')$, and by Lemma 4(3) this contradicts $i \leftrightarrow i''$. ■

Lemma 10 ($\leftrightarrow$ transitive on consistent irreducibles) *Let D be a weak prime domain and $i, i', i'' \in ir(D)$ such that $i \leftrightarrow i'$, $i \leftrightarrow i''$, and i', i'' consistent. Then $i' \leftrightarrow i''$.*

Proof: Since $i \leftrightarrow i'$ and $i' \leftrightarrow i''$, by Lemma 3 i, i' and i', i'' are consistent and by Lemma 4 $p(i) \sqcup i' = i \sqcup p(i')$ and $p(i') \sqcup i'' = i' \sqcup p(i'')$. Moreover i, i'' is consistent by hypothesis and thus, by coherence, i, i', i'' is consistent.

Therefore $i \sqsubseteq i \sqcup p(i') \sqcup p(i'') = p(i) \sqcup i' \sqcup p(i'') = p(i) \sqcup p(i') \sqcup i''$. Since D is a weak prime domain, irreducibles are weak primes and there must exist $i_1 \leftrightarrow i$ such that either $i_1 \sqsubseteq p(i)$ or $i_1 \sqsubseteq p(i')$ or $i_1 \sqsubseteq i''$. The first possibility $i_1 \sqsubseteq p(i) \sqsubseteq i$ contradicts Lemma 3, while the second one $i_1 \sqsubseteq p(i') \sqsubseteq i'$ contradicts Lemma 9.

Therefore it must be $i_1 \sqsubseteq i''$. We also note that it cannot be $i_1 \sqsubseteq p(i'')$, and thus, by Lemma 1, $i'' = i_1 \leftrightarrow i$, as desired. Otherwise, if $i_1 \sqsubseteq p(i_2)$, by Lemma 4(2) we would have

$$
\begin{aligned}
i \sqcup p(i) \sqcup p(i') \sqcup p(i'') &= i_1 \sqcup p(i) \sqcup p(i') \sqcup p(i'') \\
&= p(i) \sqcup p(i') \sqcup p(i'')
\end{aligned}
$$

In turn, the above equality could be used to prove that

$$
\begin{aligned}
i' &\sqsubseteq i' \sqcup p(i) \sqcup p(i') \sqcup p(i'') \\
&= i \sqcup p(i) \sqcup p(i') \sqcup p(i'') \\
&= p(i) \sqcup p(i') \sqcup p(i'')
\end{aligned}
$$

Again, by the fact that D is a weak prime domain, there should exist $i'_1 \leftrightarrow i'$ such that $i'_1 \sqsubseteq p(i)$ or $i'_1 \sqsubseteq p(i')$ or $i'_1 \sqsubseteq p(i'')$. Note that $i'_1 \sqsubseteq p(i') \sqsubset i'$ contradicts Lemma 3. Moreover, recalling that $i' \leftrightarrow i$ and $i' \leftrightarrow i'_1$, $i'_1 \sqsubseteq p(i) \sqsubset i$ contradicts Lemma 9. The same applies to the case $i'_1 \sqsubseteq p(i'') \sqsubset i''$. Hence we conclude. ■

Domains are irreducible algebraic, hence any element is determined by the irreducibles under it. The difference between two elements is thus somehow captured by the irreducibles that are under one element and not under the other. This motivates the following definition.

Definition 16 (irreducible difference) Let D be a domain and $d, d' \in \mathsf{K}(D)$ such that $d \sqsubseteq d'$. Then we define $\delta(d', d) = ir(d') \setminus ir(d)$.

The immediate precedence relation intuitively relates domain elements corresponding to configurations which differ for the execution of a single event. We next study how this relates to irreducibles.

Lemma 11 (immediate precedence and irreducibles/1) *Let D be a weak prime domain, $d \in \mathsf{K}(D)$, and $i \in ir(D)$ such that d, i are consistent and $p(i) \sqsubseteq d$. Then*

1) *for all $i' \in \delta(d \sqcup i, d)$ minimal, it holds $i \leftrightarrow i'$;*
2) *$d \preceq d \sqcup i$.*

Proof:

1) First note that if $d = d \sqcup i$, hence $ir(d \sqcup i) = ir(d)$ and the property trivially holds. Assume $d \neq d \sqcup i$ and take $i' \in \delta(d \sqcup i, d)$ minimal. Note that minimality implies that $p(i') \sqsubseteq d$. From $i' \sqsubseteq d \sqcup i$, since D is a weak prime domain and thus irreducibles are weak primes, there must be $i'' \in ir(D)$, $i'' \leftrightarrow i'$ such that $i'' \sqsubseteq d$ or $i'' \sqsubseteq i$. We first note that it cannot be $i'' \sqsubseteq d$, otherwise $d = d \sqcup i'' = d \sqcup i'$, the last equality motivated by Lemma 4(2). Thus we would have $i' \sqsubseteq d$, contradicting the hypotheses. Hence it must be $i'' \sqsubseteq i$. This, in turn, by the minimality of i in $\delta(d \sqcup i, d)$ implies $i = i''$, hence $i = i'' \leftrightarrow i'$, as desired.
2) Consider d' such that $d \prec d' \sqsubseteq d \sqcup i$, and let us prove that $d' = d \sqcup i$. Since $d \prec d'$, hence $d \neq d'$, we know that $\delta(d', d)$ is not empty. Take a minimal $i' \in \delta(d', d)$. Thus i' is minimal also in $\delta(d \sqcup i, d)$, and thus, by point (1), $i \leftrightarrow i'$. By minimality of i' we deduce also that $p(i') \sqsubseteq d$. Since also $p(i) \sqsubseteq d$ by hypothesis, using Lemma 4(2), we have $d \sqcup i = d \sqcup i'$. Observing that $d \sqcup i' \sqsubseteq d' \sqsubseteq d \sqcup i$ we conclude that $d' = d \sqcup i$, as desired.

■

Lemma 12 (immediate precedence and irreducibles/2) *Let D be a weak prime domain and $d, d' \in D$ such that $d \prec d'$. Then for any $i \in \delta(d', d)$*

1) *$d' = d \sqcup i$;*

Moreover, for all i, i' minimal in $\delta(d', d)$

2) *there is no $i'' \in ir(d)$ such that $i \leftrightarrow i''$;*
3) *$i \leftrightarrow i'$.*

Proof:

1) Let $i \in \delta(d', d)$. Then $d \sqsubseteq d \sqcup i \sqsubseteq d'$. It follows that either $d \sqcup i = d$ or $d \sqcup i = d'$. The first possibility can be excluded for the fact that it would imply $i \sqsubseteq d$, while we know that $i \not\in ir(d)$. Hence we get the thesis.
2) Let $i \in \delta(d', d)$ minimal. Therefore $p(i) \sqsubseteq d$. If there were $i'' \in ir(d)$ such that $i \leftrightarrow i''$ then, since also $p(i'') \sqsubseteq d$, by Lemma 4(2) we would get $d = d \sqcup i'' = d \sqcup i = d'$, the last equality following from point (1). But this contradicts the hypothesis that $d \prec d'$.
3) Let $i, i' \in \delta(d', d)$ be minimal irreducibles. By (1) we have $d' = d \sqcup i$. Therefore $i' \in \delta(d \sqcup i, d)$ minimal. Therefore $i \leftrightarrow i'$ by Lemma 11(1).

■

We next show that chains of immediate precedence are generated in essentially a unique way by sequences of irreducibles.

Lemma 13 (chains) *Let D be a weak prime domain, $d \in \mathsf{K}(D)$ and $ir(d) = \{i_1, \ldots, i_n\}$ such that the sequence $i_1, \ldots, i_n$ is compatible with the order (i.e., for all h, k if $i_h \sqsubseteq i_k$ then $h \leq k$). If we let $d_k = \bigsqcup_{h=1}^{k} i_h$ for $k \in \{1, \ldots, n\}$ we have*

$$\bot = d_0 \preceq d_1 \preceq \ldots \preceq d_n = d$$

Vice versa, given a chain $\bot = d_0 \prec d_1 \prec \ldots \prec d_n$ and taking $i_h \in \delta(d_h, d_{h-1})$ minimal for $h \in \{1, \ldots, n\}$ we have

$$d_n = \bigsqcup_{h=1}^{n} i_h \quad \textit{and} \quad \forall i \in ir(d_n) \,.\, \exists h \in [1, n].\ i \leftrightarrow i_h.$$

Therefore $[\{i_1, \ldots, i_n\}]_{\leftrightarrow^*} = [ir(d_n)]_{\leftrightarrow^*}$.

Proof: For the first part, observe that for $k \in \{1, \ldots, n\}$ we have that

$$p(i_k) \sqsubseteq d_{k-1}$$

In fact, recalling that $ir(i_k) \subseteq ir(d)$, we have that irreducibles in $ir(p(i_k)) = ir(i_k) \setminus \{i_k\}$, which are smaller than i_k, must occur before in the list hence

$$ir(p(i_k)) = ir(i_k) \setminus \{i_k\} \subseteq \{i_1, \ldots, i_{k-1}\}.$$

Therefore $p(i_k) = \bigsqcup ir(p(i_k)) \sqsubseteq \bigsqcup \{i_1, \ldots, i_{k-1}\} = d_{k-1}$. Thus we use Lemma 11(2) to infer $d_{k-1} \preceq d_{k-1} \sqcup i_k = d_k$.

For the second part, we proceed by induction on n.

- ($n = 0$) Note that $d_0 = \bigsqcup \emptyset = \bot$ and $ir(\bot) = \emptyset$, hence the thesis trivially holds.
- ($n > 0$) By induction hypothesis

$$d_{n-1} = \bigsqcup_{h=1}^{n-1} i_h \quad \text{and}$$
$$\forall i \in ir(d_{n-1}) \,.\, \exists h \in [1, n-1].\ i \leftrightarrow i_h.$$

Since by construction $i_n \in \delta(d_n, d_{n-1})$, by Lemma 12(1) we deduce

$$d_n = i_n \sqcup d_{n-1} = \bigsqcup(\{i_n\} \cup ir(d_{n-1})).$$

Moreover, for all $i \in \delta(d_n, d_{n-1})$ we have $i \sqsubseteq d_n = i_n \sqcup d_{n-1}$. By definition of weak domain domain, the exists $i' \leftrightarrow i$ such that $i' \sqsubseteq d_{n-1}$ or $i' \sqsubseteq i_n$. In the first case, since $i' \sqsubseteq d_{n-1}$, by the inductive hypothesis there is $h \in [1, n-1]$ such that $i' \leftrightarrow i_h$. Since $i \leftrightarrow i' \leftrightarrow i_h$, and $i, i', i_h \sqsubseteq d_n$ are consistent, by Lemma 10 $i \leftrightarrow i_h$, as desired. If, instead, we are in the second case, namely $i' \sqsubseteq i_n$, by minimality of i_n it follows that $i_n = i' \leftrightarrow i$, as desired.

■

In a prime domain an element admits a unique decomposition in terms of primes (see Lemma 2). Here the same holds for irreducibles but only up to interchangeability.

Proposition 7 (unique decomposition up to $\leftrightarrow$) *Let D be a weak prime domain, $d \in \mathsf{K}(D)$, and $X \subseteq ir(d)$ downward closed. Then $d = \bigsqcup X$ iff $[X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$.*

Proof: ($\Rightarrow$) Let $d = \bigsqcup X$. Clearly $X \subseteq [ir(d)]_{\leftrightarrow^*}$. Hence we only need to prove that $ir(d) \subseteq [X]_{\leftrightarrow^*}$. Let $i \in ir(d)$. Hence $i \sqsubseteq d = \bigsqcup X$. By definition of weak prime domain, this implies that there exists $i' \leftrightarrow i$ and $x \in X$ such that $i' \sqsubseteq x$. Since X is downward closed, necessarily $i' \in X$ and thus $i \in [X]_{\leftrightarrow^*}$, as desired.

($\Leftarrow$) Let $[X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$. We can prove that $\bigsqcup X = d$ by induction on $k = |\, ir(d) \setminus X|$. If $k = 0$ then $X = ir(d)$ and thus $d = \bigsqcup X$ by Proposition 1. If $k > 0$ take a minimal element $i \in ir(d) \setminus X$. Therefore $X' = X \cup \{i\}$ is downward closed. Since $[X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$, there is $i' \in X$ such that $i \leftrightarrow i'$. Therefore

$$\bigsqcup X' = \bigsqcup X \cup \{i\} = \bigsqcup X \cup \{i'\} = \bigsqcup X. \qquad (2)$$

Since $[X']_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$ and $|\, ir(d) \setminus X'| = |\, ir(d) \setminus X| - 1 < k$, by inductive hypothesis $\bigsqcup X' = d$. Hence, using (2) $\bigsqcup X = d$, as desired. ■

We now have the tools for mapping our domains to an es.

Definition 17 (es for a weak prime domain) Let D be a weak prime domain. The es $\mathcal{E}(D) = \langle E, \#, \vdash \rangle$ is defined as follows

- $E = [ir(D)]_{\leftrightarrow^*}$
- $e \# e'$ if there is no $d \in \mathsf{K}(D)$ such that $e, e' \in [ir(d)]_{\leftrightarrow^*}$;
- $X \vdash e$ if there exists $i \in e$ such that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} \subseteq X$.

Given a morphism $f : D_1 \to D_2$, its image $\mathcal{E}(f) : \mathcal{E}(D_1) \to \mathcal{E}(D_2)$ is defined for $[i_1]_{\leftrightarrow^*} \in E$ as $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*}$, where $i_2 \in \delta(f(i_1), f(p(i_1)))$ is minimal in the set, and $\mathcal{E}(f)([i_1]_{\leftrightarrow^*})$ is undefined if $f(p(i_1)) = f(i_1)$.

The definition above is well-given. In particular, there is no ambiguity in the definition of the image of a morphism, since one can show that for all $i_2, i_2' \in \delta(f(i_1), f(p(i_1)))$ minimal, $i_2 \leftrightarrow i_2'$.

We first need a technical lemma holding in any domain.

Lemma 14 *Let D be a domain and $a, b, c \in D$ such that $c \sqsubseteq a$ and $c \prec b$. Then either $b \sqsubseteq a$ or $c = a \sqcap b$.*

Proof: Recall that in a domain the meet of consistent sets exists. Since c is a lower bound for a and b, necessarily $c \sqsubseteq a \sqcap b \sqsubseteq b$. If it were $c \neq a \sqcap b$ then we would have $a \sqcap b = b$, hence $b \sqsubseteq a$, as desired. ■

Lemma 15 *Let D be a weak prime domain. Then $\mathcal{E}(D)$ is an es. Moreover, for a morphism $f : D_1 \to D_2$, its image $\mathcal{E}(f) : \mathcal{E}(D_1) \to \mathcal{E}(D_2)$ is an es morphism.*

Proof:

We need to show that if $X \vdash e$ and $X \subseteq Y \in Con$ implies $Y \vdash e$. In fact, by definition, if $X \vdash e$ then there exists $i \in e$ such that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} \subseteq X$. Hence if $X \subseteq Y$ it immediately follows that $Y \vdash e$. Moreover the es is live. The fact that conflict is saturated follows immediately buy definition of conflict and the characterisation of configurations provided later in Lemma 16. Conflict is irreflexive since for any $e \in \mathcal{E}(D)$, if $e = [i]_{\leftrightarrow^*}$ then $e \in [ir(i)]_{\leftrightarrow^*}$, which is a configuration by the same lemma.

