A Characterization of Distance between 1-Bounded Compact Ultrametric Spaces through a Universal Space

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Abstract

The category of 1-bounded compact ultrametric spaces and non-distance increasing functions (KUM's) have been extensively used in the semantics of concurrent programming languages. In this paper a universal space U for KUM's is introduced, such that each KUM can be isometrically embedded in it. U consists of a suitable subset of the space of functions from $[0,1)$ to $I\!N$, endowed with a "prefix-based" ultrametric. U allows to characterize the distance between KUM's in terms of the Hausdorff distance between its compact subsets. As applications, it is proved how to derive the existence of limits for Cauchy towers of spaces without using the classical categorical construction and how to find solutions of recursive domain equations inside $\mathcal{P}_{nco}(U)$.

1 Introduction

In the recent past metric spaces have often been used successfully in the semantics of concurrent programming languages. Since [3], where the technique of [12] for solving domain equations is adapted to the metric context, several categories of metric spaces have been introduced in the literature. Apart from technical differences, all the approaches follow a common pattern which guarantees the existence of categorical limits that provide solutions of recursive equations. We give an outline of this pattern.

1. Given a category $\mathcal{C}$, a new category $\mathcal{C}'$ is introduced, which has the same objects as $\mathcal{C}$ and whose morphisms from X to Y are pairs $\langle f,g\rangle$ of morphisms in $\mathcal{C}$, $f : X \to Y$, $g : Y \to X$ which satisfy suitable conditions.

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The pairs play the same role as *embedding-projection* pairs in the order-theoretic approach.

2. Differently from the order-theoretic approach, a number, $\delta(\langle f, g\rangle)$, is associated with each morphism $\langle f, g\rangle : X \to Y$ in $\mathcal{C}'$, which roughly speaking expresses the similarity between X and Y when comparing them via f and g.

3. These numbers allow to introduce the notion of *Cauchy towers* of spaces (a sequence $(X_n, \langle f_n, g_n\rangle)_{n\in I\!N}$ is Cauchy if for each $\epsilon > 0$ the δ's of compositions of morphisms are eventually less than ϵ) and it is proved that each Cauchy tower has a categorical limit.

4. Classes of functors (*contracting* [1, 2, 3, 13, 7] *cut-contracting* [8], *hom-contracting* [3], *locally contracting* [13, 11]) are singled out that generate Cauchy towers when iteratively applied to an initial space. This allows to solve those domain equations which involve such functors.

An important remark is that all the categories considered in the cited papers have complete or compact metric spaces as objects. Since they differ essentially in morphisms, the common pattern suggests the possibility of finding solutions to domain equations independently from the particular choice of morphisms in the category. This idea is developed in [2], where it is shown that in the compact case it is possible to get rid of the categorical setting, work in the *class* of compact metric spaces and there solve domain equations. The key idea consists in the introduction of a mapping $\Delta : \mathcal{K} \times \mathcal{K} \to [0, 1]$, where $\mathcal{K}$ is the class of compact metric spaces, which turns out to satisfy the metric axioms (provided that one works up to isometry). Since $\mathcal{K}$ is complete in the usual sense of Cauchy sequences convergence, it is possible to obtain a generalized version of the Banach-Caccioppoli's theorem on fixed points of contractions, stating that each (functorial or non-functorial) operator $F : \mathcal{K} \to \mathcal{K}$ which is contracting with respect to Δ has a unique (up to isometry) fixed point, i.e. there exists an essentially unique compact metric space X such that $X \simeq F(X)$. Since the domain constructors involved in metric domain equations in the various categories of compact metric spaces are used in such a way to define contractions on $\mathcal{K}$, the "non-functorial" fixed point result can be thought of as a generalization of the categorical ones.

In this paper we give a characterization of the metric Δ in the case of 1-bounded compact ultrametric spaces (KUM's), relating it to the Hausdorff distance d_H between compact subsets of a suitable universal space U. KUM's are considered because they are the most common framework for metric semantics.

The results of this paper can be summarized as follows. We introduce the space U and show that it is universal in the sense that each KUM can be isometrically embedded in it. A characterization of compact subsets of U is given, and it is proved that U is isometric to the space of its nonempty compact subsets endowed with the Hausdorff distance. Then we prove that $\Delta(X, Y)$ is the infimum of $d_H(i(X), i'(Y))$ computed over all possible isometric embeddings $i : X \to U$, $i' : Y \to U$.

One may wonder whether our construction generalizes to more general categories. Unfortunately this seems not to be the case. We will clarify this point at the end of Section 4.

Finally two applications of our results are presented. In the first one we show how to derive the existence of limits for Cauchy towers of KUM's without using the classical categorical construction. In the second one, following [6], we find solutions of recursive domain equations inside $\mathcal{P}_{nco}(U)$ by defining a suitable pseudo-ultrametric on it. This last application brings as a consequence the possibility of carrying out semantics in a set-theoretic framework, alternative to that of *hyperuniverses* of [4].

2 Mathematical Preliminaries

We start with recalling some standard notions and definitions (see e.g. [10]). A *metric space* is a pair (X, d) (X for short) where X is a set and $d : X \times X \to [0, \infty)$ is a mapping, called *metric*, which satisfies, for all x, y and z in X:

1. $d(x, x) = 0$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(y, z)$,
4. $d(x, y) = 0 \;\Rightarrow\; x = y$.

$B(x, r)$, where $x \in X$ and $r > 0$, denotes the open ball with centre x and radius r, i.e. the set $\{y \in X \mid d(x, y) < r\}$. If the range of d is in $[0, 1]$, X is called a 1-*bounded* metric space. If d satisfies, instead of the third condition above, the stronger one $d(x, y) \leq max\{d(x, z), d(z, y)\}$, then X is called an *ultrametric* space. A sequence $(x_n)_{n \in I\!N}$ is *Cauchy* if $\forall \epsilon > 0. \exists m. \forall n, p \geq m. d(x_n, x_p) \leq \epsilon$. X is *complete* if each Cauchy sequence $(x_n)_{n \in I\!N}$ converges to a point $\lim_n x_n$ in X. X is *compact* if for each sequence in X there exists a subsequence converging to a point of X.

In the paper we deal with compact ultrametric spaces with 1-bounded distance (KUM's). In the following X, Y will always denote KUM's.

A mapping $f : X \to Y$ is *non-distance increasing* (NDI) if for all x, x' in X $d_Y(f(x), f(x')) \leq d_X(x, x')$. The space $[X \to Y]$ of all non-distance increasing functions is endowed with the metric $d(f, g) = sup\{d_Y(f(x), g(x)) \mid x \in X\}$. $([X \to Y], d)$ is a KUM if X and Y are (see e.g. [13]).

