Course on Mean-Field Games

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Lect. May 9, 2013.

A concise introduction to STOCHASTIC CONTROL

Main reference: Fleming - Soner,

Controlled Markov proc. . . . 1992 ...2005 Springer

See also. Fleming - Rishel

Det. & Stoch opt. control Springer 1975

Bardi, ..., SONER, Visco. sols. . . . .

Springer $\geq$ 1995 ...

For deterministic control & visco. sols.

Bardi - I. Capuzzo-Dolcetta Birkhäuser 1997

Controlled diffusion process

Notation $X_s = X(s) \in \mathbb{R}^n$

(SDE) $dX_s = f(X_s, u_s)\,ds + \sigma(X_s, u_s)\,dB_s$ $\quad t \leq s \leq T$

$B_\cdot$ is a d-dimensional Brownian m.

$\forall s \quad u_s \in U$, the control set, CLOSED, $(\subseteq \mathbb{R}^m)$

$f : \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times U \to M_{n\times d}$

continuous

$$|f(x,u) - f(y,u)| \leq C_1 |x-y|$$

$$|f(x,u)| \leq C_1 (1 + |x| + |u|)$$

$\forall u \in U$

$\forall x \in \mathbb{R}^n$

$$|\sigma(x,u) - \sigma(y,u)| \leq C |x-y|$$

$$|\sigma(x,u)| \leq C_1 (1 + |x| + |u|).$$

COST FUNCTIONAL

$$J(t,x,u_\cdot) := \mathbb{E}_{tx}\left[\int_t^T \underbrace{L(X_s, u_s)\,ds}_{\text{running cost}} + \underbrace{g(X_T)}_{\text{terminal cost}}\right]$$

$\mathbb{E}_{tx}$: Conditioned to $X_t = x$

Here X_s solves (SDE) driven by $u_\cdot$, $X_t = x$

$L : \mathbb{R}^n \times U \to \mathbb{R}$ cont. $\quad |L(x,u)| \leq C(1 + |x|^k + |u|^k)$

$g : \mathbb{R}^n \to \mathbb{R} \quad |g(x)| \leq C(1 + |x|^k)$

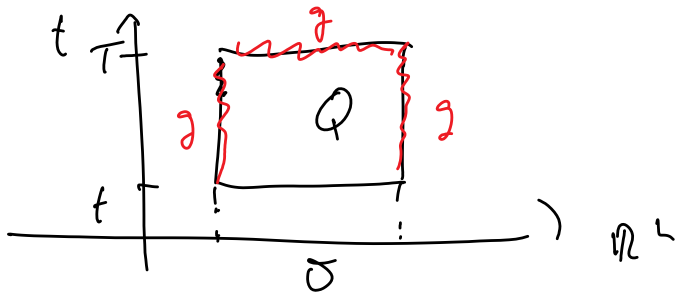
for some $k > 0$.

A more general problem:

Given $Q =]0,T[\times O$, $O \subseteq \mathbb{R}^n$ open set.

$\tau = \inf\{s : (s, X_s) \notin Q\}$ = exit time from Q
dep. on $t, x, u_\cdot$

$$J_Q(x, t, u_\cdot) = E_{tx}\left[\int_t^\tau L(X_s, u_s)\,ds + g(\tau, X_\tau)\right]$$



<u>GOAL</u> MIN J on a suitable set of "ADMISSIBLE CONTROLS"

<u>Ex. 1</u> Deterministic case ($\sigma \equiv 0$)

$u_\cdot : [0, T] \to U$ locally integrable.

$\mathcal{Q}$ = open-loop controls

<u>Ex. 2</u> FEEDBACKS or MARKOV CONTROL POLICY

$\underline{u} : [0, T] \times \mathbb{R}^n \to U$ such that

$$dX_s = f(X_s, \underline{u}(s, X_s))\,ds + \sigma(X_s, \underline{u}(s, X_s))\,dB_s$$

has a unique solution.