Given a morphism $f : D_1 \to D_2$, its image $\mathcal{E}(f) : \mathcal{E}(D_1) \to \mathcal{E}(D_2)$ is defined for $[i_1]_{\leftrightarrow^*} \in E$ as $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*}$, where $i_2 \in \delta(f(i_1), f(p(i_1)))$ is minimal in the set, and $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = \bot$ if $f(p(i_1)) = f(i_1)$. First observe that $\mathcal{E}(f)([i_1]_{\leftrightarrow^*})$ does not depend on the choice of the representative. In fact, let $i_2, i_2' \in \delta(f(i_1), f(p(i_1)))$ be minimal. Since $p(i_1) \prec i_1$, by definition of domain morphism, $f(p(i_1)) \prec f(i_1)$. Thus, by Lemma 12(3), $i_2 \leftrightarrow i_2'$.

We next show that $\mathcal{E}(f)$ is an es morphism.

- If $\mathcal{E}(f)(e_1) \# \mathcal{E}(f)(e_1')$ then $e_1 \# e_1'$.
  We prove the contronominal, namely if e_1, e_1' consistent then $\mathcal{E}(f)(e_1), \mathcal{E}(f)(e_1')$ consistent.
  The fact that e_1, e_1' consistent means that there exists $d_1 \in \mathsf{K}(D_1)$ such that $e_1, e_1' \in [ir(d_1)]_{\leftrightarrow^*}$. We show that $\mathcal{E}(f)(e_1), \mathcal{E}(f)(e_1') \in [ir(f(d_1))]_{\leftrightarrow^*}$ (note that $f(d_1)$ is a compact, since f is a domain morphism).
  Let us show, for instance, that $\mathcal{E}(f)(e_1) \in [ir(f(d_1))]_{\leftrightarrow^*}$. The fact that $e_1 \in [ir(d_1)]_{\leftrightarrow^*}$ means that $e_1 = [i_1]_{\leftrightarrow^*}$ for some $i_1 \sqsubseteq d_1$. By definition $\mathcal{E}(f)(e_1) = [i_2]_{\leftrightarrow^*}$, where $i_2 \in \delta(f(i_1), f(p(i_1)))$ minimal (since $\mathcal{E}(f)(e_1)$ is defined the difference cannot be empty). Now, since $i_1 \sqsubseteq d_1$ we have that $f(i_1) \sqsubseteq f(d_1)$, whence $i_2 \sqsubseteq f(i_1) \sqsubseteq f(d_1)$ and $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*} \in [ir(f(d_1))]_{\leftrightarrow^*}$, as desired.
- If $\mathcal{E}(f)(e_1) = \mathcal{E}(f)(e_1')$ and $e_1 \neq e_1'$ then $e_1 \# e_1'$.
  We prove the contronominal, namely if e_1, e_1' consistent and $\mathcal{E}(f)(e_1) = \mathcal{E}(f)(e_1')$ then $e_1 = e_1'$.
  Assume e_1, e_1' consistent and $\mathcal{E}(f)(e_1) = \mathcal{E}(f)(e_1')$. By the first condition there must be $d_1 \in \mathsf{K}(D_1)$ such that $e_1, e_1' \in [ir(d_1)]_{\leftrightarrow^*}$, namely $e_1 = [i_1]_{\leftrightarrow^*}$ and $e_1 = [i_1']_{\leftrightarrow^*}$ with $i_1, i_1' \sqsubseteq d_1$.
  Moreover, $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*}$ and $\mathcal{E}(f)([i_1']_{\leftrightarrow^*}) = [i_2']_{\leftrightarrow^*}$ where i_2 and i_2' are minimal elements in $\delta(f(i_1), f(p(i_1)))$ and $\delta(f(i_1'), f(p(i_1')))$, respectively, and $[i_2]_{\leftrightarrow^*} = [i_2']_{\leftrightarrow^*}$, which means $i_2 \leftrightarrow^* i_2'$, and in turn, being i_2 and i_2' consistent, implies $i_2 \leftrightarrow i_2'$.
  We distinguish two cases.

– If i_1 and i'_1 are comparable, e.g., if $i_1 \sqsubseteq i'_1$, then $i_1 = i'_1$ and we are done. In fact, otherwise, if $i_1 \neq i'_1$ we have $p(i_1) \prec i_1 \sqsubseteq p(i'_1) \prec i'_1$. By monotonicity of f we have $f(p(i_1)) \prec f(i_1) \sqsubseteq f(p(i'_1)) \prec f(i'_1)$ (where strict inequalities $\prec$ are motivated by the definition of $\mathcal{E}(f)$, since both $\mathcal{E}(f)([i_1]_{\leftrightarrow^*})$ and $\mathcal{E}(f)([i'_1]_{\leftrightarrow^*})$ are defined). Now notice that $p(i_2) \sqsubseteq i_2 \sqsubseteq f(i_1) \sqsubseteq f(p(i'_i))$. Hence, using the fact that $i_2 \leftrightarrow i'_2$, by Lemma 4(2) we have

$$f(p(i'_1)) = f(p(i'_1)) \sqcup i_2 = f(p(i'_1)) \sqcup i'_2 = f(i'_1)$$

contradicting the fact that $f(p(i'_1)) \prec f(i'_1)$.

– Assume now that i_1 and i'_1 are uncomparable and define $p = p(i_1) \sqcup p(i'_1)$. We can also assume $i_1, i'_1 \not\sqsubseteq p$. In fact, otherwise, e.g., if $i_1 \sqsubseteq p$, then, by the defining property of weak prime domains, we derive the existence of $i''_1 \leftrightarrow i_1$ such that $i''_1 \sqsubseteq p(i_1)$ or $i''_1 \sqsubseteq p(i'_1)$. The first possibility can be excluded because it would imply $i''_1 \sqsubseteq i_1$, contradicting $i''_1 \leftrightarrow i_1$. Hence it must be $i''_1 \sqsubseteq p(i'_1) \sqsubseteq i'_1$. Therefore, if we take i''_1 as representative of the equivalence class we are back to the previous case.

Using the fact that $i_1, i'_1 \not\sqsubseteq p$ and $p(i_1), p(i'_1) \sqsubseteq p$, by Lemma 11(2) we deduce that $p \prec p \sqcup i_1$ and $p \prec p \sqcup i'_1$.

By Lemma 12(1), since $i_2 \in \delta(f(i_1), f(p(i_1)))$ and $i'_2 \in \delta(f(i'_1), f(p(i'_1)))$, we have

$$f(p(i_1)) \sqcup i_2 = f(i_1) \qquad f(p(i'_1)) \sqcup i'_2 = f(i'_1) \quad (3)$$

Now, observe that

$$\begin{array}{ll}
f(p \sqcup i_1) = & \\
= f(p(i_1) \sqcup p(i'_1) \sqcup i_1) & \\
= f(p(i'_1) \sqcup i_1) & \\
= f(p(i'_1)) \sqcup f(i_1) & \text{[preservation of } \sqcup] \\
= f(p(i'_1)) \sqcup f(p(i_1)) \sqcup i_2 & \text{[by (3)]} \\
= f(p(i'_1)) \sqcup f(p(i_1)) \sqcup i'_2 & \text{[by Lemma 4(2),} \\
& \text{ since } i_2 \leftrightarrow i'_2] \\
= f(i'_1) \sqcup f(p(i_1)) & \text{[by (3)]} \\
= f(p(i_1) \sqcup i'_1) & \text{[preservation of } \sqcup] \\
= f(p(i_1) \sqcup p(i'_1) \sqcup i'_1) & \\
= f(p \sqcup i'_1) &
\end{array}$$

Since $p \prec p \sqcup i_1$ and $p \prec p \sqcup i'_1$, by Lemma 14 $(p \sqcup i_1) \sqcap (p \sqcup i'_1) = p$. Therefore, on the one hand $f((p \sqcup i_1) \sqcap (p \sqcup i'_1)) = f(p)$. On the other hand, since the meet is an immediate predecessor, it is preserved: $f((p \sqcup i_1) \sqcap (p \sqcup i'_1)) = f(p \sqcup i_1) \sqcap f(p \sqcup i'_1) = f(p \sqcup i_1) = f(p \sqcup i'_1)$. Putting things together, $f(p) = f(p \sqcup i_1) = f(p \sqcup i'_1)$, contradicting the fact that $f(p) \prec f(p \sqcup i_1)$.

• if $X_1 \vdash_1 [i_1]_{\leftrightarrow^*}$ and $\mathcal{E}(f)([i_1]_{\leftrightarrow^*})$ is defined then $\mathcal{E}(f)(X_1) \vdash_2 \mathcal{E}(f)([i_1]_{\leftrightarrow^*})$

Recall that $X_1 \vdash_1 [i_1]_{\leftrightarrow^*}$ means that $[ir(i'_1) \setminus \{i'_1\}]_{\leftrightarrow^*} = [ir(p(i'_1))]_{\leftrightarrow^*} \subseteq X_1$ for some $i'_1 \leftrightarrow i_1$.

By definition, $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*}$ where i_2 is a minimal irreducible in $\delta(f(i'_1), f(p(i'_1)))$. We show that $\mathcal{E}(f)(X) \vdash_2 [i_2]_{\leftrightarrow^*}$, namely that

$$[ir(i_2) \setminus \{i_2\}]_{\leftrightarrow^*} = [ir(p(i_2))]_{\leftrightarrow^*} \subseteq \mathcal{E}(f)(X) \quad (4)$$

Observe that since i_2 is minimal in $\delta(f(i'_1), f(p(i'_1)))$, it holds $p(i_2) \sqsubseteq f(p(i'_1))$. Therefore, we have

$$ir(p(i_2)) \subseteq ir(f(p(i'_1)))$$

Hence, in order to conclude (4), it suffices to show that

$$[ir(f(p(i'_1)))]_{\leftrightarrow^*} \subseteq \mathcal{E}(f)(X) \quad (5)$$

In order to reach this result, first note that, by Lemma 13, if $ir(p(i'_1)) = \{j^1_1, \ldots, j^n_1\}$ is a sequence of irreducibles compatible with the order, we can obtain a $\preceq$-chain

$$\bot = d^0_1 \preceq d^1_1 \preceq \ldots \preceq d^n_1 = p(i'_1) \prec i'_1$$

We can extract a strictly increasing subsequence

$$\bot = d'^0_1 \prec d'^1_1 \prec \ldots \prec d'^m_1 = p(i'_1) \prec i'_1$$

and, if we take irreducibles $j'^1_1, \ldots, j'^m_1$ in $\delta(d'^i_1, d'^{i-1}_1)$, again by Lemma 13 we know that

$$[ir(p(i'_1))]_{\leftrightarrow^*} = [\{j'^1_1, \ldots, j'^m_1\}]_{\leftrightarrow^*} \quad (6)$$

Since f is a domain morphism, it preserves $\preceq$, namely

$$\bot = f(d'^0_1) \preceq f(d'^1_1) \preceq \ldots \preceq f(d'^m_1) = f(p(i'_1)) \prec f(i'_1)$$

where the last inequality is strict since $\mathcal{E}(f)([i'_1]_{\leftrightarrow^*}) = [e_2]_{\leftrightarrow^*}$ is defined. Moreover, whenever $f(d'^{h-1}_1) \prec f(d'^h_1)$, then $\mathcal{E}(f)([j'^h_1]_{\leftrightarrow^*}) = [\ell^h_2]_{\leftrightarrow^*}$ where ℓ^h_2 is any minimal irreducible in $\delta(f(d'^h_1), f(d'^{h-1}_1))$, otherwise $\mathcal{E}(f)([j'^h_1]_{\leftrightarrow^*})$ is undefined.

Once more by Lemma 13 we know that

$$[ir(f(p(i'_1)))]_{\leftrightarrow^*} = [\{\ell^1_2, \ldots, \ell^m_2\}]_{\leftrightarrow^*} = \mathcal{E}(f)([\{j'^1_1, \ldots, j'^m_1\}]_{\leftrightarrow^*}),$$

thus, using (6)

$$[ir(f(p(i'_1)))]_{\leftrightarrow^*} = \mathcal{E}(f)([ir(p(i'_1))]_{\leftrightarrow^*}). \quad (7)$$

Hence, recalling that, by hypothesis, $[ir(p(i'_1))]_{\leftrightarrow^*} \subseteq X$, we conclude the desired inclusion (5).

■

Since in a prime domain irreducibles coincide with primes (Proposition 2), $\leftrightarrow$ is the identity (Lemma 5) and $\delta(d', d)$ is a singleton when $d \prec d'$, the construction above produces the prime es $\mathsf{pES}(D)$ as defined in Section II.

Given a domain D, the configurations of the es $\mathcal{E}(D)$ exactly correspond to the elements in $\mathsf{K}(D)$. Moreover, in such es we have a minimal enabling $C \vdash_0 e$ when there is an irreducible in e (recall that events are equivalence classes of irreducibles) such that C contains all and only (the equivalence classes of its) predecessors.

Lemma 16 (compacts vs. configurations) *Let D be a weak prime domain and $C \subseteq \mathcal{E}(D)$ a set of events. Then C is a configuration in the es $\mathcal{E}(D)$ iff there exists a (unique) $d \in \mathsf{K}(D)$ such that $C = [ir(d)]_{\leftrightarrow^*}$. Moreover, for any $e \in \mathcal{E}(D)$ we have that $C \vdash_0 e$ iff $C = [ir(i) \setminus \{i\}]_{\leftrightarrow^*}$ for some $i \in e$.*

Proof: The left to right implication follows by proving that, given a configuration $C \in \mathit{Conf}(\mathcal{E}(D))$, there exists $X \subseteq$

$ir(D)$ downward closed and consistent such that $[X]_{\leftrightarrow^*} = C$. Hence, if we let $d = \bigsqcup X$, by Proposition 7, we have that $C = [X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$. Moreover, d is uniquely determined, since, by the same proposition we have that for any other X' such that $[X']_{\leftrightarrow^*} = C$, since $[X']_{\leftrightarrow^*} = C = [X]_{\leftrightarrow^*} = [ir(d)]_{\leftrightarrow^*}$, necessarily $d = \bigsqcup X'$.