Pairs of non-distance increasing functions (NDI pairs) provide a tool for defining a distance between KUM's. More precisely, given a pair of NDI functions $f : X \to Y$, $g : Y \to X$, the number

$$\delta(\langle f, g\rangle) =^{\text{def}} max\{d(Id_X, g \circ f), d(Id_Y, f \circ g)\},$$

is a measure of the quality with which X approximates Y, and vice-versa, via $\langle f, g\rangle$. Hence

$$\Delta(X, Y) =^{\text{def}} min\{\delta(\langle f, g\rangle) \mid \langle f, g\rangle \text{ NDI-pair between } X \text{ and } Y\}$$

($min\ \emptyset$ is set equal to 1) expresses the degree to which the spaces mutually approximate each other. Notice that in the definition of Δ the existence of the minimum is guaranteed by the compactness of $[X \to Y]$ and $[Y \to X]$. The mapping $\Delta : C \times C \to [0, 1]$, where C is a suitable class of metric spaces, is studied in details in [2]. In particular, (working up to isometry) if C is the class of compact [complete] 1-bounded (ultra)metric spaces, then Δ satisfies the axioms for a metric [pseudo-metric, i.e. the fourth condition in the definition of metric is dropped] and (C, Δ) is complete w.r.t. Δ, in the usual sense that each Cauchy sequence of metric spaces has a limit. Moreover, in the 1-bounded compact case, if $F : C \to C$ is a contraction then there exists a unique (up to isometry) X in C such that $X \simeq F(X)$.

In order to characterize Δ we recall the notion of Hausdorff distance. Let $\mathcal{P}_{nco}(X)$ denote the family of nonempty compact subsets of X. For all A, B in $\mathcal{P}_{nco}(X)$ we define

$$d_H(A, B) = max\{max_{x\in A}\{d(x, B)\}, max_{y\in B}\{d(y, A)\}\},$$

where $d(x, B) = min\{d(x, y) \mid y \in B\}$ and $d(y, A)$ is defined similarly[1].

$d_H(A, B)$ can be characterized as the smallest value r such that

$$\forall x \in A.\exists y \in B.d(x, y) \leq r \ \wedge\ \forall y \in B.\exists x \in A.d(x, y) \leq r.$$

We recall (see e.g. [13]) that $(\mathcal{P}_{nco}(X), d_H)$ is compact if (X, d) is so. The next lemma gives a characterization of the Hausdorff metric for KUM's.

Lemma 2.1 *Let (X, d) be a KUM. For all $A, B \in (\mathcal{P}_{nco}(X), d_H)$ and $r > 0$,*

$$d_H(A, B) < r \Leftrightarrow A[r] = B[r],$$

where $A[r] = \bigcup\{B(x, r) \mid x \in A\}$.

Proof. We prove ($\Rightarrow$) by showing $A \subseteq B$. Let $x \in A[r]$. Then there exists $a \in A$ such that $d(a, x) < r$. Since $d_H(A, B) < r$ there exists $b \in B$ such that $d(a, b) < r$. Because $d(x, b) \leq max\{d(a, b), d(a, x)\}$, we can conclude $x \in B[r]$.

($\Leftarrow$) Let $A[r] = B[r]$ and $a \in A$. Clearly, $a \in B[r]$. Hence there exists $b \in B$ such that $d(a, b) < r$. Similarly, for each $b \in B$ there exists $a \in A$ such that $d(b, a) < r$. Therefore $d_H(A, B) < r$. □

[1] We can define Hausdorff distance by using the *max* and *min* instead of the standard *sup* and *inf* since we are dealing with compact spaces.

Corollary 2.2 *For all* $A, B \in \mathcal{P}_{nco}(X)$, $d_H(A, B) = inf\{r \mid A[r] = B[r]\}$.

We now give some properties of KUM's. They will be useful for proving the existence of isometric embeddings from KUM's into the universal space U which we will introduce later on.

For any $r > 0$, let $\mathcal{B}_r(X)$ denote the set $\{B(x, s) \mid r \leq s, x \in X\}$. For each KUM X and $r > 0$, fix a subset $C_r(X) \subseteq X$ such that:

1. $\forall c, c' \in C_r(X).d(c, c') \geq r \ (c \neq c')$;
2. $\forall x \in X.\exists c_x^r \in C_r(X).d(x, c_x^r) < r$.

Lemma 2.3 *For all* $0 < r \leq 1$, $C_r(X)$ *and* $\mathcal{B}_r(X)$ *are finite sets.*

Proof. Finiteness of $C_r(X)$ follows immediately from the fact that a metric space is compact if and only if it is complete and totally bounded (see e.g. [5]), hence $C_r(X)$ can be obtained by choosing one point in each open ball of a finite minimal covering of X with balls of radius r. As regards $\mathcal{B}_r(X)$, consider that for each $s \geq r$, $B(x, s) = \bigcup\{B(c, r) \mid d(c, x) < s \ \& \ c \in C_r(X)\}$ and $C_r(X)$ is finite, as we have just proved. Therefore $\mathcal{B}_r(X)$ is finite. $\square$

The following corollary is an immediate consequence of the last lemma.

Corollary 2.4 *(i) For each* $0 < r \leq 1$ *let* $D_r(X) = \{s \mid r \leq s \ \& \ \exists x, x' \in X.d(x, x') = s\}$. *Then* $D_r(X)$ *is finite.*
(ii) If X *is infinite then the elements of* $D_0(X) =_{def} \bigcup_{r>0} D_r(X)$ *form a sequence in* $(0, 1]$ *decreasing to* 0.

3 The Universal Space

In this section we introduce a *universal space* for KUM's. We characterize compact subsets of U and show that U is isometric to $\mathcal{P}_{nco}(U)$. Finally we prove the embedding result, namely that each KUM can be isometrically embedded in U.

We fix some notations. Given r and s such that $0 < s < r \leq 1$ and $f : [0, 1 - s) \to I\!N$, $f_{[r]}$ denotes the restriction of f to the interval $[0, 1 - r]$, and $N(f) = \{x \mid f(x) \neq 0\}$.

Here is the definition of the universal space.

Definition 3.1 *Let* $U =_{def} \{f : [0, 1) \to I\!N \mid \forall r > 0.N(f_{[r]})$ *is finite*$\}$[2], *equipped with the distance* $\hat{d}(f, g) = 1 - min\{x \in [0, 1) \mid f(x) \neq g(x)\}$.

[2] $I\!N$ can be replaced by any pointed countable set, i.e. a countable set with a distinguished element. In the present case the distinguished element is 0.