Then $u_s = \underline{u}(s, X_s)$

<u>Definition</u> of <u>ADMISSIBLE</u> <u>CONTROLS</u>. ($\sigma \not\equiv 0$)

Fix a REFERENCE PROB. SYSTEM

$$\left(\Omega, \{\mathcal{F}_s\}, \underline{P}, B_\cdot\right) = \nu$$

- a set (SAMPLE space)
- $\forall s \in [0,T]$ $\mathcal{F}_s$ is σ-algebra of subsets of Ω

  INCREASING $\mathcal{F}_s \subseteq \mathcal{F}_{s'}$ if $s \le s'$
- $(\Omega, \mathcal{F}_T, \underline{P})$ probability space
- $B_\cdot$ is a $\mathcal{F}_s$-ADAPTED Brownian motion

  (i.e. $B_s : \Omega \to \mathbb{R}^d$ $\mathcal{F}_s$-measurable $\forall s \in [0,T]$)

Adm. controls w.r.t. ν in $[t,T]$ are

$u : [t,T] \times \Omega \to U$, $\mathcal{F}_s$-progressively meas.le

(i.e. NONANTICIPATING w.r.t. $\mathcal{F}_s$)

<u>i.e.</u> $\forall s \in [t,T]$ $(r, \omega) \mapsto u_r(\omega)$ is

$$[t,s] \times \Omega \to U$$

$\mathcal{B}_s \times \mathcal{F}_s$ measurable, $\mathcal{B}_s$ = Borel σ-alg on $[t,s]$

<u>Meaning</u>: $\forall s$ u_s "depends only on data from the past $[t,s]$" possibly in a more complicated way than $u_s = \underline{u}(s, X_s)$.

Moreover, if U is NOT COMPACT, to be

admissible $\mathbb{E}\int_t^T |u_s|^m ds < \infty \quad \forall m = 1, 2, \dots$

<u>Notation</u>: $\mathcal{U}_{t\nu}$ = admiss. controls on $[t, T]$

<u>FACTS</u> on (SDE) with $u_\cdot \in \mathcal{U}_{t\nu}$

see Ch. V [FR] or App. D [FS].

- $\exists$ SOLN. to $\begin{cases} (SDE) \\ \underline{X}_t = x \end{cases}$

  X_s is $\mathcal{F}_s$-progressively measurable

  a.s. $s \mapsto X_s(\omega)$ continuous

- SOLN. is PATHWISE UNIQUE

- (Gikhman-Skorokhod) $\forall m = 1, 2, \dots$

$$\mathbb{E}_{xt} |X_s|^m < C_m \quad \forall s \in [t, T]$$

- Then $J(t, x, u_\cdot) < +\infty \quad \forall t, x$

$$\forall u_\cdot \in \mathcal{U}_{t\nu}$$

<u>PROBLEM</u>: minimize $J(t, x, \cdot)$ over $\mathcal{U}_{t\nu}$

———— o ————

<u>Recall</u> Diff. op. ass. to SDE $u \in U$

$$A^u \varphi(s,x) := \mathrm{trace}\Big(\underbrace{\frac{\sigma\sigma^T}{2}(x,u)\, D_x^2 \varphi(s,x)}_{=\frac{1}{2}\sum_{i,j} a_{ij}(x,u) u_{x_i x_j}}\Big) + f(x,u)\cdot D_x \varphi(s,x)$$

$$\forall \varphi \in C^{1,2}(\underbrace{]0,T[\times \mathbb{R}^n}_{=:Q_0})$$

$$(a_{ij}(x,u)) = (\sigma\sigma^T)(x,u) \geq 0 \Rightarrow A^u \text{ is}$$

DEGENERATE ELLIPTIC.