Let us prove the existence of a $X \subseteq ir(D)$ consistent and downward closed such that $X = [C]_{\leftrightarrow^*}$. We know that C is consistent, i.e., there is $d \in \mathsf{K}(D)$ such that $C \subseteq [ir(d)]_{\leftrightarrow^*}$. Then we can define $X = \{j \in ir(D) \mid [j]_{\leftrightarrow^*} \in C \ \wedge \ j \sqsubseteq d\}$

The set X is clearly consistent and $[X]_{\leftrightarrow^*} = C$. The fact that it is downward closed can be proved by induction on the cardinality of C, as follows

- if $|C| = 0$, namely $C = \emptyset$ then $X = \emptyset$, hence the thesis is trivial.
- if $|C| > 0$, since C is secured there is $[i]_{\leftrightarrow^*} \in C$ such that $C' = C \setminus \{[i]_{\leftrightarrow^*}\} \vdash [i]_{\leftrightarrow^*}$. We can assume without loss of generality that $i \in X$. Observe that
$$X' = X \setminus \{i\} = \{j \mid [j]_{\leftrightarrow^*} \in C' \ \wedge \ j \sqsubseteq d\}$$
hence, by inductive hypothesis, X' is closed.
The fact that $C' = C \setminus \{[i]_{\leftrightarrow^*}\} \vdash [i]_{\leftrightarrow^*}$ means that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} = [ir(p(i))]_{\leftrightarrow^*} \subseteq C'$, namely that for some $j \sqsubseteq i$, $j \neq i$, and $j \in X$ it holds that $[j]_{\leftrightarrow^*} \in C'$. Since $j \sqsubseteq i \sqsubseteq d$, by construction $j \in X'$. Recalling that X' is downward closed, this is sufficient to conclude that also $X = X' \cup \{i\}$ is downward closed.

For the converse, let $C = [ir(d)]_{\leftrightarrow^*}$. We prove by induction on $k = |\,ir(d)\,|$ that $C \in \mathit{Conf}(\mathcal{E}(D))$. If $k = 0$ then $C = \emptyset$ and we conclude. If $k > 0$ let $\bot = d_0 \prec d_1 \prec \ldots d_{n-1} \prec d_n = d$ be a chain of immediate precedence. By inductive hypothesis $[ir(d_{n-1})]_{\leftrightarrow^*} \in \mathit{Conf}(\mathcal{E}(D))$. Moreover, if we take $i \in \delta(d, d_{n-1})$ minimal, $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} \subseteq [ir(d_{n-1})]_{\leftrightarrow^*}$ hence $[ir(d_{n-1})]_{\leftrightarrow^*} \vdash [ir(i)]_{\leftrightarrow^*}$. Therefore $C = [ir(d_{n-1})]_{\leftrightarrow^*} \cup [ir(i)]_{\leftrightarrow^*} \in \mathit{Conf}(\mathcal{E}(D))$ and by the second part of Lemma 13, $C = [ir(d)]_{\leftrightarrow^*}$.

The second part follows immediately by Definition 17. ■

Now we show how weak prime domains relate to connected es.

Proposition 8 *Let D be a weak prime domain. Then $\mathcal{E}(D)$ is a connected es.*

Proof: We have to show that if $X \vdash_0 e$ and $X' \vdash_0 e$, then $X \overset{e}{\frown}{}^* X'$. Note that, by Lemma 16, from $X \vdash_0 e$ and $X' \vdash_0 e$, we deduce that there exists $i, i' \in e$ such that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} = X$ and $[ir(i') \setminus \{i'\}]_{\leftrightarrow^*} = X'$. Since $i, i' \in e$ we have that $i \leftrightarrow^* i'$, namely $i = i_0 \leftrightarrow i_1 \leftrightarrow \ldots \leftrightarrow i_n = i'$. We proceed by induction on n. The base case $n = 0$ is trivial. If $n > 0$ then from $i \leftrightarrow i_1 \leftrightarrow^* i'$, we have that $i_1 \in e$ and, if we let $X_1 = [ir(i_1) \setminus \{i_1\}]_{\leftrightarrow^*}$, then $X_1 \vdash_0 e$. By inductive hypothesis, we know that $X_1 \overset{e}{\frown}{}^* X'$. Moreover, since $i \leftrightarrow i_1$, by Lemma 3 i and i_1 are consistent. Hence, by definition of conflict in $\mathcal{E}(D)$, also $X \cup X_1 \cup \{e\}$ is consistent and hence $X \overset{e}{\frown} X_1$. Therefore $X \overset{e}{\frown}{}^* X'$, as desired. ■

D. Relating categories of models

At a categorical level, the constructions taking a weak prime domain to an es and an es to a domain (the domain of its configurations) establish a coreflection between the corresponding categories. This becomes an equivalence when it is restricted to the full subcategory of connected es.

Theorem 2 (coreflection of ES and wDom) *The functors $\mathcal{D} : \mathsf{ES} \to \mathsf{wDom}$ and $\mathcal{E} : \mathsf{wDom} \to \mathsf{ES}$ form a coreflection. It restricts to an equivalence between wDom and cES.*

Proof: Let E be an es. Recall that the corresponding domain of configurations is $\mathcal{D}(E) = \langle \mathit{Conf}(E), \subseteq \rangle$. Then, $\mathcal{E}(\mathcal{D}(E)) = \langle E', \#', \vdash' \rangle$. The set of events is defined as
$$E' = [ir(\mathcal{D}(E))]_{\leftrightarrow^*} = \{[\langle C, e\rangle]_{\leftrightarrow^*} \mid C \vdash_0 e\}$$

By Lemma 8(4), the equivalence class of an irreducible $\langle C, e\rangle$ consists of all minimal enablings of event e which are connected. Therefore we can define a morphism
$$\begin{array}{rccc} \theta_E : & \mathcal{E}(\mathcal{D}(E)) & \to & E \\ & \langle C, e\rangle & \mapsto & e \end{array}$$
Observe that θ_E is surjective. In fact E is live and thus for any event $e \in E$ it has at least a minimal history. Take any $I = \langle C, e\rangle$. Then $[I]_{\leftrightarrow^*} \in \mathcal{E}(\mathcal{D}(E))$ and $\theta_E(I) = e$. This is clearly a morphism of event structures. In fact, observe that

- for $I_1, I_2 \in ir(\mathcal{D}(E))$, with $[I_1]_{\leftrightarrow^*} \neq [I_2]_{\leftrightarrow^*}$, we have that $\theta_E([I_1]_{\leftrightarrow^*}) = \theta_E([I_2]_{\leftrightarrow^*})$ implies $[I_1]_{\leftrightarrow^*} \#' [I_2]_{\leftrightarrow^*}$.
In fact, by Lemma 8(2) the irreducibles will be of the kind $I_1 = \langle C_1, e_1\rangle$ and $I_2 = \langle C_2, e_2\rangle$. We show that if $[I_1]_{\leftrightarrow^*}$ and $[I_2]_{\leftrightarrow^*}$ are consistent and $\theta_E([I_1]_{\leftrightarrow^*}) = \theta_E([I_2]_{\leftrightarrow^*})$ then $[I_1]_{\leftrightarrow^*} = [I_2]_{\leftrightarrow^*}$.
Assume $\theta_E([I_1]_{\leftrightarrow^*}) = \theta_E([I_2]_{\leftrightarrow^*})$, hence $e_1 = e_2$. Moreover the fact that $[I_1]_{\leftrightarrow^*}$ and $[I_2]_{\leftrightarrow^*}$ are consistent means that there exists $k \in \mathsf{K}(\mathcal{D}(E))$ such that $[I_1]_{\leftrightarrow^*}, [I_2]_{\leftrightarrow^*} \in [ir(k)]_{\leftrightarrow^*}$. Since compacts in $\mathcal{D}(E)$ are configurations, the condition amounts to the existence of $C \in \mathit{Conf}(E)$ such that $[I_1]_{\leftrightarrow^*}, [I_2]_{\leftrightarrow^*} \in [ir(C)]_{\leftrightarrow^*}$, i.e., there are are I'_1, I'_2 with $I_i \leftrightarrow^* I'_i$ for $i \in \{1, 2\}$, such that $I'_1, I'_2 \subseteq C$. Since the choice of the representative is irrelevant, we can assume that $I_1 = I'_1$ and $I_2 = I'_2$. Summing up, I_1 and I_2 are consistent minimal histories of the same event, hence by Lemma 8(4), $I_1 \leftrightarrow I_2$, i.e., $[I_1]_{\leftrightarrow^*} = [I_2]_{\leftrightarrow^*}$, as desired.
- for $I_1, I_2 \in ir(\mathcal{D}(E))$, if $\theta_E([I_1]_{\leftrightarrow^*}) \# \theta_E([I_2]_{\leftrightarrow^*})$ then $[I_1]_{\leftrightarrow^*} \#' [I_2]_{\leftrightarrow^*}$.
Let $I_1 = \langle C_1, e_1\rangle$ and $I_2 = \langle C_2, e_2\rangle$. If $\theta_E([I_1]_{\leftrightarrow^*}) = e_1 \# e_2 = \theta_E([I_2]_{\leftrightarrow^*})$, then there cannot be any configuration $C \in \mathit{Conf}(E)$ such that $I_1, I_2 \subseteq C$. Hence $[I_1]_{\leftrightarrow^*} \#' [I_2]_{\leftrightarrow^*}$.
- For the enabling relation, observe that according to the definition of the functor $\mathcal{E}$, it holds that $X \vdash' [\langle C, e\rangle]_{\leftrightarrow^*}$ whenever there exists $i \in [\langle C, e\rangle]_{\leftrightarrow^*}$ such that $[ir(\langle C, e\rangle) \setminus \{\langle C, e\rangle\}]_{\leftrightarrow^*} \subseteq X$. Take $i \in [\langle C, e\rangle]_{\leftrightarrow^*}$, namely $i = \langle C', e\rangle$ such that $C' \vdash_0 e'$. We have $ir(\langle C', e\rangle) \setminus \{\langle C', e\rangle\} = ir(C') = \{[\langle C'', e''\rangle]_{\leftrightarrow^*} \mid C'' \subseteq C' \ \wedge \ C'' \vdash_0 e''\}$. Requiring

$[ir(\langle C', e'\rangle) \setminus \{\langle C', e'\rangle\}]_{\leftrightarrow^*} \subseteq X$ is equivalent to require that $C' \subseteq \theta_E(X)$, i.e., $X \vdash' [\langle C, e\rangle]_{\leftrightarrow^*}$ if there exists C' such that $C' \vdash_0 e$ and $C' \subseteq \theta_E(X)$. This in turn means that

$$X \vdash' [\langle C, e\rangle]_{\leftrightarrow^*} \text{ if } \theta_E(X) \vdash e.$$

Finally, we prove the naturality of θ by showing that the diagram below commutes.

$$\begin{array}{ccc} \mathcal{E}(\mathcal{D}(E_1)) & \xrightarrow{\theta_{E_1}} & E_1 \\ {\scriptstyle \mathcal{E}(\mathcal{D}(f))}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{E}(\mathcal{D}(E_2)) & \xrightarrow[\theta_{E_2}]{} & E_2 \end{array}$$

Consider $[\langle C_1, e_1\rangle]_{\leftrightarrow^*} \in \mathcal{E}(\mathcal{D}(E_1))$. Recall that $\mathcal{E}(\mathcal{D}(f))([\langle C_1, e_1\rangle]_{\leftrightarrow^*})$ is computed by considering the image of the irreducible $\langle C_1, e_1\rangle$ and of its predecessor, namely

$$\mathcal{D}(f)(C_1) = f(C_1) \text{ and } \mathcal{D}(f)(\langle C_1, e_1\rangle) = f(C_1 \cup \{e_1\})$$

If $f(e_1)$ is defined, then $f(C_1) \prec f(C_1 \cup \{e_1\})$ and $\mathcal{E}(\mathcal{D}(f))([\langle C_1, e_1\rangle]_{\leftrightarrow^*}) = f(e_1)$, otherwise $\mathcal{E}(\mathcal{D}(f))([\langle C_1, e_1\rangle]_{\leftrightarrow^*})$ is undefined. This means that in all cases, as desired

$$\mathcal{E}(\mathcal{D}(f))([\langle C_1, e_1\rangle]_{\leftrightarrow^*}) = f(e_1) = f(\eta_{E_1}([\langle C_1, e_1\rangle]_{\leftrightarrow^*})).$$

Vice versa, let D be a weak prime domain. Consider $\mathcal{E}(D) = \langle E, \#, \vdash\rangle$ defined as:

- $E = [ir(D)]_{\leftrightarrow^*}$
- $e\#e'$ if there is no $d \in \mathsf{K}(D)$ such that $e, e' \in [ir(d)]_{\leftrightarrow^*}$;
- $X \vdash e$ if there exists $i \in e$ such that $[ir(i) \setminus \{i\}]_{\leftrightarrow^*} \subseteq X$.

and consider $\mathcal{D}(\mathcal{E}(D))$. Elements of $\mathsf{K}(\mathcal{D}(\mathcal{E}(D)))$ are configurations of $C \in \mathit{Conf}(\mathcal{E}(D))$. We can define a function

$$\begin{array}{rccc} \eta_D: & \mathsf{K}(\mathcal{D}(\mathcal{E}(D))) & \to & \mathsf{K}(D) \\ & C & \mapsto & d \end{array}$$

where $d \in \mathsf{K}(D)$ is the unique element such that $C = [d]_{\leftrightarrow^*}$, given by Lemma 16. The function is monotone and bijective with inverse $\eta^{-1} : \mathsf{K}(D) \to \mathsf{K}(\mathcal{D}(\mathcal{E}(D)))$ given by $\eta^{-1}(d) = \{[i]_{\leftrightarrow^*} \mid i \in ir(D) \ \wedge i \sqsubseteq d\}$. By algebraicity of the domains, this function thus uniquely extends to an isomorphism $\eta_D : \mathcal{D}(\mathcal{E}(D)) \to D$.

Finally, we prove the naturality of η, showing that the diagram below commutes.

$$\begin{array}{ccc} \mathcal{E}(\mathcal{D}(D_1)) & \xrightarrow{\eta_{D_1}} & D_1 \\ {\scriptstyle \mathcal{E}(\mathcal{D}(f))}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{E}(\mathcal{D}(D_2)) & \xrightarrow[\eta_{D_2}]{} & D_2 \end{array}$$

Let $C_1 \in \mathsf{K}(\mathcal{D}(\mathcal{E}(D_1)))$, namely $C_1 \in \mathit{Conf}(\mathcal{E}(D_1))$, and let $\eta_{D_1}(C_1) = d_1$ be the element such that $C_1 = [ir(d_1)]_{\leftrightarrow^*}$.