We introduce some further notations. For each $r > 0$, $X \subseteq U$, $X_{[r]} =_{def} \{f_{[r]} \mid f \in X\}$. The following equivalences, which hold for any $f, g \in U$, $X \subseteq U$, show how the operators $(\cdot)_{[r]}$ are related to the topology of U.

$$f_{[r]} = g_{[r]} \Leftrightarrow f \in B(g,r),$$
$$f_{[r]} \in X_{[r]} \Leftrightarrow f \in X[r].$$

Let $0 < s \leq r \leq 1$. If $f \in U_{[r]}, g \in U_{[s]}$, we write $f \sqsubseteq g$ when $f = g_{[r]}$. If moreover $g(t) = 0$ for each t such that $1 - r < t < 1 - s$, we write $f \sqsubseteq^* g$. If $A \subseteq^{\text{fne}} U_{[r]}$, $B \subseteq^{\text{fne}} U_{[s]}$ (i.e. they are finite nonempty subsets), we write $A \sqsubseteq^* B$ if the following two conditions are satisfied:
- $\forall f \in A. \exists g \in B. f \sqsubseteq^* g$;
- $\forall g \in B. \exists f \in A. f \sqsubseteq^* g$.[3]

$0 \sqsubseteq^* B$ abbreviates "$\forall f \in B. \forall t \in [0, 1-s). f(t) = 0$".

Let K be either $I\!N$ or a initial segment of $I\!N$, and $(r_k)_{k \in K}$ be a decreasing sequence of elements in $(0,1]$, converging to 0 if $K = I\!N$. Suppose, for each $k \in K$, $f_k \in U_{[r_k]}$ and $f_k \sqsubseteq f_{k+1}$.
- If $K = I\!N$, $\bigsqcup_{k \in K} f_k$ denotes the unique element g of U such that $\forall s. g(s) = f_{k'}(s)$, where k' is any index such that $1 - r_{k'} \geq s$.
- If $K = \{0, 1, \ldots, i\}$, $\bigsqcup_{k \in K} f_k$ denotes the unique element g of U such that $g(s) = f_i(s)$, if $s \leq 1 - r_i$, otherwise $g(s) = 0$.
- If $K = \emptyset$, then $\bigsqcup_{k \in K} f_k = \lambda t.0$.

In the following, K will always denote either $I\!N$ or some initial segment of $I\!N$. In order to keep notation uniform, if $K = \{0, 1, \ldots, i\}$, $\lim_{k \in K} x_k$ stands for x_i. If we write $(r_k)_k$ it is intended that k ranges over $I\!N$.

Lemma 3.2 *$(U, \hat{d})$ is a complete ultrametric space.*

Proof. The proof that $\hat{d}$ is an ultrametric easily follows from the equivalence $\hat{d}(f,g) < r \Leftrightarrow f_{[r]} = g_{[r]}$. As for completeness, let $(f_n)_n$ a Cauchy sequence in U. Fix a decreasing sequence to 0, say $(r_k)_k$. Then $\forall k. \exists n_k. \forall n, m \geq n_k. (f_n)_{[r_k]} = (f_m)_{[r_k]}$. It is not restrictive to suppose $\forall k. n_k \leq n_{k+1}$. Hence $(f_{n_k})_{[r_k]} \sqsubseteq (f_{n_{(k+1)}})_{[r_{(k+1)}]}$ and thus we can define $\bar{f} = \bigsqcup_k (f_{n_k})_{[r_k]}$. $\bar{f}$ is an element of U and is the limit of $(f_n)_n$, since $\hat{d}(\bar{f}, f_n) \leq r_k$ if $n \geq n_k$. □

Before showing that each KUM can be isometrically embedded in U, we focus on the characterization of compact subsets of U, and show that U is isometric to $\mathcal{P}_{nco}(U)$. This digression seems useful for several reasons.

Firstly, compact subsets of U are the ranges of isometric embeddings $i : X \to U$, X being any KUM.

Secondly, the result of isometry between U and $\mathcal{P}_{nco}(U)$ is interesting since, as shown in the second application, it is possible to develop in $\mathcal{P}_{nco}(U)$ a set-theoretic approach to domain equations alternative to that provided by *hyper-universes* in [4].

[3]The definition of "$\sqsubseteq^*$" corresponds to that of the *Egli-Milner* preorder over nonempty finite subsets of compact elements in ω-algebraic cpo's.

Thirdly, characterization of $\mathcal{P}_{nco}(U)$ casts light on the structure of U and provides the proof of Theorem 4.3 with some intuition.

Let us fix some notation. Let $X \subseteq U$ be any subset of U. Then, for any $r > 0$, define:
- $X(1-r) = \{f(1-r) \mid f \in X\}$;
- $D^U_r(X) = \{s \mid s \geq r \ \& \ X(1-s) \supset \{0\}\}$;
- $D^U_0(X) = \bigcup_{r>0} D^U_r(X)$. Notice that for each $r \in D^U_0(X)$, the set $X(1-r)$ is nonempty.

Proposition 3.3 *$X \subseteq U$ is compact if and only if the following three conditions are satisfied:*

1. *$\forall r > 0. D^U_r(X)$ is finite;*
2. *$\forall r \in D^U_0(X). X(1-r)$ is finite;*
3. *for each sequence $(r_k)_k$ decreasing to 0 and $g_k \in U_{[r_k]}$ satisfying $g_k \sqsubseteq g_{k+1}$ it holds:*
   $(*)$ $(\forall k. \exists f_k \in X. g_k \sqsubseteq f_k) \Rightarrow \bigsqcup_k g_k \in X$.

The proof of the proposition above follows immediately from the following two lemmata.

Lemma 3.4 *$X \subseteq U$ is totally bounded if and only if conditions 1 and 2 of Proposition 3.3 hold.*

Proof. ($\Rightarrow$) Consider the covering $\mathcal{C} = \{B(f,r) \mid f \in X\}$ of X. Since X is totally bounded, we can extract a finite subcovering $\mathcal{C}' = \{B(f_i, r) \mid i \in I\}$. We have $\forall i \in I. N((f_i)_{[r]})$ is finite. This fact immediately implies 3.3.1 and 3.3.2.

($\Leftarrow$) Let $\mathcal{C} = \{B(g,r) \mid g \in Y\}$ any covering of X. Let $r' < r$. Define $H = \prod_{s \in D^U_{r'}(X)} X(1-s)$. From the hypotheses H is finite. Consider $H' = \{h \in H \mid \exists f \in X. \forall s > r'. f(1-s) = h_s\}$. We have $f \in X \Rightarrow (\exists h \in H'. \forall s \geq r. f(1-s) = h_s)$. We get a finite subcovering $\mathcal{C}' \subseteq \mathcal{C}$ of X by choosing, for each $h \in H'$, any $g \in Y$ such that $g(1-s) = h_s$, for each $s \in D^U_r(X)$. $\square$

Lemma 3.5 *$X \subseteq U$ is closed if and only if condition 3 of Proposition 3.3 holds.*

Proof. ($\Rightarrow$) If the premise of $(*)$ holds, then $(f_k)_k$ is a sequence in X converging to $\bigsqcup_k g_k$. Since X is a closed subset, $\bigsqcup_k g_k \in X$.

