Analytic Discs Attached to Manifolds with Boundary

By

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§1. Introduction

Let $X$ be a complex manifold of dimension $n$, $M^+$ a closed half-space with boundary $M$, $A$ an analytic disc of $X$ "attached" to $M^+$, tangent to $M$ at some point $z_0$ of $\partial A \cap M$, and intersecting $\tilde{M}^+$ in any neighbourhood of $z_0$. Then holomorphic functions extend from $\tilde{M}^+$ to a full neighborhood of $z_0$. This theorem refines the results of [1] where the boundary $\partial A$ (instead of the whole $A$) was supposed to intersect $\tilde{M}^+$. The argument of the proof consists in constructing a (closed) manifold with boundary $W$, contained in the envelop of holomorphy of $\tilde{M}^+$ and such that $A \subset W$ but $A \not\subset \partial W$. In this situation it is easy to find a new small disc $A_1 \subset A$ with $\partial A_1 \not\subset \partial W$. We are therefore in a situation similar to [1], and get the conclusion by exhibiting a disc transversal to $\partial W$ at $z_0$.

Extension of holomorphic functions by the aid of tangent discs attached to $M$ and of "defect 0" is a particular case of a general theorem of "wedge extendibility" of CR-functions by A. Tumanov; the new part of our theorem is that no assumptions on "defect" are made.

This paper is tightly inspired to the results and the techniques by A. Tumanov [7]. We also owe to A. Tumanov a great help during private communications.

§2.

Let $X$ be a complex manifold of dimension $n$, $M$ a real submanifold of...
one of the two closed half-spaces with boundary $M$, $A = A(\tau)$, $\tau \in \Delta$ an analytic disc of $X$, $z_0 = A(1)$ a point of $\partial A \cap M$, $\{B\}$ the system of spheres of center $z_0$. Let $C^{k,\alpha}$ be the functions whose derivatives up to the order $k$ are $\alpha$-Lipschitz continuous. We assume $M$ to be $C^{2,\alpha}$ and $A$ to be $C^{1,\alpha}$ up to the boundary and small. We recall the result by [1].

**Theorem A.** ([1, Theorem 1]) Assume

(i) $\partial_1 A(1) \in T_{z_0} C M$,

(ii) $\partial A \subset M^+$,

(iii) there exists $z_1 \in \partial A$ with $z_1 \in \hat{M}^+$.

Then for any $B \supset A$ there is $B' \subset B$ such that holomorphic functions extend from $\hat{M}^+$ to $B'$.

It is essential in the previous statement that $z_1 \in \partial A$ (in addition to $z_1 \in \hat{M}^+$). As for the case $z_1 \in \text{int } A$, we can reduce to the former case when we strengthen (ii) to "$A \subset M^+$". In this case extension from $B \cap \hat{M}^+$ to a suitable $B'$ holds provided that $B$ contains $z_1$ ([1, Corollary 4]).

What happens when $z_1 \notin \partial A$ and $A \notin M^+$? Holomorphic extension seems not to take place. However it holds when a sequence converging to $z_0$ of such points $z_1 \notin \partial A$ does exist.

**Theorem 1.** Let $M$ be $C^{3,\alpha}$, and assume that $\partial A$ is a $C^{2,\alpha}$ curve with $A(1) = z_0$. Suppose

(i) $\partial_1 A(1) \in T_{z_0} C M$

(ii) $\partial A \subset M^+$

(iii) $A \cap \hat{M}^+ \cap B' \neq \emptyset \forall B'$.

Then if $B \supset A$ there exists $B'$ such that any holomorphic function extends from $B \cap \hat{M}^+$ to $B'$.

**Proof.** When $\partial A \notin M$, the statement is the same as in Theorem A. For completeness, we shall treat it at the end of the proof. Assume therefore $\partial A \subset M$.

(a) Construction of a manifold with boundary $W$ such that $W \supset A$. We
assume $M$ is described by $y_1 = h(x_1, z')$, $z' \in \mathbb{C}^{n-1} = T_{z_0}^C M$, $h(0, 0) = 0$, $\partial h(0, 0) = 0$.

Set $v_2 = i\partial_a A(1) \in T_{z_0}^C M$, choose $v_1 \in T_{z_0} M$ transversal to $T_{z_0}^C M$, e.g. $v_1 = e_1$, and write:

$$T_{z_0} M = Rv_1 \oplus Cv_2 \oplus \mathbb{C}^{n-2}.$$ 

Let $w' = z' \circ A$ take $w_0' \in \mathbb{C}^{n-2}$, $s \in \mathbb{R}$ and consider the equation

$$(1) \quad u(\tau) = -T_1 (h(u(\tau)), w'(\tau) + (0, w_0'^s)) + s \quad \tau \in \partial \Delta$$

in the unknown $u$. Note that for $s = 0$, $w_0'^s = 0$, the "$x_1$-component" $u$ of $A$ satisfies (1). We need the following technical tool.