The construction offered by Lemma 13 provides a chain

$$d_0^1 = \bot \prec d_1^1 \prec d_1^2 \prec \ldots \prec d_1^n = d_1$$

and, by the same lemma, if we take an irreducible i_1^h minimal in $\delta(d_1^h, d_1^{h-1})$ for $1 \le h \le n$ we have that $C_1 = [ir(d_1)]_{\leftrightarrow^*} = [\{i_1^1, \ldots, i_1^n\}]_{\leftrightarrow^*}$. Therefore the image

$$\mathcal{D}(\mathcal{E}(f))(C_1) = \{\mathcal{E}(f)([j_1]_{\leftrightarrow^*}) \mid [j_1]_{\leftrightarrow^*} \in C_1\} = \{\mathcal{E}(f)([i_1^h]_{\leftrightarrow^*}) \mid h \in [1, n]\}$$

is the set of equivalence classes of irreducibles $i_2^1, \ldots, i_2^k$ corresponding to

$$f(d_1^0) = \bot \prec f(d_1^1) \prec f(d_1^2) \prec \ldots \prec f(d_1^n) = f(d_1)$$

namely i_2^j minimal in $\delta(f(d_1^j), f(d_1^{j-1}))$, and, again, by Lemma 13, $[\{i_2^1, \ldots, i_2^k\}]_{\leftrightarrow^*} = [ir(f(d_1))]_{\leftrightarrow^*}$. Summing up

$$\eta_{D_2}(\mathcal{D}(\mathcal{E}(f))(C_1)) = \eta_{D_2}(\{[i_2^h]_{\leftrightarrow^*} \mid 1 \le h \le k\}) = f(d_1) = f(\eta_{D_1}(C_1))$$

as desired.

Now, just observe that in the proof above, when E is a connected es, then the morphism θ_E defined as

$$\begin{array}{rccc} \theta_E: & \mathcal{E}(\mathcal{D}(E)) & \to & E \\ & [\langle C, e\rangle]_{\leftrightarrow^*} & \mapsto & e \end{array}$$

is an isomorphism. In fact, it is a bijection. We already know that it is surjective, and it is also injective. In fact, if $\theta_E([I]_{\leftrightarrow^*}) = \theta_E([I']_{\leftrightarrow^*})$ then I and I' are minimal enablings of the same event, i.e., $I = [\langle C, e\rangle]_{\leftrightarrow^*}$ and $I' = [\langle C', e\rangle]_{\leftrightarrow^*}$. Since E is a weak prime domain, $C \overset{e}{\frown}{}^* C'$ and thus, by Lemma 8(4), $I \leftrightarrow^* I'$, i.e., $[I]_{\leftrightarrow^*} = [I']_{\leftrightarrow^*}$. Proving that also the inverse is an es morphism is immediate, by exploiting the fact that the es is live. ■

IV. A CONNECTED ES SEMANTICS FOR GRAPH REWRITING

In this section we consider non-linear graph rewriting systems where rules are left-linear but possibly not right-linear and thus, as an effect of a rewriting step, some items can be merged. We argue that weak prime domains and connected es are the right tool for providing a concurrent semantics to this class of rewriting systems. More precisely, we show that the domain associated with a graph rewriting system by a generalisation of a classical construction is a weak prime domain and vice versa that each connected es arises as the semantics of some graph rewriting system.

A. Graph rewriting and concatenable traces

We start by reviewing the basic definitions about graph rewriting in the double-pushout approach [19]. We recall graph grammars and then introduce a notion of trace, providing a representation of a sequence of rewriting steps that abstracts from the order of independent rewrites. Traces are then turned into a category $\mathsf{Tr}(\mathcal{G})$ of concatenable derivation traces [22].

Definition 18 A *(directed, unlabelled) graph* is a tuple $G = \langle N, E, s, t\rangle$, where N and E are sets of nodes and edges, and $s, t : E \to N$ are the source and target functions. The components of a graph G are often denoted by N_G, E_G, s_G, t_G. A *graph morphism* $f : G \to H$ is a pair of functions $f = \langle f_N : N_G \to N_H, f_E : E_G \to E_H\rangle$ such that $f_N \circ s = s' \circ f_E$

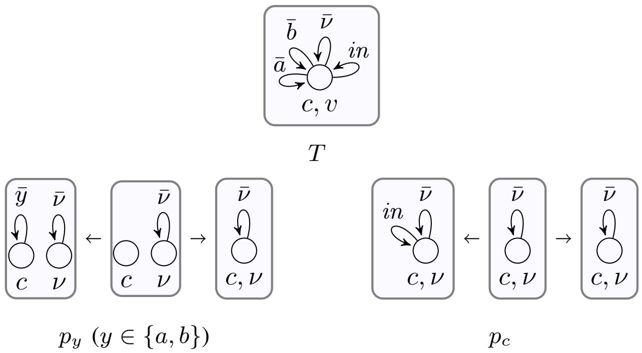


Fig. 5: The type graph of the grammar in Fig. 2a and its rules as spans.

$$
\begin{tikzcd}
L \arrow[d, "m^L"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "m^K"] & R \arrow[d, "m^R"] \\
G & D \arrow[l, "l^*"] \arrow[r, "r^*"'] & H
\end{tikzcd}
$$

Fig. 6: A direct derivation.

and $f_N \circ t = t' \circ f_E$. We denote by Graph the category of graphs and graph morphisms

An *abstract graph* $[G]$ is an isomorphism class of graphs. We work with typed graphs, i.e., graphs which are "labelled" over some fixed graph. Formally, given a graph T, the category of *graphs typed over* T, as introduced in [23], is the slice category $(\mathsf{Graph} \downarrow T)$, also denoted Graph_T.

Definition 19 (graph grammar) A *(T-typed graph) rule* is a span $(L \stackrel{l}{\leftarrow} K \stackrel{r}{\rightarrow} R)$ in Graph_T where l is mono and not epi. The typed graphs L, K, and R are called the *left-hand side*, the *interface*, and the *right-hand side* of the rule, respectively. A *(T-typed) graph grammar* is a tuple $\mathcal{G} = \langle T, G_s, P, \pi \rangle$, where G_s is the *start (typed) graph*, P is a set of *rule names*, and π maps each rule name in P into a rule.

Sometimes we write $p : (L \stackrel{l}{\leftarrow} K \stackrel{r}{\rightarrow} R)$ for denoting the rule $\pi(p)$. When clear from the context we omit the word "typed" and the typing morphisms. Note that we consider only *consuming* grammars, namely grammars where for each rule $\pi(p)$ the morphism l is not epi. This corresponds to the requirement on non-empty preconditions for Petri nets.

An example of graph grammar has been discussed in the introduction (see Fig. 2a). The type graph was left implicit: it can be found in the top part of Fig. 5. The typing morphisms for the start graph and the rules are implicitly represented by the labelling. Also observe that for the rules only the left-hand side L and right-hand side R were reported. The same rules with the interface graph explicitly represented are in Fig. 5.

Definition 20 (direct derivation) Given a typed graph G, a rule $p : (L \stackrel{l}{\leftarrow} K \stackrel{r}{\rightarrow} R)$, and a match, i.e., a typed graph morphism $g : L \to G$, a *direct derivation* δ from G to H via p (based on g) is a diagram as in Fig. 6, where both squares are pushouts in Graph_T. We write $\delta : G \stackrel{p/m}{\Longrightarrow} H$, where $m = \langle m^L, m^K, m^R \rangle$, or simply $\delta : G \Longrightarrow H$.

Since pushouts are defined only up to isomorphism, given an isomorphisms $\kappa : G' \to G$ and $\nu : H \to H'$, also $G' \stackrel{p/m'}{\Longrightarrow} H$ with $m' = \langle \kappa^{-1} \circ m^L, m^K, m^R \rangle$ and $G' \stackrel{p/m''}{\Longrightarrow} H'$ with $m'' = \langle m^L, m^K, \nu \circ m^R \rangle$ are direct derivations, that we denote respectively by $\kappa \cdot \delta$ and $\delta \cdot \nu$. Informally, the rewriting step removes (the image of) the left-hand side from the graph G and replaces it by (the image of) the right-hand side R. The interface K (the common part of L and R) specifies what is preserved. For example, the transitions in Fig. 2b are all direct derivations.

Definition 21 (derivations) Let $\mathcal{G} = \langle T, G_s, P, \pi \rangle$ be a graph grammar. A *derivation* over $\mathcal{G}$ is a sequence of direct derivations $\rho = \{G_{i-1} \stackrel{q_{i-1}}{\Longrightarrow} G_i\}_{i \in [1,n]}$ where $p_i \in P$ for $i \in [1, n]$. The derivation is written $\rho : G_0 \Longrightarrow^*_{\mathcal{G}} G_n$ or simply $\rho : G_0 \Longrightarrow^* G_n$. The graphs G_0 and G_n are called the *source* and the *target* of ρ, and denoted by $\mathsf{s}(\rho)$ and $\mathsf{t}(\rho)$, respectively. The *length* of ρ is $|\rho| = n$. Given two derivations ρ and ρ' such that $\mathsf{t}(\rho) = \mathsf{s}(\rho')$, their sequential composition $\rho\,;\rho' : \mathsf{s}(\rho) \Longrightarrow^* \mathsf{t}(\rho')$ is defined in the obvious way.

If $\rho : G \Longrightarrow^* H$ is a derivation, with $|\rho| > 0$, and $\kappa : G' \to G$, $\nu : H \to H'$ are graph isomorphisms, then $\kappa \cdot \rho : G' \Longrightarrow^* H$ and $\rho \cdot \nu : G \Longrightarrow H'$ are defined in the expected way.

In the double pushout approach to graph rewriting, it is natural to consider graphs and derivations up to isomorphism. However some structure must be imposed on the start and end graph of an abstract derivation in order to have a meaningful notion of sequential composition. In fact, isomorphic graphs are, in general, related by several isomorphisms, while in order to concatenate derivations keeping track of the flow of causality one must specify how the items of the source and target isomorphic graphs should be identified. We follow [2], inspired by the theory of Petri nets [24]: we choose for each class of isomorphic typed graphs a specific graph, named the *canonical graph*, and we decorate the source and target graphs of a derivation with a pair of isomorphisms from the corresponding canonical graphs to such graphs.

Let C denote the operation that associates to each (T-typed) graph its *canonical graph*, thus satisfying $\mathsf{C}(G) \simeq G$ and if $G \simeq G'$ then $\mathsf{C}(G) = \mathsf{C}(G')$.

Definition 22 (decorated derivation) A *decorated derivation* $\psi : G_0 \Longrightarrow^* G_n$ is a triple $\langle \alpha, \rho, \omega \rangle$, where $\rho : G_0 \Longrightarrow^* G_n$ is a derivation and $\alpha : \mathsf{C}(G_0) \to G_0$, $\omega : \mathsf{C}(G_n) \to G_n$ are isomorphisms. We define $\mathsf{s}(\psi) = \mathsf{C}(\mathsf{s}(\rho))$, $\mathsf{t}(\psi) = \mathsf{C}(\mathsf{t}(\rho))$ and $|\psi| = |\rho|$. The derivation is called *proper* if $|\psi| > 0$.

Definition 23 (sequential composition) Let $\psi = \langle \alpha, \rho, \omega \rangle$, $\psi' = \langle \alpha', \rho', \omega' \rangle$ be decorated derivations such that $\mathsf{t}(\psi) = \mathsf{s}(\psi')$. Their *sequential composition* $\psi; \psi'$ is defined, if ψ and ψ' are proper, as $\langle \alpha, (\rho \cdot \omega^{-1}); (\alpha' \cdot \rho'), \omega' \rangle$. Otherwise, if $|\psi| = 0$ then $\psi; \psi' = \langle \alpha' \circ \omega^{-1} \circ \alpha, \rho', \omega' \rangle$, and similarly, if $|\psi'| = 0$ then $\psi; \psi' = \langle \alpha, \rho, \omega \circ \alpha' \circ \omega^{-1} \rangle$.

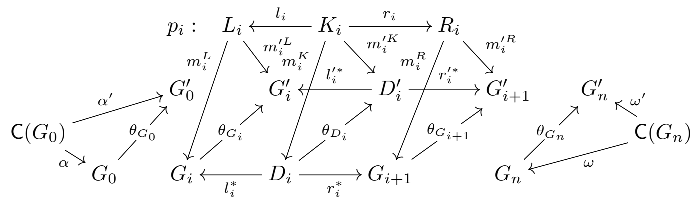


Fig. 7: Abstraction equivalence of decorated derivations.

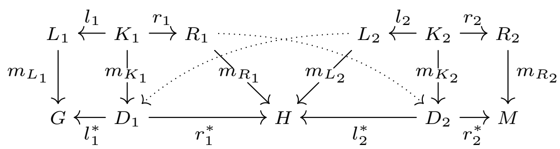


Fig. 8: Sequential independence for $\rho = G \overset{p_1/m_1}{\Longrightarrow} H \overset{p_2/m_2}{\Longrightarrow} M$.

We next define an abstraction equivalence that identifies derivations that differ only in representation details.

Definition 24 (abstraction equivalence) Let $\psi = \langle \alpha, \rho, \omega \rangle$, $\psi' = \langle \alpha', \rho', \omega' \rangle$ be decorated derivations with $\rho : G_0 \Longrightarrow^* G_n$ and $\rho' : G'_0 \Longrightarrow^* G'_{n'}$ (whose i^{th} step is depicted in the lower rows of Fig. 7). They are *abstraction equivalent*, written $\psi \equiv^a \psi'$, if $n = n'$, $q_{i-1} = q'_{i-1}$ for all $i \in [1, n]$, and there exists a family of isomorphisms $\{\theta_{X_i} : X_i \to X'_i \mid X \in \{G, D\},\ i \in [1, n]\} \cup \{\theta_{G_0}\}$ between corresponding graphs in the two derivations such that (1) the isomorphisms relating the source and target commute with the decorations, i.e., $\theta_{G_0} \circ \alpha = \alpha'$ and $\theta_{G_n} \circ \omega = \omega'$; and (2) the resulting diagram (whose i^{th} step is represented in Fig. 7) commutes.

Equivalence classes of decorated derivations with respect to $\equiv^a$ are called *abstract derivations* and denoted by $[\psi]_a$, where ψ is an element of the class.

From a concurrent perspective, derivations that only differ for the order in which two independent direct derivations are applied should not be distinguished. This is formalised by the classical shift equivalence on derivations.

Definition 25 (sequential independence) Let $G \overset{p_1/m_1}{\Longrightarrow} H \overset{p_2/m_2}{\Longrightarrow} M$ be a derivation as in Fig. 8. Then, its components are *sequentially independent* if there exists an *independence pair* among them, i.e., two graph morphisms $i_1 : R_1 \to D_2$ and $i_2 : L_2 \to D_1$ such that $l_2^* \circ i_1 = m_{L_2}$, $r_1^* \circ i_2 = m_{R_1}$.

Proposition 9 (interchange operator) *Let $\rho = G \overset{p_1/m_1}{\Longrightarrow} H \overset{p_2/m_2}{\Longrightarrow} M$ be a derivation whose components are sequentially independent via an independence pair ξ. Then, a derivation $IC_\xi(\rho) = G \overset{p_2/m_2^*}{\Longrightarrow} H^* \overset{p_1/m_1^*}{\Longrightarrow} M$ can be uniquely chosen, such that its components are sequentially independent via a canonical independence pair ξ^*.*

The interchange operator can be used to formalise a notion of shift equivalence [13], identifying (as for the analogous permutation equivalence of λ-calculus) those derivations which differ only for the scheduling of independent steps.