($\Leftarrow$) X is closed if whichever converging sequence in X, say $(f_k)_k$, has its limit $\bar{f}$ in X. Let $r_k = \hat{d}(f_k, \bar{f})$. It is not restrictive to suppose that $(r_k)_k$ is a decreasing sequence. For each k define $g_k = (f_k)_{[r_k]}$. The hypotheses of $(*)$ are satisfied and therefore $\bigsqcup_k g_k = \bar{f}$ belongs to X. $\square$

We give now a second characterization of compact subsets of U, which is inspired by that of the Plotkin powerdomain in [9]. Consider the set $\mathcal{E}$ consisting

of all the sequences of pairs $\langle r_k, A_k \rangle_{k\in K}$ (K may be empty) such that:
- $(r_k)_{k\in K}$ is a decreasing sequence in $(0,1]$, converging to 0 whenever $K = I\!N$;
- for each k, $A_k \subseteq^{\text{fne}} U_{[r_k]}$, $0 \sqsubseteq^* A_0$ and $A_k \sqsubseteq^* A_{k+1}$ hold (therefore for each s in $[0, 1-r_k]$, if $f \in A_k$ and $f(s) \neq 0$ then $s = 1 - r_{k'}$ for some $k' \leq k$).

We have the following result, which says that $\mathcal{E}$ is in one-to-one correspondence with compact subsets of U.

Proposition 3.6 *There is a bijection ϕ from $\mathcal{P}_{nco}(U)$ to $\mathcal{E}$.*

Proof. For each $X \in \mathcal{P}_{nco}(U)$ define $\phi(X) = \langle r_k, A_k \rangle_{k\in K}$, where $(r_k)_{k\in K}$ is the sequence consisting of the elements of $D_0^U(X)$ (if X is finite so is K, otherwise $K = I\!N$) and for each $k \in K$, $A_k = X_{[r_k]}$. ¿From condition 1 and 2 of Proposition 3.3 it follows that $(r_k)_{k\in K}$ converges to 0 whenever $D_0^U(X)$ is infinite and that each A_k is nonempty and finite. Trivially $0 \sqsubseteq^* A_0$ and $A_k \sqsubseteq^* A_{k+1}$. Hence $\phi : \mathcal{P}_{nco}(U) \to \mathcal{E}$ is well-defined (notice that $\phi(\{\lambda t.0\})$ is the empty sequence)

For each $e = \langle r_k, A_k \rangle_{k\in K} \in \mathcal{E}$ define $\psi(e)$ as the set of all $f \in U$ such that

$$\exists (g_k)_{k\in K}.(\forall k. g_k \in A_k) \ \&\ g_k \sqsubseteq g_{k+1} \ \&\ \bigsqcup_{k\in K} g_k = f.$$

Notice that ψ maps the empty sequence to $\lambda t.0$. We prove that $\psi(e)$ is compact. Definition of $\psi(e)$ and finiteness of A_k immediately imply 1 and 2 of Proposition 3.3. Thus $\psi(e)$ is totally bounded. When K is finite $\psi(e)$ is trivially closed since it is finite. Consider the case $K = I\!N$. Let $(f_p)_p$ be a sequence of elements in $\psi(e)$, converging to $\bar{f}$. We have to prove $\bar{f} \in \psi(e)$. ¿From a Koenig's Lemma argument the following conditions are equivalent:
- $\forall k \in I\!N. \exists g_k \in A_k. g_k \sqsubseteq g_{k+1} \ \&\ \bigsqcup_k g_k = f$;
- $\forall k \in I\!N. \exists g_k \in A_k. g_k \sqsubseteq f$. (†)

Fix any $k \in I\!N$. Let p be such that $\hat{d}(\bar{f}, f_p) \leq r_k$ and say $f_p = \bigsqcup_{k'} g'_{k'}$. Then $g'_k \sqsubseteq \bar{f}$. Therefore (†) is satisfied and we conclude that $\psi(e)$ is closed.

Finally a routine check shows that $\phi \circ \psi(e) = e$ and $\psi \circ \phi(X) = X$. □

We now prove that U is isometric to $\mathcal{P}_{nco}(U)$. For each $0 < s < r \leq 1$, $A \subseteq^{\text{fne}} U_{[r]}$, fix bijections

$$\alpha_{A,s} : \{B \subseteq^{\text{fne}} U_{[s]} \mid A \sqsubseteq^* B\} \to I\!N,$$
$$\alpha_{0,s} : \{B \subseteq^{\text{fne}} U_{[s]} \mid 0 \sqsubseteq^* B\} \to I\!N.$$

Let now $e = \langle r_k, A_k \rangle_{k\in K} \in \mathcal{E}$. We define $\nu(e) \in U$ as $\bigsqcup_{k\in K} f_k$, where f_k are inductively defined as follows:

$$f_0(t) = \begin{cases} 0 & \text{if } t < 1 - r_0, \\ \alpha_{0,r_0}(A_0) & \text{if } t = 1 - r_0. \end{cases}$$

$$f_{k+1}(t) = \begin{cases} f_k(t) & \text{if } t \leq 1 - r_k, \\ 0 & \text{if } 1 - r_k < t < 1 - r_{k+1}, \\ \alpha_{A_k, r_{k+1}}(A_{k+1}) & \text{if } t = 1 - r_{k+1}. \end{cases}$$

Now define $\sigma = \nu \circ \phi : \mathcal{P}_{nco}(U) \to U$. σ is the required isometry. Before proving this, we need a lemma.