**Lemma 2.** Let $h$ be $C^{k, \alpha}$. Then for any $w'$ in $C^{k-1, \alpha}$ small, there is an unique solution to (1) $u(\tau) = u_{s, w_0'}(\tau)$, $\tau \in \partial \Delta$, which belongs to $C^{k-1, \alpha} \mid _{s, w_0'}$.

**Proof.** For $j \leq k - 1$, let

$$F: C^{j, \alpha}(\partial \Delta, \mathbb{R}) \times \mathbb{C}^{n-2} \times \mathbb{R} \to C^{j, \alpha}(\partial \Delta, \mathbb{R})$$

$$(u, w_0'^s, s) \mapsto u + T_1 (h(u, w' + (0, w_0'^s))) - s.$$ 

Let $v = h(u, w' + (0, w_0'^s))$ in $\partial \Delta$, and define

$$D_{w_0'^s, s} = (u + iv, w' + (0, w_0'^s)) \quad \text{in} \ \partial \Delta.$$ 

We have:

$$D_{w_0'^s, s} \text{ extends holomorphically to } \Delta \text{ with } D_{w_0'^s, s}(1) = s$$

if and only if

$$F = 0.$$ 

If $h$ is $C^{k, \alpha}$, then $F$ is $C^1$ (as application between functional spaces) and we have

$$F'(\dot{u}, w_0'^s, s) = \dot{u} + T_1 (\partial_x h \dot{u} + \partial_{w'} h w_0'^s + \partial_{w''} h \ddot{w}'') - s.$$ 

We have

$$\begin{cases} F(0, 0, 0) = 0 \\ F'(\dot{u}) = \dot{u} + T_1 (\partial_x h \dot{u}). \end{cases}$$

Thus the equation $F = 0$ has solution $u$ in $C^{j, \alpha}(\partial \Delta, \mathbb{R})$ (and this depends $C^{k-1-j, \alpha}$.
on data $w$, $x$, $z$ and on $w_0$, $s$. (Cf. [5]).

In particular, since $h$ is $C^{3,a}$ and $A$ is $C^{2,a}$ up to the boundary, then setting $w'=z'\circ A$, (1) has an unique solution

$$u = u_{w_0,a}(z) \in C^{2,a} |_{r,w_0,x}.$$ 

With $v = T(z) + h(s,w_0)$, define $D=D_{w_0,a}$ as in the proof of Lemma 2. Clearly for $s=0$ and $w_0=0$, $D$ equals the initial disc $A$. Note also that for any $s$ and $w_0$, $D$ is attached to $M$.

We have that

$$D: C^{n-2}_w \times z_s \times R_s \to C^n$$

$$(w_0, \tau, s) \mapsto D_{w_0,a}(\tau)$$

verifies

$$\text{rank}^R_{\partial w_0,a} D = 2n - 1$$

and therefore the range of $D$ is a manifold $S$ of class $C^{1,a}$ with boundary $\partial S = D(C^{n-2} \times \partial z_s \times R_s)$ in a neighborhood of $z_0 = D_{0,0}(1)$. Note that $\partial S$ is generic and $\partial S \subset M$. Note also that $A \subset S$. Define

$$W = \bigcup_{\zeta_1} -i\epsilon_1 \zeta_1 + S.$$ 

It is clear that any holomorphic function extends from $\hat{M}^+$ to $W \cup \hat{M}^+$. In a neighborhood $B_1$ of $z_0=0$, possibly smaller than $B$, we can describe $W$ by:

$$(2) \quad y_1 = g(x_1, x_2 + i\epsilon_2, z_0) + \zeta_1 \quad \zeta_1 \leq 0, \quad \zeta_2 \leq 0,$$

and

$$(3) \quad S: W \cap \{\zeta_1 = 0\}.$$

(b) Perturbation of $W$ and of $A$ such that $\partial A \notin W$. We perturb $g$ for $\zeta_2 > \epsilon$ to a new function $\bar{g} \geq g$ such that $\{y_1 < \bar{g}\} \subset W \cup \hat{M}^+$ and such that the initial disc $A = (u + iv, w)$ verifies in a point $\tau_1 \in \Delta$:

$$(4) \quad \psi(\tau_1) < \bar{g}(u(\tau_1), w(\tau_1)).$$

We also suppose $z_1 = A(\tau_1)$ close to $z_0$. Let us still write $g$ instead of $\bar{g}$ and
denote by $S$ and $W$ the manifolds with boundary defined by (2) and (3) for this new $g$. Thus (4) is equivalent to