Definition 26 (shift equivalence) Two derivations ρ and ρ' are *shift equivalent*, written $\rho \equiv^{sh} \rho'$, if ρ' can be obtained from ρ by repeatedly applying the interchange operator.

For instance, in Fig. 2b it is easy to see that the derivations $\rho = G_s \overset{p_a}{\Longrightarrow} G_b \overset{p_b}{\Longrightarrow} G_{ab}$ consists of sequential independent direct derivations. It is shift equivalent to $\rho' = G_s \overset{p_b}{\Longrightarrow} G_a \overset{p_a}{\Longrightarrow} G_{ab}$.

Two decorated derivations are said to be shift equivalent when the underlying derivations are, i.e., $\langle \alpha, \rho, \omega \rangle \equiv^{sh} \langle \alpha, \rho', \omega \rangle$ if $\rho \equiv^{sh} \rho'$. Then the equivalence of interest arises by joining abstraction and shift equivalence.

Definition 27 (concatenable traces) We denote by $\equiv^c$ the equivalence on decorated derivations arising as the transitive closure of the union of the relations $\equiv^a$ and $\equiv^{sh}$. Equivalence classes of decorated derivations with respect to $\equiv^c$ are denoted as $[\psi]_c$ and are called *concatenable (derivation) traces*.

Several proofs concerning concatenable traces exploit a characterization of equivalence $\equiv^c$ presented in [2, Sec. 3.5], that we summarize and adapt here to our framework.

If ψ and ψ' are decorated derivations, then a *consistent permutation* between their steps relates two direct derivations if they consume and produce the same items, up to an isomorphism that is consistent with the decorations.

Definition 28 (consistent permutation) Given a decorated derivation $\psi = \langle \alpha, \rho, \omega \rangle : G_0 \Longrightarrow^* G_n$, we denote by $\mathsf{col}(\psi)$ the colimit of the corresponding diagram in category Graph_T, and by $in^X_{\mathsf{col}(\psi)}$ the injection of X into the colimit, for any graph X in ρ. Given two such decorated derivations ψ and ψ' of equal length n, a *consistent permutation* σ from ψ to ψ' is a permutation σ on $[0, n-1]$ such that

1) there exists an isomorphism $\xi : \mathsf{col}(\psi) \to \mathsf{col}(\psi')$;
2) for each $i \in [0, n-1]$ the direct derivations δ_i of ψ and $\delta_{\sigma(i)}$ of ψ' use the same rule;
3) for each $i \in [0, n-1]$, let $p : (L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R)$ be the rule used by direct derivations $\delta_i : G_i \overset{p/m}{\Longrightarrow} G_{i+1}$ and $\delta'_{\sigma(i)} : G'_{\sigma(i)} \overset{p/m'}{\Longrightarrow} G'_{\sigma(i)+1}$; then
   - $\xi \circ in^{G_i}_{\mathsf{col}(\psi)} \circ m^L = in^{G_{\sigma(i)}}_{\mathsf{col}(\psi')} \circ m'^L$, and
   - $\xi \circ in^{G_{i+1}}_{\mathsf{col}(\psi)} \circ m^R = in^{G_{\sigma(i)+1}}_{\mathsf{col}(\psi')} \circ m'^R$;
4) [α-consistency] $\xi \circ in^{G_0}_{\mathsf{col}(\phi)} \circ \alpha = in^{G'_0}_{\mathsf{col}(\phi')} \circ \alpha'$;
5) [ω-consistency] $\xi \circ in^{G_n}_{\mathsf{col}(\phi)} \circ \omega = in^{G'_n}_{\mathsf{col}(\phi')} \circ \omega'$;

A permutation σ from ψ to ψ' is called *left-consistent* if it satisfies conditions (1)-(4), but possibly not ω-consistency. It is easy to show, by induction on the length of derivations, that the isomorphism $\xi : \mathsf{col}(\psi) \to \mathsf{col}(\psi')$ is uniquely determined by conditions (2)-(4), if it exists.

The next result shows that consistent permutations characterize equivalence $\equiv^c$ in a strong sense.

Lemma 17 *Let ψ, ψ' be decorated derivations.*

1) $\psi \equiv^c \psi'$ iff $|\psi| = |\psi'|$ and there is a consistent permutation σ on $[0, |\psi| - 1]$ between them. We write

$\psi \equiv^c_\sigma \psi'$ *in this case.*

2) *If* $\psi;\psi_1 \equiv^c_\sigma \psi';\psi'_1$ *and* $\psi \equiv^c_{\sigma_0} \psi'$, *then* σ_0 *is the restriction of* σ *to* $[0, |\psi|-1]$. *In this case it also holds* $\psi_1 \equiv^c_{\sigma_1} \psi'_1$, *with* $\sigma_1(i) = \sigma(i+|\psi|) - |\psi|$.
3) *If* $\psi \equiv^c \psi'$, *then there is a unique consistent permutation* σ *such that* $\psi \equiv^c_\sigma \psi'$.

Proof: [sketch]

1) This holds by [2, Thm. 3.5.3], which does not use linearity of rules.
2) Suppose by absurd that j be the smallest index in $[0, |\psi_1|-1]$ such that $\sigma(j) \neq \sigma_1(j)$. Let $p : (L \xleftarrow{l} K \xrightarrow{r} R)$ be the rule used in δ_j and let $x \in L \setminus l(K)$ be an item consumed by it, which exists because all rules are consuming. By Definition 28 we deduce that both direct derivations $\delta'_{\sigma(j)}$ and $\delta'_{\sigma_1(j)}$ of $\psi'_1;\psi_2$ use the same rule p (say, with matches m' and m''), and that the items ${m'}^L(x) \in G'_{\sigma(j)}$ and ${m''}^L(x) \in G'_{\sigma_1(j)}$ which are consumed by $\delta'_{\sigma(j)}$ and $\delta'_{\sigma_1(j)}$, respectively, are identified in the colimit $\mathsf{col}(\psi'_1;\psi'_2)$ (actually, from $\psi_1 \equiv^c_{\sigma_1} \psi'_1$ we know that there is a morphism $G'_{\sigma(j)} \to \mathsf{col}(\psi'_1)$, but we can compose it with the obvious (possibly not injective) morphism $\mathsf{col}(\psi'_1) \to \mathsf{col}(\psi'_1;\psi'_2)$). But given the shape of the derivation diagram determined by the left-linearity of rules, and the properties of colimits in Graph, this is not possible, because there is no undirected path of morphisms relating the images of element $x \in L$ in $G'_{\sigma(j)}$ and $G'_{\sigma_1(j)}$ respectively. Therefore σ and σ_1 must coincide on $[0, |\psi_1|-1]$.
For the second part, by the fact just proved clearly σ_2 is a well-defined permutation on $[0, |\psi_2|-1]$. For consistency, most conditions of Definition 28 follow from the fact that it is a projection of σ: only α-consistency is not obvious, but it follows from ω-consistency of σ_1 and from Definition 23. Therefore $\psi_2 \equiv^c_{\sigma_2} \psi'_2$ by point (1).
3) Direct consequence of the previous point, considering zero-length decorated derivations ψ'_1 and ψ'_2.

■

The sequential composition of decorated derivations lifts to composition of derivation traces so that we can consider the corresponding category.

Definition 29 (category of concatenable traces) Let $\mathcal{G}$ be a graph grammar. The *category of concatenable traces* of $\mathcal{G}$, denoted by $\mathsf{Tr}(\mathcal{G})$, has abstract graphs as objects and concatenable traces as arrows.

B. A weak prime domain for a grammar

Given a grammar $\mathcal{G}$ we can obtain a partially ordered representation of the derivations in $\mathcal{G}$ starting from the initial graph by considering the concatenable traces ordered by prefix. Formally, as already done in [2], [3] for linear grammars, we first consider the structure of the category $([G_s] \downarrow \mathsf{Tr}(\mathcal{G}))$, which, by definition of sequential composition between traces, can be easily shown to be a preorder.

Proposition 10 *Let* $\mathcal{G}$ *be a graph grammar. Then the category* $([G_s] \downarrow \mathsf{Tr}(\mathcal{G}))$ *is a preorder.*

Proof: Let $[\psi] : [G_s] \to [G]$, $[\psi'] : [G_s] \to [G']$ be concatenable traces and let $[\psi_1], [\psi_2] : [\psi] \to [\psi']$ be arrows in the slice category. Spelled out, this means that $\psi_1, \psi_2 : G \to G'$ are such that $\psi;\psi_1 \equiv^c \psi;\psi_2 \equiv^c \psi'$. By point (2) of Lemma 17, using the fact that $\psi \equiv^c \psi$ we can conclude that $\psi_1 \equiv^c \psi_2$, as desired. ■

Explicitly, elements of the preorder are concatenable traces $[\psi]_c : [G_s] \to [G]$ and, for $[\psi']_c : [G_s] \to [G']$, we have $[\psi]_c \sqsubseteq [\psi']_c$ if there is $\psi'' : G \to G'$ such that $\psi;\psi'' \equiv_c \psi'$. Therefore, given two concatenable traces $[\psi]_c : [G_s] \to [G]$ and $[\psi']_c : [G_s] \to [G']$, if $[\psi]_c \sqsubseteq [\psi']_c \sqsubseteq [\psi]_c$ then ψ can be obtained from ψ' by composing it with a zero-length trace. Hence the elements of the partial order induced by $([G_s] \downarrow \mathsf{Tr}(\mathcal{G}))$ intuitively consist of classes of concatenable traces whose decorated derivations are related by an isomorphism that has to be consistent with the decoration of the source. This construction, applied to the grammar in Fig. 2a produces a domain isomorphic to that in Fig. 2c.

Lemma 18 *Let* $\mathcal{G}$ *be a graph grammar. The partial order induced by* $([G_s] \downarrow \mathsf{Tr}(\mathcal{G}))$, *denoted* $\mathcal{P}(\mathcal{G})$, *has as elements* $\langle\psi\rangle_c = \{[\psi \cdot \nu]_c \mid \nu : \mathsf{t}(\psi) \xrightarrow{\sim} \mathsf{t}(\psi)\}$ *and* $\langle\psi\rangle_c \sqsubseteq \langle\psi'\rangle_c$ *if* $\psi;\psi'' \equiv^c \psi'$ *for some decorated derivation* ψ''.

Proof: Immediate. ■

Lemma 19 *Let* $\mathcal{G}$ *be a graph grammar and* $\langle\psi\rangle_c \in \mathcal{P}(\mathcal{G})$. *Then* $[\psi']_c, [\psi'']_c \in \langle\psi\rangle_c$ *iff there is a left-consistent permutation from* ψ' *to* ψ''.

Proof: Immediate. ■

The domain of interest is then obtained by completion of the preorder $([G_s] \downarrow \mathsf{Tr}(\mathcal{G}))$, with the elements in $\mathcal{P}(\mathcal{G})$ as compact elements. In order to prove this, we need a preliminary technical lemma that essentially provides existence and shape of the least upper bounds in the domain of traces.

Lemma 20 (properties of $\equiv^c$) *1) Let* ψ, ψ' *be decorated derivations, and* ψ_1, ψ'_1 *such that* $\psi;\psi_1 \equiv^c_\sigma \psi';\psi'_1$ *and* $n = |\{j \in [|\psi|, |\psi;\psi_1|-1] \mid \sigma(j) < |\psi'|\}|$. *Then for all* ϕ_2, ϕ'_2 *such that* $\psi;\phi_2 \equiv^c \psi';\phi'_2$ *it holds* $|\phi_2| \geq n$ *and there are* ψ_2, ψ'_2, ψ_3 *such that*

- $\psi;\psi_1 \equiv^c \psi;\psi_2;\psi_3$
- $\psi;\psi_2 \equiv^c \psi';\psi'_2$
- $|\psi_2| = n$

2) *Let* ψ, ψ' *be derivation traces and* ψ_1, ψ'_1, ψ_2, ψ'_2 *such that* $\psi;\psi_1 \equiv^c_{\sigma_1} \psi';\psi'_1$ *and* $\psi;\psi_2 \equiv^c_{\sigma_2} \psi';\psi'_2$ *with* ψ_1, ψ_2 *of minimal length. Then* $\psi_1 \equiv^c_\sigma \psi_2 \cdot \nu$, *where* $\nu : \mathsf{t}(\psi_2) \to \mathsf{t}(\psi_2)$ *is some iso and* $\sigma(j) = \sigma_2^{-1}(\sigma_1(j+|\psi|)) - |\psi|$ *for* $j \in [0, |\psi_1|-1]$.

Proof:

1) We first observe that if ψ, ψ' are derivation traces and ψ_1, ψ'_1 are such that $\psi;\psi_1 \equiv^c_\sigma \psi';\psi'_1$, with $|\psi| = k$, $|\psi'| = k'$, $|\psi;\psi_1| = |\psi';\psi'_1| = h$ then there is a ϕ_1 such that $\psi;\psi_1 \equiv^c \psi;\phi_1 \equiv^c_{\sigma_1} \psi';\psi'_1$ and

for $i, j \in [|\psi|, h-1]$, $i \leq j$ implies $\sigma_1(i) \leq \sigma_1(j)$. (†)

In order to prove this, we can proceed by induction on the number of inversions $x = |\{(i,j) \in [|\psi|, h-1] \mid i \leq j \ \wedge \ \sigma(i) > \sigma(j)\}|$, i.e., on the number of pairs (i,j) in the interval of interest that do not respect the monotonicity condition. When $x = 0$ the thesis immediately holds. Assume that $x > 0$. Then there are certainly indices $j \in [|\psi|, h-2]$ such that $\sigma(j) > \sigma(j+1)$. Among these, take the index i such that $\sigma(i+1)$ is minimal. Then it can be shown that direct derivations at position i and $i+1$ in ψ_1 are sequential independent, and thus they can be switched, i.e., there is ϕ_2 such that $\psi; \phi_2 \equiv^c_{id[i \mapsto i+1, i+1 \mapsto i]} \psi; \psi_1$. Therefore $\psi; \phi_2 \equiv^c_{\sigma \circ id[i \mapsto i+1, i+1 \mapsto i]} \psi'; \psi'_1$. This reduces the number of inversions and thus the inductive hypothesis allows us to conclude.