Lemma 3.7 *Let $X, Y \in \mathcal{P}_{nco}(U)$, $\phi(X) = \langle r_k, A_k\rangle_{k\in K}$, $\phi(Y) = \langle s_k, B_k\rangle_{k\in H}$. For $r > 0$ let $p = max\{k \in K \mid r_k \geq r\}$, $q = max\{k \in H \mid s_k \geq r\}$. Then the following conditions are equivalent:*

1. $X_{[r]} = Y_{[r]}$;
2. $\hat{d}_H(X, Y) < r$;
3. $p = q$ & $\forall k \leq p. r_k = s_k$ & $A_k = B_k$;

Proof. The equivalence between 1 and 2 follows from $X_{[r]} = Y_{[r]} \Leftrightarrow X[r] = Y[r]$ and Corollary 2.1. In order to prove 1 $\Rightarrow$ 3, consider that $X_{[r]} = Y_{[r]}$ implies both $D^U_r(X) = D^U_r(Y)$ and $\forall t \geq r. X_{[t]} = Y_{[t]}$. These two conditions clearly imply the three of 3. Hence 1 $\Rightarrow$ 3. In order to show 3 $\Rightarrow$ 1, we prove that $\forall f \in X. \exists g \in Y. f_{[r]} = g_{[r]}$. Since $A_p = B_p$ there exists $g \in Y$ such that $f_{[r_p]} = g_{[r_p]}$. Since, in the case $p+1 \in K$, we have $r_{p+1} > r, s_{p+1} > r$ and $f(t) = g(t) = 0$ for each $t \in (1 - r_p, 1 - r]$, we get $f_{[r]} = g_{[r]}$. Similarly one proves that $\forall g \in Y. \exists f \in X. f_{[r]} = g_{[r]}$. □

Theorem 3.8 *σ is an isometry between $(\mathcal{P}_{nco}(U), \hat{d}_H)$ and $(U, \hat{d})$.*

Proof. It is easy to show that σ is a surjection. In fact arrange $N(f)$ into a decreasing sequence $(r_k)_{k\in K}$. Then define the sequence $(A_k)_{k\in K}$ by induction on k:

$$\begin{array}{rcl} A_0 & = & \alpha^{-1}_{0,r_0}(f(r_0)), \\ A_{k+1} & = & \alpha^{-1}_{A_k, r_{k+1}}(f(r_{k+1})). \end{array}$$

We have $A_k \sqsubseteq^* A_{k+1}$, hence $e = \langle r_k, A_k\rangle_{k\in K} \in \mathcal{E}$. By definition it follows $\nu(e) = f$. Therefore the compact subset $X \subseteq U$ defined as $X = \psi(e)$ satisfies $\sigma(X) = f$. We now prove that σ preserves distances. Let $X, Y \in \mathcal{P}_{nco}(U)$, $r > \hat{d}_H(X, Y)$ and $\phi(X) = \langle r_k, A_k\rangle_{k\in K}$, $\phi(Y) = \langle s_k, B_k\rangle_{k\in K'}$. By Lemma 2.1 it follows $X[r] = Y[r]$, hence $X_{[r]} = Y_{[r]}$. Define $p = max\{k \in K \mid r_k \geq r\}$. Then the thesis of Lemma 3.7 ensures $r_k = s_k$ and $A_k = B_k$ for any $k \leq p$. By definition of σ it follows $\sigma(X)_{[r]} = \sigma(Y)_{[r]}$ and therefore $\hat{d}(\sigma(X), \sigma(Y)) < r$, hence $\hat{d}(\sigma(X), \sigma(Y)) \leq \hat{d}_H(X, Y)$.
Let now $\hat{d}(\sigma(X), \sigma(Y)) < r$, for some r. This is equivalent to $\sigma(X)_{[r]} = \sigma(Y)_{[r]}$, which implies (since $\langle r_k, A_k\rangle_{k\in K} = \nu^{-1}(\sigma(X)), \langle s_k, B_k\rangle_{k\in K'} = \nu^{-1}(\sigma(Y))$) $r_k = s_k, A_k = B_k$ for any k such that $r_k \geq r$. This implies $\hat{d}_H(X, Y) < r$ by Lemma 3.7. Thus $\hat{d}_H(X, Y) \leq \hat{d}(\sigma(X), \sigma(Y))$ and we conclude. □

This section ends with the proof of the embedding result.

Theorem 3.9 *Let X be a KUM. Then there exist isometric embeddings $i : X \to U$.*

Proof. Arrange elements of $D_0(X)$ into a decreasing sequence $(r_k)_{k\in K}$. We define injections $\rho_k : C_{r_k}(X) \to U_{[r_k]}$ inductively on k, as follows:

(a) ρ_0 is any injection such that $\forall c \in C_{r_0}(X).0 \sqsubseteq^* \rho_0(c)$;

(b) ρ_{k+1} is any injection such that $\forall c \in C_{r_k}(X), c' \in C_{r_{k+1}}(X).d_X(c,c') < r_k \Rightarrow \rho_k(c) \sqsubseteq^* \rho_{k+1}(c')$.

Given $x \in X$, we have $x = \lim_{k\in K} c_x^{r_k}$. We define $i(x) = \bigsqcup_{k\in K} \rho_k(c_x^{r_k})$. i is well-defined (if $K = I\!N$, $(r_k)_{k\in K}$ converges to 0 by Corollary 2.4). The range of i is $\phi^{-1}(\langle r_k, A_k\rangle_{k\in K})$, where $A_k = \rho_k(C_{r_k}(X))$. Notice that each A_k is finite by Lemma 2.3, hence $\langle r_k, A_k\rangle_{k\in K} \in \mathcal{E}$. We state that i is an isometry. In fact let $d_X(x,y) = r_k$. Then, if $k > 0$, $c_x^{r_j} = c_y^{r_j}$ for each $j \in K, j < k$, while $c_x^{r_k} \neq c_y^{r_k}$. By definition of i and ρ_k it follows $i(x)_{[s]} = i(y)_{[s]}$ for each $s > r_k$, and $i(x)(1-r_k) \neq i(y)(1-r_k)$. Therefore $\hat{d}(i(x), i(y)) = r_k$. $\square$

4 The Result

If we consider two isometries $i : X \to U$, $i' : Y \to U$, we can compute the Hausdorff distance between $i(X)$ and $i'(Y)$ as compact subsets of U. The aim of this section is to study the relation between $\Delta(X,Y)$ and such Hausdorff distances. This will lead to the characterization of Δ.

As an application we will show how it is possible to derive the existence of limits for Cauchy towers without reference to the (categorical) limit construction.

We start with a technical result.

Proposition 4.1 *For each* $r \geq 0$, $\Delta(X,Y) < r \Leftrightarrow$
(i) $D_r(X) = D_r(Y)$,
(ii) $\forall s \in D_r(X).\exists g_s : C_s(X) \to C_s(Y)$ *bijection such that*
$\forall s' \in D_s(X), c \in C_s(X), c' \in C_{s'}(X).d(c,c') < s' \Rightarrow d(g_s(c), g_{s'}(c')) < s'$.