$$z_1 \in A \cap \bar{W}.$$

Take $\tilde{\Delta} \subset \Delta$ such that $\tau_1 \in \partial \tilde{\Delta}$ and $\partial \tilde{\Delta} = \partial \Delta$ at 1. Set $\tilde{A} = A \circ \Phi$ where $\Phi$ is an analytic diffeomorphism $\Delta \to \tilde{\Delta}$. Let us write (2) as

$$(5) \begin{cases} y_1 = g(x_1, x_2, z') + \zeta_1 \\ y_2 = \zeta_2 \end{cases}$$

with $\zeta_1, \zeta_2 \leq 0$. In these coordinates $S$ is defined by $\zeta_1 = 0$ and $\partial S$ by $\zeta_1 = \zeta_2 = 0$. Note that $\partial \tilde{\Delta} = \partial A$ at $z_0$ and that $\partial A \subset \partial S$; it follows that $\zeta_1 = 0$, $\zeta_2 = 0$ at $\tau = 1$. We shall now use only the new disc $\tilde{A}$, and call again $A$.

**(c) Construction of a transversal disc.** Let $\zeta_1(\tau), \zeta_2(\tau)$ be defined by (5) over $A_1$. Consider the system:

$$(6) \begin{cases} u_1 = - T_1(g(u_1(\tau), u_2(\tau) + i\zeta_2(\tau), w''(\tau)) + (1 - \eta)\zeta_1(\tau)), \quad \tau = e^{i\theta} \\ u_2(\tau) = - T_1(\zeta_2(\tau)), \quad \tau = e^{i\theta} \end{cases}$$

There exists an unique solution $(u_1, u_2) = (u_{1\eta}, u_{2\eta})$ in $C^{1,\alpha}|_{\eta}$. Moreover if we set $v_1 = T_1(u_1)$, $v_2 = T_1(u_2)$, and

$$A_\eta = (u_{1\eta} + iv_{1\eta}, u_{2\eta} + iv_{2\eta}, w')$$

we have that

$$\partial_\tau A_\eta \quad \text{is} \quad C^1|_{\eta}.$$

In fact $\partial_\eta u_{2\eta} \equiv 0$, whereas $\partial_\eta u_{1\eta}$ is a solution of:

$$\partial_\eta u_{1\eta} + T_1(\partial_\eta g \partial_\eta u_{1\eta} - \zeta_1) = 0.$$

Since $\partial g$ and $\zeta_1$ are $C^{2,\alpha}$, then $\partial_\eta u_{1\eta}$ is $C^{1,\alpha}(\partial \Delta, R)$; in particular, with $\tau = e^{i\theta}$, $\partial_\eta \partial_\eta u_{1\eta}$ exists and is continuous.

As for $\partial_\eta \partial_\eta v_{1\eta}$, we begin by remarking that

$$v_{1\eta} = g(u_{1\eta}, u_{2} + i\zeta_{2}, w') + (1 - \eta)\zeta_1 \quad \text{on} \quad \partial \Delta.$$

It follows that
\[ \partial_{\eta} v_{1\eta} = \partial_{x_1} g(u_{1\eta}, w) \partial_{\eta} u_{\eta} - \zeta \] belongs to \( C^{1,\alpha}(\partial \Delta, \mathbb{R}) \).

We derive with respect to \( \theta \) and obtain:

\[ \partial_{\eta} \partial_{\eta} v_{1\eta} = \partial_{\theta} (\partial_{x_1} g(u_{1\eta}, w) \partial_{\eta} u_{\eta} - \zeta) = \partial_{x_1} g \partial_{\eta}^2 u_{1\eta} \partial_{\eta} u_{1\eta} + \partial_{x_1} g \partial_{\theta} (\partial_{\eta} u_{1\eta} - \partial_{\theta} \zeta). \]

It is easy to check that all the terms on the right have at least class \( C^0 \); thus also \( \partial_{\theta} \partial_{\eta} v_{1\eta} \) is \( C^0 \). Finally Cauchy-Riemann equations yield:

\[ \partial_{\theta} (u_{1\eta} + iv_{1\eta}) = i e^{i \theta} \partial_{\theta} (u_{1\eta} + iv_{1\eta}) \]

and we are done. It follows from (7):

(8) \[ \partial_{\eta} A_{\eta} = \partial_{\eta} A + \eta \partial_{\eta} \partial_{\eta} A_n |_{\eta=0} + o(\eta). \]

Set \( r = y_1 - g \). We prove now that

(9) \[ \Re \langle \partial r \circ A, \partial_{\eta} \partial_{\eta} A_n \rangle |_{\tau = 1} < 0. \]