In the same way, we can prove that there is a ϕ'_1 such that $\psi; \phi_1 \equiv^c_{\sigma_2} \psi'; \phi'_1 \equiv^c \psi'; \psi'_1$ and

for $i, j \in [|\psi'|, h-1]$, if $i \leq j$ then $\sigma_2^{-1}(i) \leq \sigma_2^{-1}(j)$ (‡)

Putting conditions (†) and (‡) together we derive that $\psi; \psi_1 \equiv^c \psi; \phi_1 \equiv^c_{\sigma'} = \psi'; \phi'_1 \equiv^c \psi'; \psi'_1$. Now let $y \in [|\psi|, h-1]$ be the largest index such that $\sigma'(y) < |\psi'|$ (or $y = |\psi|$ if it does not exist), let $l_3 = h - y$ and consider decorated derivations $\psi_2, \psi_3, \psi'_2, \psi'_3$ such that $|\psi_3| = |\psi'_3| = l_3$ and $\psi; \psi_2; \psi_3 = \psi; \phi_1 \equiv^c_{\sigma'} \psi'; \phi'_1 = \psi'; \psi'_2; \psi'_3$. By construction we obtain that $|\psi_2| = n$ and that σ' restricts to a permutation σ'_2 on $[0, |\psi; \psi_2| - 1]$ which can be made consistent, if necessary, by changing the ω decoration of ψ_2, affecting only the α decoration of ψ_3. Thus by Lemma 17(1) we conclude that $\psi; \psi_2 \equiv^c \psi'; \psi'_2$. Finally, notice that by the definition of y and the properties of σ', it follows that $\sigma'(j) < |\psi'|$ for all $j \in [|\psi|, |\psi; \psi_2| - 1]$ and $\sigma'(j) \geq |\psi'|$ for all $j \in [|\psi; \psi_2|, h-1]$. That is, the direct derivations in ψ_2 match all direct derivations of ψ' that are not matched in ψ. This implies that there cannot exist a derivation ϕ_2 shorter than n such that $\psi; \phi_2 \equiv^c \psi'; \phi'_2$ for some ϕ'_2.

2) Let $n = |\psi|$ and $m = |\psi_1| = |\psi_2|$, which must have the same length. By the last part of the proof of the previous point, since both ψ_1 and ψ_2 are of minimal length, we have that for all $j \in [n, n+m-1]$ it holds $\sigma_1(j) < |\psi'|$ and $\sigma_2(j) < |\psi'|$. Furthermore, $\sigma_1([n, n+m-1]) = \sigma_2([n, n+m-1])$, because both ψ_1 and ψ_2 consist of direct derivation that match those of ψ' which are not matched in ψ Thus $\sigma(j) = \sigma_2^{-1}(\sigma_1(j + |\psi|)) - |\psi|$ is a well-defined permutation on $[0, |\psi_1| - 1]$ from ψ_1 to ψ_2. Conditions (1)-(3) of Definition 28 are guaranteed by the corresponding properties of σ_1 and σ_2, and α-consistency holds because both ψ_1 to ψ_2 start from the same graph $(\mathsf{t}(\psi))$. Therefore σ is a left-consistent permutation from ψ_1 to ψ_2.

■

Relying on the results above we can easily prove that the ideal completion of the partial order of traces is a domain.

Proposition 11 (domain of traces) *Let $\mathcal{G}$ be a graph grammar. Then $\mathcal{D}(\mathcal{G}) = \mathsf{Idl}(\mathcal{P}(\mathcal{G}))$ is a domain.*

Proof: By Lemma 6 it is sufficient to prove (1) that $\downarrow\langle\psi\rangle_c$ is finite for every $\langle\psi\rangle_c \in \mathcal{P}(\mathcal{G})$, and (2) that if $\{\langle\psi_1\rangle_c, \langle\psi_2\rangle_c, \langle\psi_3\rangle_c\}$ is pairwise consistent then $\langle\psi_1\rangle_c \sqcup \langle\psi_2\rangle_c$ exists and is consistent with $\langle\psi_3\rangle_c$.

1) Let $\langle\psi'\rangle_c \sqsubseteq \langle\psi\rangle_c$. By Lemma 18 and by Lemma 17(1) we know that $\psi'; \psi'' \equiv^c_\sigma \psi$ for some decorated derivation ψ'' and a consistent permutation σ. Now suppose that ψ'_1 and ψ'_2 are decorated derivations such that $\psi'_1; \psi''_1 \equiv^c_{\sigma_1} \psi$ and $\psi'_2; \psi''_2 \equiv^c_{\sigma_2} \psi$ for some ψ''_1, ψ''_2, and that $\sigma_1([0, |\psi'_1|]) = \sigma_2([0, |\psi'_2|]) \subseteq [0, |\psi|]$. Then $\sigma_2^{-1} \circ \sigma_1$ is a permutation on $[0, |\psi'_1|]$ from ψ'_1 to ψ'_2 which satisfies conditions (1)-(4) of Definition 28. Therefore by Lemma 19 $\langle\psi'_1\rangle_c = \langle\psi'_2\rangle_c$. As a consequence, the cardinality of $\downarrow\langle\psi\rangle_c$ is bound by $2^{|\psi|}$.

2) Given two consistent elements $\langle\psi_1\rangle_c$ and $\langle\psi_2\rangle_c$ of $\mathcal{P}(\mathcal{G})$, there exists $\langle\psi\rangle_c = \langle\psi_1\rangle_c \sqcup \langle\psi_2\rangle_c$, where ψ is the minimal common extension of ψ_1 and ψ_2, provided by Lemma 20(1). Uniqueness of $\langle\psi\rangle_c$ follows by Lemmas 20(2) and 19 because minimal common extensions are unique, up to left-consistent permutations. Suppose further that $\langle\psi_3\rangle_c$ is compatible with both $\langle\psi_1\rangle_c$ and $\langle\psi_2\rangle_c$: we have to show that it is compatible with $\langle\psi\rangle_c$. Let $\langle\psi'\rangle_c = \langle\psi_2\rangle_c \sqcup \langle\psi_3\rangle_c$. Then there exist ϕ_1, ϕ and ϕ' such that $\psi_1; \phi_1 \equiv^c_{\sigma_1} \psi_2; \phi \equiv^c_\sigma \psi$ and $\psi_2; \phi' \equiv^c_{\sigma'} \psi'$ for consistent permutations σ_1, σ and σ'.

We conclude by showing that either $\langle\psi\rangle_c$ and $\langle\psi'\rangle_c$ are compatible, or $\langle\psi_1\rangle_c \sqcup \langle\psi_3\rangle_c$ and $\langle\psi'\rangle_c$ are compatible, both of which are equivalent and imply the thesis. We proceed by induction on $k = |\psi_1| + |\psi_3|$. If $|\psi_1| = 0$, i.e. ψ_1 is a zero-length decorated derivation, hence, by Lemma 20, also ϕ is so and thus $\langle\psi\rangle_c = \langle\psi_2\rangle_c$, and the latter is compatible with $\langle\psi'\rangle_c$. If $|\psi_3| = 0$ we conclude analogously. If $k > 0$, let δ be the last derivation step in ψ_1, i.e., $\psi_1 = \psi'_1; \delta$. If $\sigma_1(|\psi_1| - 1) < |\psi_2|$, namely if step δ is already in ψ_2, then by Lemma 20 we get that $\langle\psi\rangle_c = \langle\psi'_1\rangle_c \sqcup \langle\psi_2\rangle_c$. Since $|\psi'_1| < k$ we conclude by inductive hypothesis that ψ and ψ' are compatible. If instead, $\sigma_1(|\psi_1| - 1) \geq |\psi_2|$ then, again by Lemma 20, we can write ψ as $\psi \equiv^c_{\sigma''} \psi_2; \phi''; \delta'$, where $\langle\psi_2; \phi''\rangle_c = \langle\psi'_1\rangle_c \sqcup \langle\psi_2\rangle_c$ and $\sigma''(|\psi_1| - 1) = |\psi| - 1$, i.e., δ is mapped to δ'. Hence, by inductive hypothesis $\psi_2; \phi''$ and ψ' are compatible.

Now, since $\langle\psi_1\rangle_c$ and $\langle\psi_3\rangle_c$ are compatible (thus $\psi_1; \phi'_1 \equiv^c_{\sigma_3} \psi_3; \phi'_3$ for suitable derivations ϕ'_1, ϕ'_3 and permutation σ_3), either step δ is already in ψ_3 (thus $\sigma_3(|\psi_1| - 1) < |\psi_3|$), or it isn't, and $\sigma_3(|\psi_1| - 1) \geq |\psi_3|$. In the first case δ is related to a step in ψ', and it follows that $\langle\psi'\rangle_c \sqcup \langle\psi_2; \phi''\rangle_c = \langle\psi'\rangle_c \sqcup \langle\psi_2; \phi''; \delta'\rangle_c$ and we conclude. If instead δ is not a step in ψ_3, we can write $\psi_3; \phi'_3$ as $\psi_3; \phi''_3; \delta''$, where step δ'' matches step δ of ψ_1. By inductive hypothesis we have that $\psi_3; \phi''_3$ and ψ' are compatible, and we get $\langle\psi_3; \phi''_3\rangle_c \sqcup \langle\psi'\rangle_c = \langle\psi_2; \phi''\rangle_c \sqcup \langle\psi'\rangle_c$. Since both steps δ' and δ'' are related by consistent permutations to step δ of ψ_1, we can extend uniformly the two derivations preserving consistency, obtaining $\langle\psi_3; \phi_3''; \delta''\rangle_c \sqcup \langle\psi'\rangle_c = \langle\psi_2; \phi''; \delta'\rangle_c \sqcup \langle\psi'\rangle_c = \langle\psi\rangle_c \sqcup \langle\psi'\rangle_c$, as desired.

■

We can show that $\mathcal{D}(\mathcal{G})$ is a weak prime domain. The proof relies on the fact that irreducibles are elements of the form $\langle\epsilon\rangle_c$, where $\epsilon = \psi; \delta$ is a decorated derivation such that its last direct derivation δ cannot be switched back, i.e., minimal traces enabling some direct derivation. These are called *pre-events* in the work of [2], [3], where graph grammars are linear and thus, consistently with Lemma 2, such elements provides the primes of the domain. Two irreducibles $\langle\epsilon\rangle_c$ and $\langle\epsilon'\rangle_c$ are interchangeable when they are different minimal traces for the same direct derivation.

Theorem 3 (weak prime domain from a graph grammar) *Let $\mathcal{G}$ be a graph grammar. Then $\mathcal{D}(\mathcal{G})$ is a weak prime domain.*

Proof: We know by Proposition 11 that $\mathcal{D}(\mathcal{G})$ is a domain. Hence, recalling Definition 12, we have to show that $\mathcal{D}(\mathcal{G})$ is weak prime algebraic.

We will exploit the characterisation in Lemma 6. First provide a characterisation of irreducibles and of the interchangeability relation among them. As usual, we confuse compact elements of $\mathcal{D}(\mathcal{G})$ with the corresponding generators in $\mathcal{P}(\mathcal{G})$.

As mentioned above, irreducibles in $\mathcal{D}(\mathcal{G})$ are, in the terminology of [2], [3], *pre-events*, namely elements of the form $\langle\epsilon\rangle_c$, where $\epsilon = \psi; \delta$ is a decorated derivation such that its last direct derivation δ cannot be switched back. Formally, $\langle\epsilon\rangle_c$ is a pre-event if letting $n = |\epsilon|$ then for all $\epsilon = \psi; \delta \equiv^c_\sigma \psi'$ it holds $\sigma(n) = n$.

In fact, assume that $\langle\epsilon\rangle_c = \langle\psi_1\rangle_c \sqcup \langle\psi_2\rangle_c$, and let $\epsilon \equiv^c_\sigma \psi_1; \psi_1' \equiv^c_{\sigma'} \psi_2; \psi_2'$ for suitable ψ_1', ψ_2' of minimal length. Since ϵ is a pre-event, we have that if $n = |\psi; \delta| = |\psi_1; \psi_1'| = |\psi_2; \psi_2'|$, then $\sigma'(n) = n$. This implies that $|\psi_1'| = 0$ (and thus $\langle\epsilon\rangle_c = \langle\psi_1\rangle_c$) or $|\psi_2'| = 0$ (and thus $\langle\epsilon\rangle_c = \langle\psi_2\rangle_c$), as desired.

Two irreducibles $\langle\epsilon\rangle_c$ and $\langle\epsilon'\rangle_c$ are interchangeable iff the corresponding traces are compatible and whenever $\epsilon; \psi_1 \equiv^c_\sigma \epsilon'; \psi_1'$ with ψ_1, ψ_1' of minimal length (thus $\langle\epsilon; \psi_1\rangle_c = \langle\epsilon'; \psi_1'\rangle_c = \langle\epsilon\rangle_c \sqcup \langle\epsilon'\rangle_c$), then $\sigma(|\epsilon|) = |\epsilon'|$.

In fact, assume that $\langle\epsilon\rangle_c = \langle\psi; \delta\rangle_c$ and $\langle\epsilon'\rangle_c = \langle\psi'; \delta'\rangle_c$ are interchangeable, and $\epsilon; \psi_1 \equiv^c_\sigma \epsilon'; \psi_1'$ with ψ_1, ψ_1' of minimal length. By the proof of Lemma 20(1) we have that σ maps steps in ψ_1 to ϵ' and, analogously, σ^{-1} maps steps in ψ_1' to ϵ (formally, $\sigma(j) < |\epsilon'|$ for $j \geq |\epsilon|$ and, dually, if $\sigma(j) \geq |\epsilon'|$ then $j < |\epsilon|$). By Lemma 4(3) we have that $\langle\epsilon\rangle_c \sqcup \langle\epsilon'\rangle_c = \langle\psi\rangle_c \sqcup \langle\epsilon'\rangle_c = \langle\epsilon\rangle_c \sqcup \langle\psi'\rangle_c$. Hence we can view the previous equivalence of decorated derivations as $\psi; (\delta; \psi_1) \equiv^c_\sigma (\psi'; \delta'); \psi_1'$, with $\delta; \psi_1$ and ψ_1' of minimal length. This means that σ maps steps in $\delta; \psi_1$ to ϵ' and, with a dual argument, steps in $\delta'; \psi_1'$ to ϵ. Putting all this together we get that necessarily $\sigma(|\epsilon|) = |\epsilon'|$, as desired.

For the converse, assume that $\langle\epsilon\rangle_c$, $\langle\epsilon'\rangle_c$ are compatible, that $\langle\psi\rangle_c = \langle\epsilon\rangle_c \sqcup \langle\epsilon'\rangle_c$, and that $\psi \equiv^c \epsilon; \psi_1 \equiv^c_\sigma \epsilon'; \psi_1'$ where $\sigma(|\epsilon|) = |\epsilon'|$. Then, reverting the reasoning above, we get that $\langle\psi\rangle_c \sqcup \langle\epsilon'\rangle_c = \langle\epsilon\rangle_c \sqcup \langle\psi'\rangle_c$, and thus we conclude that $\langle\epsilon\rangle_c, \langle\epsilon'\rangle_c$ are interchangeable by Lemma 4(3).