Proof: ($\Rightarrow$) We prove first that *(i)* holds.
Let $i : X \to Y$, $j : Y \to X$ such that $\delta(\langle i,j\rangle) < r$. Suppose $D_r(X) \neq D_r(Y)$. Then there exist $x_1, x_2 \in X$ (or $y_1, y_2 \in Y$ etc.) such that $d(x_1,x_2) = s \geq r$, while for all $y, y' \in Y$ $d(y,y') \neq s$. We get the contradiction $s = d(x_1,x_2) \leq max\{d(x_1, ji(x_1)), d(ji(x_1), ji(x_2)), d(x_2, ji(x_2))\} < s$ (notice that $d(ji(x_1), ji(x_2)) < s$ since s is not a value of distance in Y and i, j are NDI-functions). We prove now that *(ii)* holds. Define $g_s : C_s(X) \to C_s(Y)$ as $g_s(c) = c^s_{i(c)}$ (it is the unique point e in $C_s(Y)$ such that $d(i(c), e) < s$). We prove that g_s is a bijection by giving the inverse mapping. For $c' \in C_s(Y)$ let $h_s(c') = c^s_{j(c')}$. Then, for each $c \in C_s(X)$,

$$d(c, h_s g_s(c)) \leq max\{d(c, ji(c)), d(ji(c), jg_s(c)), d(jg_s(c), h_s g_s(c))\} < s$$

. By definition of $C_s(X)$ it follows that $c = h_s g_s(c)$. Analogously, for each $c' \in C_s(Y)$, $c' = g_s h_s(c')$. Thus $h_s = g_s^{-1}$. Let now $s' \geq s, c_1 \in C_s(X), c_2 \in C_{s'}(X)$. Then $d(g_s(c_1), g_{s'}(c_2)) \leq max\{d(g_s(c_1), i(c_1)), d(g_{s'}(c_2), i(c_2)), d(i(c_1), i(c_2))\} < s'$.

($\Leftarrow$) We will prove that *(ii)* is enough to conclude $\Delta(X,Y) < r$ (hence *(ii)* implies *(i)*). First we extend the domain of g_s and h_s to the whole X and Y respectively, by defining $\overline{g}_s(x) = g_s(c_x^s)$, $\overline{h}_s(y) = h_s(c_y^s)$. $\overline{g}_s : X \to C_s(Y)$ and $\overline{h}_s : Y \to C_s(X)$ are NDI functions. In fact, consider $\overline{g}_s$. If $d(x,x') < s$, then $\overline{g}_s(x) = \overline{g}_s(x')$. If $d(x,x') = s' \geq s$, let $t > s'$. Then $d(\overline{g}_s(x), \overline{g}_s(x')) \leq max\{d(g_s(c_x^s), g_t(c_x^t)), d(g_s(c_{x'}^s), g_t(c_{x'}^t))\} < t$ (in the first inequality we use the fact that $c_x^t = c_{x'}^t$), hence $d(\overline{g}_s(x), \overline{g}_s(x')) \leq s'$. The proof that $\overline{h}_s$ is NDI is similar. Now consider $\overline{g}$ and $\overline{h}$. $d(x, \overline{h}_r\overline{g}_r(x)) = d(x, c_x^r) < r$. Similarly $d(y, \overline{g}_r\overline{h}_r(y)) = d(y, c_y^r) < r$. Since we have just proved $\delta(\langle \overline{g}_r, \overline{h}_r \rangle) < r$, it follows $\Delta(X,Y) < r$. □

Remark 4.2 *Notice that for each isometric embedding $i : X \to U$ there exists $\rho_k : C_{r_k}(X) \to U_{[r_k]}$ such that $i(x) = \bigsqcup_{k \in K} \rho_k(c_x^{r_k})$. It is sufficient to define $\rho_k(c) = i(c)_{[r_k]}$ for each $c \in C_{r_k}(X)$.*

We now give the main result by characterizing the distance Δ between KUM's in terms of Hausdorff distance in U.

Theorem 4.3 *Let X, Y be KUM's and $i : X \to U$ any isometric embedding. Then*

$$\Delta(X,Y) = min\{\hat{d}_H(i(X), j(Y)) \mid j : Y \to U \text{ isometric embedding}\}.$$

Proof. Let $(r_k)_{k \in K}$ be the decreasing sequence built on the elements in $D_0(X)$, and let $\rho_k : C_{r_k}(X) \to U_{[r_k]}$ defined as in the previous remark. If $\Delta(X,Y) < r$, then we have, for each $r_k \geq r$, bijections $g_k : C_{r_k}(Y) \to C_{r_k}(X)$ as in Proposition 4.1. We build an isometric embedding $j_r : Y \to U$ as in the proof of Theorem 3.9. We define, for each k such that $r_k \geq r$, $\rho'_k : C_{r_k}(Y) \to U_{[r_k]}$, by $\rho'_k = \rho_k \circ g_k$. Condition (a) in the proof of Theorem 3.9 holds trivially for ρ'_0. Moreover, for each k such that $r_{k+1} \geq r$, condition (b) is guaranteed by *(ii)* of Proposition 4.1. For each k such that $r_k < r$ we simply define ρ'_k according to (b) of Theorem 3.9. Now define j_r according to $j_r(x) = \bigsqcup_{k \in K} \rho'_k(c_x^{r_k})$. By definition of j_r it follows $i(X)_{[r_k]} = j_r(Y)_{[r_k]}$ for each k such that $r_k \geq r$. Therefore, by Lemma 3.7 we get $\hat{d}_H(i(X), j_r(Y)) < r$. This proves $\Delta(X,Y) \geq inf\{\hat{d}_H(i(X), j(Y)) \mid j : Y \to U$ isometric embedding$\}$.
In order to prove the converse, let $\hat{d}_H(i(X), j(Y)) < r$. By Remark 4.2 we have, for suitable $\rho_k : C_{r_k}(X) \to U_{[r_k]}$, $\rho'_k : C_{r'_k}(Y) \to U_{[r'_k]}$, ($r_k \in D_0(X)$, $r'_k \in D_0(Y)$):

$$i = \bigsqcup_{k \in K} \rho_k \quad \phi(i(X)) = \langle r_k, \rho_k(C_{r_k}(X)) \rangle_{k \in K},$$
$$j = \bigsqcup_{k \in K'} \rho'_k \quad \phi(j(Y)) = \langle r'_k, \rho'_k(C_{r'_k}(Y)) \rangle_{k \in K'}.$$

Let $p = max\{k \in K \mid r_k \geq r\}$, $q = max\{k \in K' \mid r'_k \geq r\}$. By 3.7 $p = q$ and $i(X)_{[r_p]} = j(Y)_{[r_p]}$, that is $\rho_p(C_{r_p}(X)) = \rho'_p(C_{r_p}(Y))$. This enables us to define two mappings,

$$u : C_{r_p}(X) \to C_{r_p}(Y), \quad u = (\rho'_p)^{-1} \circ \rho_p;$$
$$v : C_{r_p}(Y) \to C_{r_p}(X), \quad v = \rho_p^{-1} \circ \rho'_p.$$