In fact one finds a real function \( \lambda \) on \( \partial \Delta \) such that \( \lambda \partial r \circ A \) extends holomorphically to \( \Lambda \) and notice that then:

\[ \langle \lambda \partial r \circ A, \partial_{\eta} \partial_{\eta} A_n \rangle = \langle \lambda \partial_{\eta} r \circ A, \partial_{\eta} A_n \rangle \]

is a holomorphic function. The real part \( \phi \) of this holomorphic function verifies \( \phi |_{\partial \Delta} = - \zeta \lambda \) whence: \( \phi |_{\partial \Delta} \geq 0, \phi(1) = 0, \phi(\tau) > 0 \). (Here \( \tau_1 \) is such that \( A(\tau_1) = z_1 \).) Then Hopf Lemma implies (9). Plugging together (8) and (9) we get

(10) \[ \Re \langle \partial r, \partial_{\eta} A_n \rangle (1) < 0. \]

(d) Construction of a dihedron. We denote again by \( \zeta_1, \zeta_2, w'' \) the components of this new disc \( A_{\eta} \) and solve the equations:

\[ \begin{align*}
    u_1 &= -T_1 (g(u_1(\tau), u_2(\tau) + i \zeta_2'(\tau), w'(\tau) + w''_0) + \zeta_1(\tau)) + s \\
    u_2 &= -T_1 \zeta_2(\tau) \\
    v_1 &= -T_1 u_1 + g(s, 0, w''_0) \\
    v_2 &= T_1 u_2.
\end{align*} \]

Let \( A = A_{s, w''_0} \) be defined by (11). Note that
\[
\begin{align*}
\text{(12)} & \quad \begin{cases} \partial A \subset W \text{ and } \partial A \subset S \text{ at } z_0 \\ A \text{ is transversal to } M \text{ (for } s, w' \text{ small)} \end{cases} \\
\text{Define} & \\
\text{(13)} & \quad S_1 = \{ A_{s,w_0'}(\tau) ; \tau \in \Delta, s \text{ and } w_0'' \in \mathbb{R} \times \mathbb{C}^{n-2} \}.
\end{align*}
\]

We have
\[
\begin{align*}
\text{(14)} & \quad \begin{cases} \partial S_1 \text{ generic} \\ \partial S_1 = \partial S \text{ at } z_0 \text{ (hence } \partial S_1 \subset M) \end{cases}
\end{align*}
\]
where the latter of (14) follows from the former of (12). By using \(S_1 \) and \(M \) one gets a dihedron \(V \) with non-proper tangent cone, such that any \(f \) holomorphic on \(M^+ \cap B \), for \(B \supset A \), is extended to \(V \) at \(z_0 \).

(e) Conclusion of the proof in the case \(\partial A \subset M \). The dihedron \(V \) has generic edge \(\partial S_1 \). We approximate \(V \) by an increasing sequence \(V_v \) of domains with \(C^2 \) boundary such that
\[
V_v \subset V, \partial V_v \supset \partial S_1 \quad \forall v.
\]

It is obvious that for large \(v \), \(V_v \) has at least one Levi-pseudoconcavity. Then germs of holomorphic functions extends from \(V_v \) to a full neighborhood of \(z_0 \) (according to a famous theorem by Hans Lewy).

(f) The case \(\partial A \not\subset M \) (cf. [1]). In this case the proof is simpler. Let \(M \) be defined by \(y_1 = h(x,z') \) \((z' = (z_2, z_3, \cdots))\) with \(h(0) = 0, \partial h(0) = 0 \), write \(A = (u + iv, w') \), and put \(\zeta_1 = v - (h \circ A) \). Solve
\[
\text{(15)} \quad u_\eta = -T_1(h(u(\tau), w'(\tau)) + (1 - \eta)(\zeta_1(\tau)).
\]
This produces a family of discs \(A_\eta = (u_\eta + iv_\eta, w') \) \((v_\eta = T_1(u_\eta))\) which verify by the same argument as above:
\[
\Re \langle \partial r \circ A, \partial A_\eta \rangle |_{\tau = 1} < 0
\]
for any sufficiently small \(\eta \). Let \(\zeta_1, w' \) still denote the components of \(A_\eta \) and solve
\[
\text{(16)} \quad u(\tau) = -T_1(h(u(\tau), z'(\tau) + w_0') + \zeta_1(\tau)) + s.
\]
By taking the union of the discs $A_{s,w_0}(t)$ one gets a manifold $S_1$ such that $\partial S_1 < M^+$ and $S_1 \cap M$ is a generic manifold. We thus get a dihedron $V$ with edge $S_1 \cap M$ and such that functions extend from $M^+ \cap B$ to $V$ at $z_0$ ($B \supset A$). The conclusion is the same as above. Q.E.D.

References