We conclude that $\mathcal{D}(\mathcal{G})$ is a weak prime domain, relying on Lemma 6. Let $\langle\epsilon\rangle_c$ with $\epsilon = \psi; \delta$ be an irreducible, and $\langle\epsilon\rangle_c \sqsubseteq \langle\psi_1\rangle_c \sqcup \langle\psi_2\rangle_c$. Let ψ_1' and ψ_2' be decorated derivations of minimal length such that $\epsilon; \psi \equiv^c_\sigma \psi_1; \psi_1' \equiv^c_{\sigma_1} \psi_2; \psi_2'$ for some ψ. If $\sigma(|\epsilon|) \in [0, |\psi_1| - 1]$ then consider ϕ_1 such that $\psi_1; \psi_1' \equiv^c_{\sigma'} \phi_1; \psi_1'$ and $\sigma'(\sigma(|\epsilon|))$ is minimal. Then $\langle\phi_1\rangle_c$ is an irreducible, $\langle\phi_1\rangle_c$ and $\langle\epsilon\rangle_c$ are interchangeable, and clearly $\langle\phi_1\rangle_c \sqsubseteq \langle\psi_1\rangle_c$. If instead $\sigma(|\epsilon|) \geq |\psi_1|$ we have that $\sigma_1(\sigma(|\epsilon|)) < |\psi_2|$, and we can conclude, in the same way, the existence of $\langle\phi_2\rangle_c \sqsubseteq \langle\psi_2\rangle_c$ irreducible and interchangeable with $\langle\epsilon\rangle_c$. ■

Note that when the rules are right-linear the domain and es semantics specialises to the usual prime event structure semantics (see [2]–[4]), since the construction of the domain in the present paper is formally the same as in [2].

C. Any connected es is generated by some grammar

By Theorem 3, given any graph grammar $\mathcal{G}$ the domain $\mathcal{D}(\mathcal{G})$ is a weak prime domain. We next show that also the converse holds, i.e., any weak prime domain is generated by a suitable graph grammar. This shows that connected es are precisely what is needed to capture the concurrent semantics of non-linear graph grammars, and thus strengthen our claim that they represent the right structure for modelling formalisms with fusions.

Construction (graph grammar for a connected es) Let $\langle E, \#, \vdash\rangle$ be a connected es. The grammar $\mathcal{G}_E = \langle T, P, \pi, G_s\rangle$ is defined as follows.

First, for any $e \in E$, we define the following graphs, which are then used as basic building blocks

- I_e and S_e as in Fig. 9(a) and Fig. 9(b);
- let U_e denote the set-theoretical product of the minimal enablings of e, i.e., $U_e = \Pi\{X \subseteq E \mid X \vdash_0 e\}$; for any tuple $u \in U_e$ we define the graph $L_{u,e}$ as in Fig. 9(c).

Moreover, for any pair of events $e, e' \in E$ such that $e\#e'$, we define a graph $C_{e,e'}$ as in Fig. 9(d).

The set of productions is $P = E$, i.e., we add a rule for each event. For $e \in E$, we define the corresponding rule in a way that

- it deletes I_e and $C_{e,e'}$.
- it preserves the graph $S_e \cup \bigcup_{u \in U_e} L_{u,e}$
- for all $e' \in E$, for all graphs $L_{u,e'}$ such that e occurs in u, it merges the corresponding nodes into one.

The graph $S_e \cup \bigcup_{u \in U_e} L_{u,e}$ arises from S_e and $L_{u,e}$, $u \in U_e$ by merging all the nodes (we use $\bigcup$ and $\biguplus$ to denote union and disjoint union, respectively, with a meaning illustrated in Figs. 9(f) and (g).) Hence, there is a match for the rule e only if S_e and all $L_{u,e}$ for $u \in U_e$ have been merged and this happens if and only if at least a minimal enabling of e have been executed. The deletion of the graphs $C_{e,e'}$ establishes the needed conflicts. The rule is consuming since it deletes the node of graph I_e. Formally, the rule for e has as left-hand side the graph

$$I_e \cup (\bigcup_{\substack{e' \in E \\ e \# e'}} C_{e,e'}) \cup (\bigcup_{e' \in E} (S_{e'} \uplus \biguplus_{\substack{u' \in U_{e'} \\ e \in u}} L_{u',e'})) \cup (S_e \cup \bigcup_{u \in U_e} L_{u,e})$$

while the right-hand side is

$$(S_e \cup \bigcup_{u \in U_e} L_{u,e}) \cup (\bigcup_{e' \in E} (S_{e'} \cup \bigcup_{\substack{u' \in U_{e'} \\ e \in u}} L_{u',e'}))$$

The rule is schematised in Fig. 9(e), where it is intended that e occurs in $u_j^1, \ldots, u_j^{n_j}$ for $u_j^i \in U_{e_j}$, $j \in [1,k], i \in [1, n_k]$. Moreover $e'_1, \ldots, e'_h$ are the events in conflict with e and, finally, $U_e = \{u_1, \ldots, u_n\}$.

The start graph is just the disjoint union of all the basic graphs introduced above

$$G_s = (\bigcup_{e \# e'} C_{e,e'}) \cup \bigcup_{e \in E} (I_e \cup S_e \uplus \biguplus_{u \in U_e} L_{u,e})$$

Then the type graph is

$$T = (\bigcup_{e \# e'} C_{e,e'}) \cup \bigcup_{e \in E} (I_e \cup S_e \cup \bigcup_{u \in U_e} L_{u,e})$$

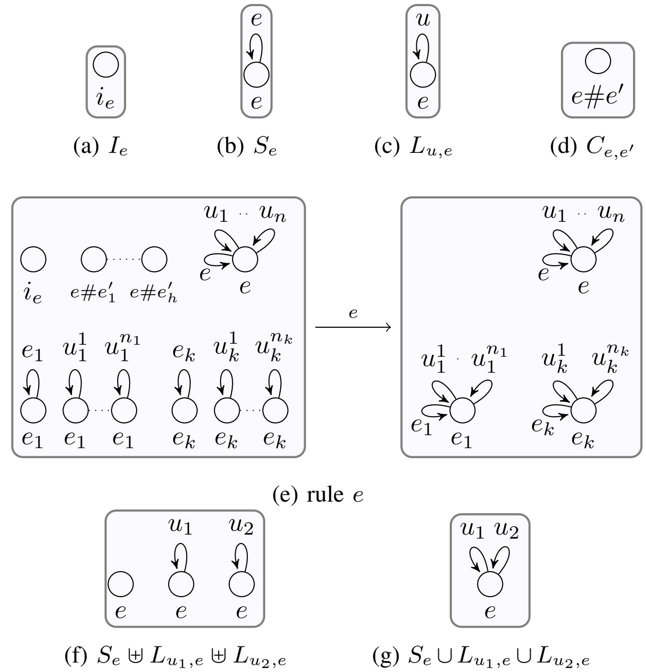


Fig. 9: Some graphs illustrating the construction of $\mathcal{G}_E$.

It is not difficult to show that the grammar $\mathcal{G}_E$ generates exactly the es E.

Theorem 4 *Let $\langle E, \#, \vdash \rangle$ be a connected es. Then, E and $\mathcal{E}(\mathcal{D}(\mathcal{G}_E))$ are isomorphic connected es.*

Proof: First observe that each rule in $\mathcal{G}_E$ is executed at most once in a derivation since it consumes an item (the node of graph I_e) which is not generated by any other rule. If we consider $\mathcal{D}(\mathcal{G}_E)$, then the irreducibles are minimal $\langle \epsilon \rangle_c$ with $\epsilon = \psi; \delta$. By the shape of rule e, the derivation ψ must contain occurrences of a minimal set of rules e' such that the graphs S_e and $L_{u,e}$ for $u \in U_e$ are merged along the common node. By construction, in order to merge all such graphs, if we denote by X_ψ the set of rules applied in ψ, it must be $X_\psi \supseteq C$ for some $C \in \mathit{Conf}(E)$ such that $C \vdash_0 e$. Therefore by minimality we conclude that $X_\psi \vdash_0 e$. Relying on this observation, a routine induction on the $|C|$ shows that minimal enablings $C \vdash_0 e$ in E are in one to one correspondence with irreducibles $\langle \epsilon \rangle_c$ in $\mathcal{D}(\mathcal{G}_E)$. Recalling, that, in turn, irreducibles in $\mathcal{D}(E)$ are again minimal enablings, i.e., $\langle C, e \rangle$ with $C \in \mathit{Conf}(E)$ such that $C \vdash_0 e$ we obtain a bijection between irreducibles in $\mathcal{D}(\mathcal{G}_E)$ and $\mathcal{D}(E)$.

The fact that the correspondence preserves and reflects the order is, again, almost immediate by construction. In fact, consider two irreducibles $\langle \epsilon \rangle_c$ and $\langle \epsilon' \rangle_c$ in $\mathcal{D}(\mathcal{G}_E)$ and the corresponding irreducibles $\langle C, e \rangle$ and $\langle C', e' \rangle$ in $\mathcal{D}(E)$. If $\langle C, e \rangle \subseteq \langle C', e' \rangle$, take $X = \langle C', e' \rangle \setminus \langle C, e \rangle$. Then ϵ can be extended with the rules corresponding to the events in X, thus showing the existence of a derivation ψ such that $\epsilon; \psi \equiv^c \epsilon'$. In fact, if this were not possible, there would be an event $e'' \in X$ such that the corresponding rule compete for deleting some item of the start graph with a rule e_1 in ϵ, hence $e_1 \in \langle C, e \rangle$. By construction, the only possibility is that the common item is C_{e'',e_1}. But this would mean that $e'' \# e_1$. This contradicts the fact that $\{e_1, e''\} \subseteq \langle C', e' \rangle$. The converse, i.e., the fact that if $\langle \epsilon \rangle_c \sqsubseteq \langle \epsilon' \rangle_c$ then $\langle C, e \rangle \subseteq \langle C', e' \rangle$ is immediate.

Recalling that domains are irreducible algebraic (Proposition 1), we conclude that $\mathcal{D}(\mathcal{E}(\mathcal{G}_E))$ and $\mathcal{D}(E)$ are isomorphic. Therefore, by Theorem 2, since E is a weak prime domain, also $\mathcal{E}(\mathcal{G}_E)$ and E are isomorphic, as desired. ■

Example 2 Consider the running example es, from Example 1, with set of events is $\{a, b, c\}$, empty conflict relation and minimal enablings $\{a\} \vdash_0 c$ and $\{b\} \vdash_0 c$. The associated grammar is depicted in Fig. 10.

As a further example, consider an es E_1 with events $\{a, b, c, d, e\}$. The conflict relation $\#$ is given by $e \# d$ and minimal enablings $\emptyset \vdash_0 a$, $\emptyset \vdash_0 b$, $\emptyset \vdash_0 e$, $\{a, b\} \vdash_0 d$ and $\{c\} \vdash_0 d$. The grammar is in Fig. 11.

V. Conclusions and related work

In the paper we provided a characterisation of a class of domains, referred to as weak prime algebraic domains, appropriate for describing the concurrent semantics of those formalisms where a computational step can merge parts of the state. We established a categorical equivalence between weak prime algebraic domains and a suitably defined class of connected event structures. We also proved that the category of Winskel's general event structures coreflects in the category of weak prime algebraic domains. The appropriateness of the class of weak prime domains is witnessed by the results in the second part of the paper that show that weak prime algebraic domains are precisely those arising from left-linear graph

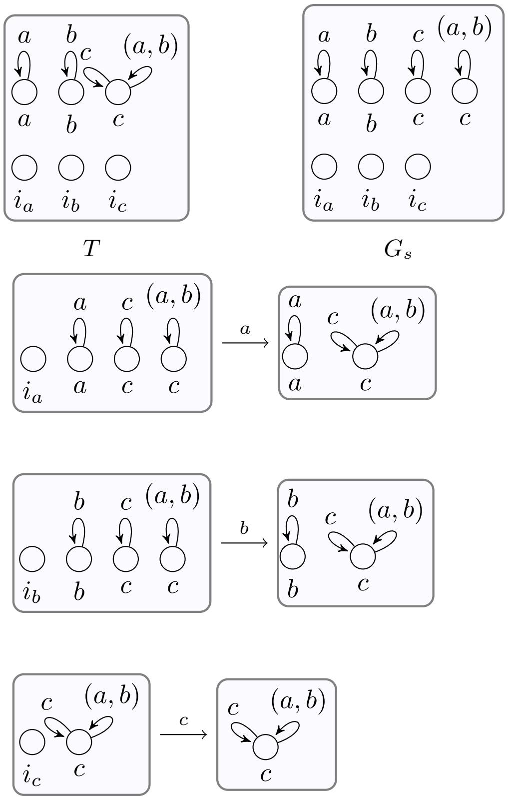


Fig. 10: The grammar associated to our running example.

rewriting systems, i.e., those systems where rules besides generating and deleting can also merge graph items.

Technically, the starting point is the relaxation of the stability condition for event structures. As already noted by Winskel in [5] "[t]he stability axiom would go if one wished to model processes which had an event which could be caused in several compatible ways [...]; then I expect complete irreducibles would play a similar role to complete primes here". Indeed, the correspondence between irreducibles and weak primes, based on the notion of interchangeability, is the ingenuous step that allowed us to obtain a smooth extension of the classical duality between prime event structures and prime algebraic domains.

The coreflection between the category of general event structures and the one of weak prime algebraic domains says that the latter are exactly the partial orders of configurations of general event structures (with binary conflict). Such class of domains has been studied originally in [20] where, generalising the work on concrete domains and sequentiality [25], a characterisation is given in terms of a set of axioms expressing properties of prime intervals. A similar characterisation for non-binary conflict is in [21]. We consider our simple characterisation of this class of domains, where weak primes give an intuitive account of events in computation, as a valuable contribution of the paper. We plan to provide an in depth comparison with these previous results in the full version of the paper.

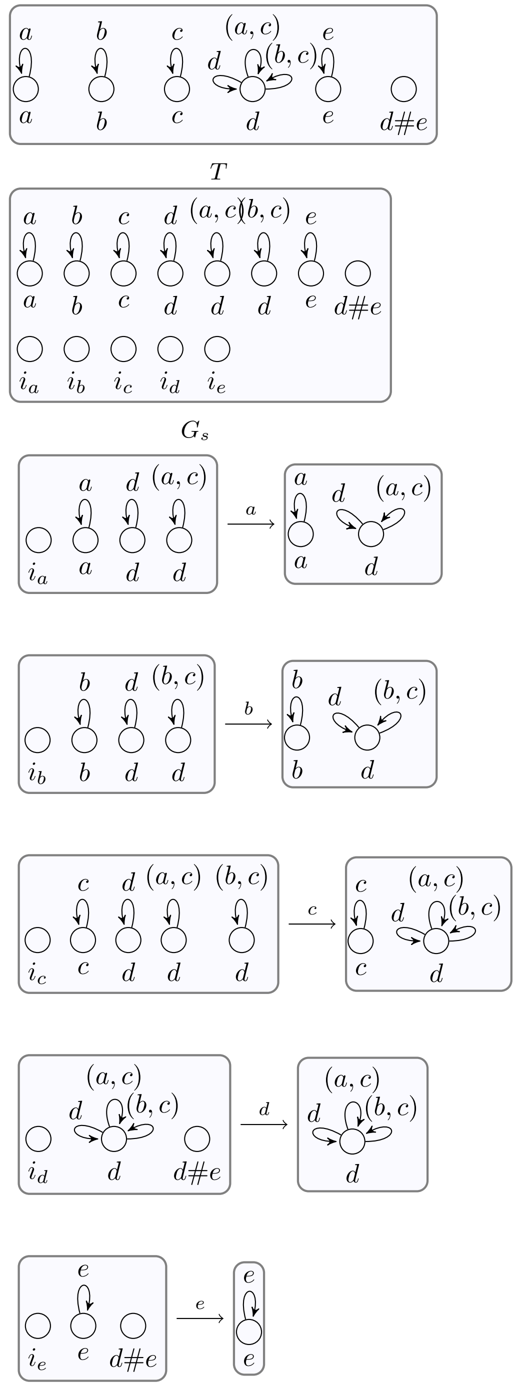


Fig. 11: The grammar for the es in example 2.