The extensions $\overline{u} : X \to C_{r_p}(Y)$, $\overline{v} : Y \to C_{r_p}(X)$, defined as in the proof of 4.1, are easily shown to satisfy $\delta(\langle \overline{u}, \overline{v} \rangle) < r$. Therefore it holds $\Delta(X, Y) \leq inf\{\hat{d}_H(i(X), j(Y)) \mid j : Y \to U \text{ isometric embedding}\}$. Finally we prove that the infimum is actually a minimum. Let $\Delta(X, Y) = r$. If $r = 0$ the thesis is trivial since X and Y are isomorphic. If $r \neq 0$, define $r' = min(D_r^U(i(X)) \setminus \{r\})$. Such a minimum exists by Proposition 3.3. Then take $j_{r'}$ as defined in the first part of the proof. By construction we have $(i(X))_{[r']} = (j_{r'}(Y))_{[r']}$. Moreover, for each $r < s < r'$ we have $(i(X))(1 - s) = 0 = (j_{r'}(Y))(1 - s)$. Thus for each $s > r$ we get $(i(X))_{[s]} = (j_{r'}(Y))_{[s]}$. By applying Lemma 3.7 we obtain $\forall s > r. \hat{d}_H(i(X), j_{r'}(Y)) < s$, hence it must be $\hat{d}_H(i(X), j_{r'}(Y)) = r$. □

As mentioned in the Introduction, we conclude the section by explaining why our construction hardly generalizes to other categories (such as complete ultrametric spaces or compact metric spaces). Actually both ultrametricity and compactness hypotheses play an essential role in the construction of U as a universal space. In fact, when proving a key result, namely Theorem 3.9, we rely on Lemma 2.1, Lemma 3.7, which both use ultrametricity hypothesis, and Lemma 2.3, which uses compactness hypothesis. On the contrary the hypothesis of 1-boundness could be dropped. With slight modifications one can extend the construction of the universal space in the case of compact ultrametric spaces with distances which take values in $[0, +\infty)$. However, the wide use of one-boundness hypothesis throughout the literature on metric semantics suggested us to maintain it.

5 Two applications

In this section we give two applications of the previous results. They are both related to the problem of solving recursive domain equations.

Consider the category $\mathcal{C}$ of [2] whose objects are KUM's and morphisms are ϵ-adjoint pairs, i.e. pairs $\langle i, j \rangle : X \to Y$ such that $i : X \to Y$, $j : Y \to X$ are NDI functions. This notion of morphism is more general than that of embedding-projection pairs in [1, 3, 11, 13], where the further condition $j \circ i = Id_X$ is imposed (there is a similar generalization in the order-theoretic framework when considering Galois-connections instead of embedding-projection pairs). Thus what we prove below holds also for the category of KUM's and embedding-projection pairs.

A crucial role for finding fixed points solutions of domain equations is played by Cauchy towers. A Cauchy tower of spaces is a sequence $(X_n, \langle u_n, v_n\rangle)_n$ such that

$\langle u_n, v_n\rangle : X_n \rightarrow X_{n+1};$
$\forall\epsilon > 0.\exists\overline{n}.m > n \geq \overline{n} \Rightarrow \delta(\langle u_m \circ u_{m-1} \ldots \circ u_n, v_n \circ v_{n+1} \ldots \circ v_m\rangle) < \epsilon.$

By using the universal space U one can derive the existence of limits for Cauchy towers of KUM's just from the completeness of $\mathcal{P}_{nco}(U)$. This approach seems more simple than that devised in [2], where the existence of limits is proved by building, as standard, the categorical limit $\lim_{\leftarrow}(X_n, \langle u_n, v_n\rangle)_n$ as a suitable subset $Y \subseteq \prod_n X_n$.

Theorem 5.1 *Let* $(X_n, \langle u_n, v_n\rangle)_n$ *be a Cauchy tower. Then there exists a unique (up to isometries)* X *such that* $\lim_n \Delta(X_n, X) = 0$*. Moreover* X *is isomorphic to* $\lim_{\leftarrow}(X_n, \langle u_n, v_n\rangle)_n$.

Proof. Let $i_0 : X_0 \rightarrow U$ any isometric embedding. Define, inductively on $I\!N$, $i_{n+1} : X_{n+1} \rightarrow U$ as any isometric embedding such that $\Delta(X_n, X_{n+1}) = \hat{d}_H(i_n(X_n), i_{n+1}(X_{n+1}))$. i_{n+1} exists by Theorem 4.3. We have that $(i_n(X_n))_n$ is a Cauchy sequence in $\mathcal{P}_{nco}(U)$. Since this space is complete we get the existence of $X \in \mathcal{P}_{nco}(U)$ such that $\lim_n \hat{d}_H(X, i_n(X_n)) = 0$. This implies $\lim_n \Delta(X, X_n) = 0$, by Theorem 4.3 again.

As to the last statement, let $Y = \lim_{\leftarrow}(X_n, \langle u_n, v_n\rangle)_n$. Then $\Delta(X, Y) \leq max\{\lim_n \Delta(X, X_n), \lim_n \Delta(Y, Y_n)\} \rightarrow 0$. By Proposition 4.7 of [2] it follows $X \simeq Y$. □

The discussion of the second application will not be given in full details. We will prove that the usual constructors over $\mathcal{C}$ can be represented (in a sense explained below) over $\mathcal{P}_{nco}(U)$, hence over U. We need some definitions. Given an element $A \in \mathcal{P}_{nco}(U)$, let $\theta(A)$ be the KUM obtained by endowing A with the subspace metric induced by U. Now we endow $\mathcal{P}_{nco}(U)$ with the mapping $\Delta_U : \mathcal{P}_{nco}(U) \times \mathcal{P}_{nco}(U) \rightarrow [0, 1]$ defined by:

$$\Delta_U(A, B) = inf\{\hat{d}_H(A, i(\theta(B))) \mid i : B \rightarrow U \text{ isometric embedding}\}.$$

The mapping Δ_U satisfies the following properties: for each $A, B, C \in \mathcal{P}_{nco}(U)$

$$\Delta_U(A, A) = 0,$$
$$\Delta_U(A, B) = \Delta_U(B, A),$$
$$\Delta_U(A, B) \leq max\{\Delta_U(A, C), \Delta_U(B, C)\},$$

hence Δ_U is a pseudo-ultrametric over $\mathcal{P}_{nco}(U)$ (see e.g. [2] or [10]). The following facts are easy to prove:

- for each X, Y KUM's and $i : X \rightarrow U$, $j : Y \rightarrow U$ isometric embeddings,

$$\Delta(X, Y) = \Delta_U(i(X), j(Y)).$$

In particular

$$\begin{array}{ll} X \simeq Y \Leftrightarrow \Delta_U(i(X), j(Y)) = 0, & (\ddagger) \\ \Delta_U(A,B) = \Delta(\theta(A), \theta(B)), \quad \text{for each } A, B \in \mathcal{P}_{nco}(U). \end{array}$$

- $(\mathcal{P}_{nco}(U), \Delta_U)$ is a complete pseudo-ultrametric space (see [10]), in the sense that each Cauchy sequence $(A_n)_n$ converges to (infinitely many) limits $\bar{A}$ such that $\Delta_U(A_n, \bar{A}) \to 0$. All such limits, considered as KUM's, are isometric, since their mutual distance is zero.