The paper [26] study a characterisation of the partial order of configurations associated with a variety of classes of event structures in terms of axiomatisability of the associated propositional theories. Despite the fact that the focus is here mainly on event structures that genereralise Winskel's ones, we believe that this work can provide interesting suggestions for further development.

The need of resorting to unstable event structures for modelling the concurrent computations of name passing process calculi has been observed by several authors. In particular, in [16] an event structure semantics for the pi-calculus is defined by relying on a notion of such structure that is tailored for parallel extrusions. These are labelled unstable event structures with the constraint that two minimal enablings can differ only for one event (intuitively, the extruder). They fail to be connected event structure since non-connected minimal enablings are admitted (roughly, because identical events in disconnected minimal enabling are identified via the labelling).

We finally remark that a possibility for recovering a notion of causality based on prime event structures also for rule-based formalisms with fusions is to introduce suitable restrictions on the concurrent applicability of rules. Indeed, the lack of stability seems to arise essentially from considering as concurrent those fusions that act on common items. Preventing fusions to act on already merged items, one may lose some concurrency, yet gaining a definite notion of causality. Technically, a prime event structure can be obtained for left-linear grammars by restricting the applicability condition: the match must be such that the pair $\langle l; m^L, r\rangle$ of Fig. 6 is jointly mono. This essentially means that those items that have been already fused, should not be fused again. This is indeed the proposal advanced in [27]. Concerning our running example, this requirement would forbid the reachability of graphs G_{ab} and G_c in Fig. 2(b), and in turn this would imply that the domain of configurations is the one depicted in Fig. 4, with the limits concerning expressiveness that we already remarked there.

REFERENCES

[1] M. Nielsen, G. Plotkin, and G. Winskel, "Petri Nets, Event Structures and Domains, Part 1," *Theoretical Computer Science*, vol. 13, pp. 85–108, 1981.

[2] P. Baldan, A. Corradini, H. Ehrig, M. Löwe, U. Montanari, and F. Rossi, "Concurrent semantics of algebraic graph transformation systems," in *Handbook of Graph Grammars and Computing by Graph Transformation*, G. Rozenberg, Ed. World Scientific, 1999, vol. III: Concurrency, pp. 107–187.

[3] P. Baldan, "Modelling concurrent computations: from contextual Petri nets to graph grammars," Ph.D. dissertation, University of Pisa, 2000.

[4] G. Schied, "On relating rewriting systems and graph grammars to event structures," in *Dagstuhl Seminar 9301 on Graph Transformations in Computer Science*, ser. LNCS, H.-J. Schneider and H. Ehrig, Eds., vol. 776. Springer, 1994, pp. 326–340.

[5] G. Winskel, "Event structure semantics for CCS and related languages," University of Aarhus, Tech. Rep. DAIMI PB-159, 1983.

[6] D. Varacca and N. Yoshida, "Typed event structures and the linear pi-calculus," *Theoretical Computer Science*, vol. 411, no. 19, pp. 1949–1973, 2010.

[7] R. Bruni, H. C. Melgratti, and U. Montanari, "Event structure semantics for nominal calculi," in *CONCUR 2006*, ser. LNCS, C. Baier and H. Hermanns, Eds., vol. 4137. Springer, 2006, pp. 295–309.

[8] G. Winskel, "Events, causality and symmetry," *Computer Journal*, vol. 54, no. 1, pp. 42–57, 2011.

[9] J. Pichon-Pharabod and P. Sewell, "A concurrency semantics for relaxed atomics that permits optimisation and avoids thin-air executions," in *POPL 2016*, R. Bodík and R. Majumdar, Eds. ACM, 2016, pp. 622–633.

[10] A. Jeffrey and J. Riely, "On thin air reads towards an event structures model of relaxed memory," in *LICS 2016*, M. Grohe, E. Koskinen, and N. Shankar, Eds. ACM, 2016, pp. 759–767.

[11] M. Dumas and L. García-Bañuelos, "Process mining reloaded: Event structures as a unified representation of process models and event logs," in *Petri Nets 2015*, ser. LNCS, R. R. Devillers and A. Valmari, Eds., vol. 9115. Springer, 2015, pp. 33–48.

[12] J. Meseguer, "Conditional rewriting logic as a unified model of concurrency," *Theoretical Computer Science*, vol. 96, no. 1, pp. 73–155, 1992.

[13] A. Corradini, U. Montanari, F. Rossi, H. Ehrig, R. Heckel, and M. Löwe, "Algebraic approaches to graph transformation I: Basic concepts and double pushout approach," in *Handbook of Graph Grammars and Computing by Graph Transformation. Volume 1: Foundations.*, G. Rozenberg, Ed. World Scientific, 1997.

[14] J. Lévy, "Optimal reductions in the lambda-calculus," in *To H.B. Curry, Essays on Combinatory Logic, Lambda Calculus and Formalism*, J. Seldin and J. Hindley, Eds. Academic Press, 1980, pp. 159–191.

[15] G. Winskel, "Event Structures," in *Petri Nets: Applications and Relationships to Other Models of Concurrency*, ser. LNCS, vol. 255. Springer, 1987, pp. 325–392.

[16] S. Crafa, D. Varacca, and N. Yoshida, "Event structure semantics of parallel extrusion in the π-calculus," in *FoSSaCS 2012*, ser. LNCS, L. Birkedal, Ed., vol. 7213. Springer, 2012, pp. 225–239.

[17] F. Gadducci, "Graph rewriting and the π-calculus," *Mathematical Structures in Computer Science*, vol. 17, no. 3, pp. 1–31, 2007.

[18] I. Phillips, I. Ulidowski, and S. Yuen, "Modelling of bonding with processes and events," in *RC 2013*, ser. LNCS, G. W. Dueck and D. M. Miller, Eds., vol. 7948. Springer, 2013, pp. 141–154.

[19] H. Ehrig, "Tutorial introduction to the algebraic approach of graph-grammars," in *TAGT 1986*, ser. LNCS, H. Ehrig, M. Nagl, G. Rozenberg, and A. Rosenfeld, Eds., vol. 291. Springer, 1987, pp. 3–14.

[20] G. Winskel, "Events in computation," Ph.D. dissertation, University of Edinburgh, 1980.

[21] M. Droste, "Event structures and domains," *Theoretical Computer Science*, vol. 68, no. 1, pp. 37–47, 1989.

[22] A. Corradini, H. Ehrig, M. Löwe, U. Montanari, and F. Rossi, "An event structure semantics for graph grammars with parallel productions," in *TAGT 1994*, ser. LNCS, J. Cuny, H. Ehrig, G. Engels, and G. Rozenberg, Eds., vol. 1073. Springer, 1996.

[23] A. Corradini, U. Montanari, and F. Rossi, "Graph processes," *Fundamenta Informaticae*, vol. 26, no. 3/4, pp. 241–265, 1996.

[24] P. Degano, J. Meseguer, and U. Montanari, "Axiomatizing the algebra of net computations and processes," *Acta Informatica*, vol. 33, no. 7, pp. 641–647, 1996.

[25] G. Kahn and G. Plotkin, "Concrete domains," *Theoretical Computer Science*, vol. 121, no. 1, pp. 187–277, 1993, based on [28].

[26] R. van Glabbeek and G. Plotkin, "Configuration structures, event structures and Petri nets," *Theoretical Computer Science*, vol. 410, no. 41, pp. 4111–4159, 2009.

[27] P. Baldan, F. Gadducci, and U. Montanari, "Concurrent rewriting for graphs with equivalences," in *CONCUR 2006*, ser. LNCS, C. Baier and H. Hermanns, Eds., vol. 4137. Springer, 2006, pp. 279–294.

[28] G. Kahn and G. Plotkin, "Domaines concretes," INRIA Paris, Tech. Rep. 336, 1978.

APPENDIX

A. Es with non-binary conflict

In the literature also es with non-binary conflict have been considered, where the binary conflict relation is replaced by a consistency predicate. The duality results of Section III easily adapt to this case.

Definition 30 (es with non-binary conflict) An *es with non-binary conflict* (*esn* for short) is a tuple $\langle E, \vdash, Con\rangle$ such that

- E is a set of events
- $Con \subseteq \mathbf{2}^E_{fin}$ is the consistency predicate, satisfying $X \in Con$ and $Y \subseteq X$ implies $Y \in Con$
- $\vdash \subseteq Con \times E$ is the *enabling* relation, satisfying $X \vdash e$ and $X \subseteq Y \in Con$ implies $Y \vdash e$.

The esn E is *stable* if whenever $X \vdash e$, $Y \vdash e$, $X \cup Y \cup \{e\} \in Con$ then $X \cap Y \vdash e$.

The definition of the category of esn is changed accordingly.

Definition 31 (category of esn) A morphism of esn $f : E_1 \to E_2$ is a partial function $f : E_1 \to E_2$ such that

- if $X_1 \in Con_1$ then $f(X) \in Con_2$;
- if $\{e_1, e_1'\} \in Con_1$ and $f(e_1) = f(e_1')$ then $e_1 = e_1'$;
- if $X_1 \vdash_1 e_1$ and $f(e_1)$ defined then $f(X_1). \vdash_2 f(e_1)$

We denote by $\mathsf{cES_n}$ the category of esn and esn morphisms.

Then in the definition of domains (Definition 6), the existence of joins is required only for consistent subsets (not for pairwise consistent).

Definition 32 (b-domains) A *bounded complete domain (b-domain)* is an algebraic finitary partial order such that any $X \subseteq D$ consistent admits a joint $\bigsqcup X$. B-domain morphisms are as in Definition 13. We denote by $\mathsf{Dom_b}$ the corresponding category.

Note that any domain is a b-domain.

Lemma 10 that shows a form of transitivity for interchangeability ceases to hold in the current formulation (an counterexample can be found in Fig. 12), but it suffices to strengthen the hypotheses by asking that i, i', i'' are consistent and the proof goes through again. The definition of weak prime algebraic domain remains formally the same, but the underlying partial order is required to be a b-domain.

Definition 33 (weak prime algebraic b-domain) A *weak prime algebraic b-domain* (or simply *weak prime b-domain*) is a b-domain D which is weak prime algebraic. We denote by $\mathsf{Dom_b}$ the corresponding category.

The proof of the fact that, given an esn E, the ideal completion of the partial order of configurations $\mathcal{D}(E) = \mathsf{Idl}(\langle Conf(E), \subseteq \rangle)$ is a weak prime b-domain, is unchanged. The same holds for the fact that if $f : D_1 \to D_2$ is a weak prime b-domain morphism then $\mathcal{D}(f) : \mathcal{D}(E_1) \to \mathcal{D}(E_2)$ is a weak prime b-domain morphism.

Vice versa the esn associated with a weak prime b-domain is defined as follows.

Definition 34 (esn for a weak prime b-domain) Let D be a weak prime domain. The corresponding event structure $\mathcal{E}(D) = \langle E, Con, \vdash \rangle$ is defined as follows

- $E = [ir(D)]_{\leftrightarrow^*}$;
- $Con = \{X \mid \exists d \in \mathsf{K}(D) \,.\, X \subseteq [ir(d)]_{\leftrightarrow^*}\}$;
- $X \vdash e$ when there is $i \in e$ such that $[ir(i) - \{i\}]_{\leftrightarrow^*} \subseteq X$.

Given a morphism $f : D_1 \to D_2$, its image $\mathcal{E}(f) : \mathcal{E}(D_1) \to \mathcal{E}(D_2)$ is defined as follows: for $[i_1]_{\leftrightarrow^*} \in E$, if $f(p(i_1)) = f(i_1)$, then $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = \bot$, else $\mathcal{E}(f)([i_1]_{\leftrightarrow^*}) = [i_2]_{\leftrightarrow^*}$, where $i_2 \in \delta(f(i_1), f(p(i_1)))$ is minimal in the set.

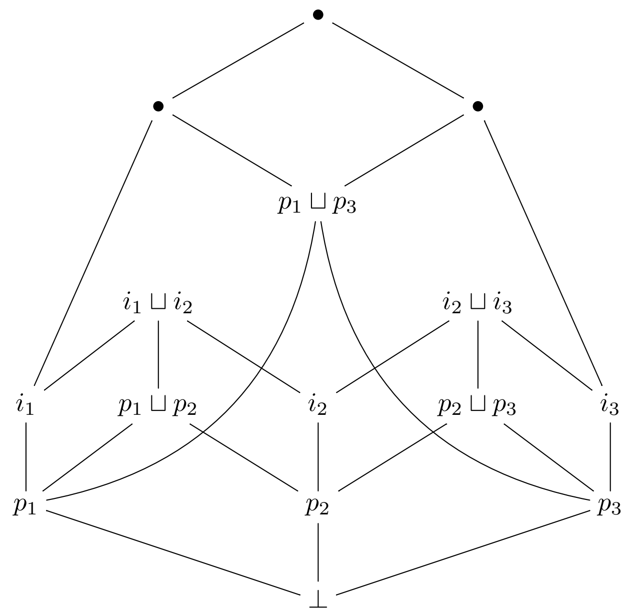


Fig. 12: A counterexample to Lemma 10: a weak prime algebraic domains where $i_1 \leftrightarrow i_2$, $i_2 \leftrightarrow i_3$, and i_2, i_3 are consistent but $i_2 \not\leftrightarrow i_3$.

We then get a result corresponding to Theorem 1 for es with non-binary conflict and weak prime b-domains.

Theorem 5 (corecflection of $\mathsf{ES_n}$ and $\mathsf{wDom_b}$) *The functors $\mathcal{D} : \mathsf{ES_n} \to \mathsf{wDom_b}$ and $\mathcal{E} : \mathsf{wDom_b} \to \mathsf{ES_n}$ form a coreflection. It restricts to an equivalence between $\mathsf{wDom_b}$ and $\mathsf{cES_n}$.*