We now give the notion of representable operator (see [6]). Given a operator $F : \mathcal{C}^n \to \mathcal{C}$, we say that F is *representable* over $\mathcal{P}_{nco}(U)$ if there exists a non-distance increasing function $\phi_F : \mathcal{P}_{nco}(U)^n \to \mathcal{P}_{nco}(U)$ such that, up to isometry,

$$F \circ \langle \theta, \ldots, \theta \rangle = \theta \circ \phi_F.$$

The next result states that all the standard constructors are representable. In the following $+$ and $\times$ denote the disjoint union and cartesian product respectively; $\to$ is the non-distance increasing function constructor and Id^ε (for $0 < \varepsilon \leq 1$) the *shrinking* constructor, which transforms a KUM (X, d) into the KUM (X, d^ε), where $d^\varepsilon(x, y) = \varepsilon \cdot d(x, y)$.

Theorem 5.2 *$+$, $\times$, $\to$, $\mathcal{P}_{nco}(U)$ and Id^ε are representable constructors over $\mathcal{P}_{nco}(U)$. Moreover composition of representable operators is representable.*

Proof: We give the proof for $\to$. Given two KUM X, Y, $[X \to Y]$ denotes the space of non-distance increasing functions from X to Y. We have to prove that there exists a non-distance increasing function $\phi_\to : \mathcal{P}_{nco}(U) \times \mathcal{P}_{nco}(U) \to \mathcal{P}_{nco}(U)$ which represents $\to$. For any $A, B \in \mathcal{P}_{nco}(U)$, fix an isometric embedding $u_{A,B} : [\theta(A) \to \theta(B)] \to U$. Consider now $A, A', B, B' \in \mathcal{P}_{nco}(U)$ and let $\Delta_U(A, A') = r$, $\Delta_U(B, B') = s$. Then $\Delta(\theta(A), \theta(A')) = r$ and $\Delta(\theta(B), \theta(B')) = s$. Let $\langle i, j \rangle : \theta(A) \to \theta(A')$ and $\langle h, k \rangle : \theta(B) \to \theta(B')$ be NDI-pairs such that $\delta(\langle i, j \rangle) = r$, $\delta(\langle h, k \rangle) = s$. As remarked in Section 2, these pairs exist by the compactness hypothesis. We consider $\langle j \to h, i \to k \rangle : [\theta(A) \to \theta(B)] \to [\theta(A') \to \theta(B')]$ defined by:

$$\begin{array}{l} \forall f \in [\theta(A) \to \theta(B)].(j \to h)(f) = h \circ f \circ j, \\ \forall g \in [\theta(A') \to \theta(B')].(i \to k)(g) = k \circ g \circ i. \end{array}$$

We have (f ranges over $[\theta(A) \to \theta(B)]$, x, x' range over $\theta(A)$)

$$\begin{array}{ll} \multicolumn{2}{l}{max_f\{d(f, (i \to k) \circ (j \to h)(f))\} =} \\ = & max_f\{d(f, k \circ h \circ f \circ j \circ i)\} \\ \leq & max_f\{max\{d(f, k \circ h \circ f), d(k \circ h \circ f, k \circ h \circ f \circ j \circ i)\}\} \\ \leq & max_f\{max\{d(Id, k \circ h), d(Id, j \circ i)\}\} \end{array}$$

$$
\begin{aligned}
&\leq \quad max_f\{max\{max_x\{d(f(x), khf(x))\}, max_{d(x,x')\leq r}\{d(khf(x), khf(x'))\}\}\} \\
&\leq \quad max_f\{max\{s, r\}\} \\
&= \quad max\{r, s\}.
\end{aligned}
$$

Similarly $max_{g\in[\theta(A')\to\theta(B')]}d(g, (j \to h)\circ(i \to k)(g)) \leq max\{r, s\}$, hence we have $\Delta([\theta(A) \to \theta(B)], [\theta(A') \to \theta(B')]) \leq max\{r, s\}$. Therefore

$$
\begin{aligned}
&\Delta_U(u_{A,B}([\theta(A) \to \theta(B)]), u_{A',B'}([\theta(A') \to \theta(B')])) \leq \\
&\leq \quad max\{\Delta_U(A, A'), \Delta_U(B, B')\}.
\end{aligned}
$$

Thus we have shown that the function

$$
\phi_{\to} = \lambda A, B \in \mathcal{P}_{nco}(U).u_{A,B}([\theta(A) \to \theta(B)])
$$

is non-distance increasing. It is immediate to prove that $\phi_{\to}$ represents $\to$ over $\mathcal{P}_{nco}(U)$.

Following similar arguments one can prove that all the above mentioned constructors are representable. Finally it is easy to show that the composition of representable operators is represented by the function obtained as composition of the representations of the original operators. □

Consider a domain equation $X \simeq F(X)$ over $\mathcal{C}$, where F is a representable contractive operator. Similarly to Theorem 7.3 of [6], we can now prove that the equation has solution, by taking the fixed point of the function which represents F. We use, without giving the easy proof, the fact that a representable contractive operator over $\mathcal{C}$ is represented by a contractive function over $\mathcal{P}_{nco}(U)$.

Theorem 5.3 *If $F : \mathcal{C} \to \mathcal{C}$ is a contractive representable functor, then the equation $X \simeq F(X)$ has a (unique up to isometry) solution.*

Proof: Let ϕ_F be the contractive function which represents F. Since $\mathcal{P}_{nco}(U)$ is complete, there exists $A \in \mathcal{P}_{nco}(U)$ such that $\Delta_U(\phi_F(A), A) = 0$, hence we have, by (‡),

$$
\theta(A) \simeq \theta(\phi_F(A)) \simeq F(\theta(A)).
$$

Uniqueness follows from contractiveness of F.

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