Ito's formula.

$$\varphi(s,X_s) = \varphi(t,X_t) + \int_t^s \Big[\frac{\partial \varphi}{\partial t}(r,X_r) + A^{u_r}\varphi(r,X_r)\Big] dr$$

$$+ \int_t^s D_x\varphi(r,X_r)\cdot \sigma(X_r,u_r)\, dB_r$$

$$\forall \quad 0 < t < s < T$$

FACT it holds $\forall\, u. \in \mathcal{U}_{t\nu}$

DYNKIN formula take $\mathbb{E}_{tx}$ in Ito's formula

Want. $\mathbb{E}_{tx} \int_t^s I_r \, dB_r = 0$

This is known if $\mathbb{E}_{tx} \int_t^s |I_r|^2 dr < \infty$

$$I_r = |D\varphi(r,X_r)|^2 |\sigma(X_r,u_r)|^2 .$$

So, if $\varphi \in C_p^{1,2}(Q_0)$ $\quad p :=$ $\varphi, \varphi_t, \varphi_{x_i x_j}$ have at most polynomial growth.

$$\mathbb{E}_{tx}\left(\int_t^s D\varphi(\ )\sigma(\ )\,dB_r\right)=0. \implies$$

$$\text{(DF)}\quad \mathbb{E}_{tx}\,\varphi(s,X_s)-\varphi(x,t)=\mathbb{E}_{tx}\int_t^s\left(\frac{\partial\varphi}{\partial t}+A^{u_r}\varphi\right)(r,X_r)\,dr$$

$$\forall\ \varphi\in C_p^{1,2}(]0,T[\times\mathbb{R}^n)$$

DYNAMIC PROGRAMMING METHOD.

Value function:

$$V(t,x):=\inf_{u_\cdot\in\mathcal{A}_{t\nu}} J(t,x,u_\cdot)$$

In principle V depends on ν, but can give assumptions under which it does NOT.

DYNAMIC PROGRAMMING PRINCIPLE

Thm. $\forall\ h>0$, $0<t<t+h<T$

$$V(t,x)=\inf_{\mathcal{A}_{t\nu}}\mathbb{E}_{tx}\left[\int_t^{t+h}L(X_s,u_s)\,ds+V(t+h,X_{t+h})\right]$$

MEANING. If you know $V(t+h,\cdot)=$ the optimal cost on $[t+h,T]$, can calculate $V(t,\cdot)$ working on $[t,t+h]$ using $V(t+h,\cdot)$ as a terminal cost.

Proof. Fare in the determ. case.

Hard in general .... see [FS]. $\mathbb{A}$

FORMAL DERIVATION OF HJB PDE

Take $u_s \equiv u \in U$ CONSTANT

$$V(t,x) \le \mathbb{E}_{tx} \int_t^{t+h} L(X_s, u)\,ds + \mathbb{E}_{tx} V(t+h, X_{t+h})$$

ASS. $V \in C_p^{1,2}(]0,T[\times\mathbb{R}^n)$. (DF) $\Rightarrow$

$$\mathbb{E}_{tx} V(t+h, X_{t+h}) - V(t,x) = \mathbb{E}_{tx} \int_t^{t+h} \left(\frac{\partial v}{\partial t} + A^u v\right)(s, X_s)\,ds$$

(DPP)

$$\ge -\mathbb{E}_{tx}\int_t^{t+h} L(X_s, u)\,ds$$

Divide by $h>0$, let $h \to 0$

R.H.S. $\lim_{h\to 0} \mathbb{E}_{tx} \frac{1}{h}\int_t^{t+h} L(X_s, u)\,ds = L(x,u)$

by cont.ty of L at (x,u), $|L(x,u)| \le C(1+|x|^k+|u|^k)$

& FACT $\mathbb{E}_{tx} \sup_{t\le s\le t+h} |X_s - x|^m \to 0$ $\forall m = 1,2,\dots$

L.H.S: Since $\frac{\partial v}{\partial t}$ & $A^u v$ have polyn. growth

$$\lim_{h\to 0} \frac{1}{h} \text{L.H.S.} = \left(\frac{\partial v}{\partial t} + A^u v\right)(t,x) \Rightarrow$$

$$-\frac{\partial v}{\partial t} - A^u v - L(x,u)\Big|_{(t,x)} \le 0 \qquad \forall t,x \quad \forall u \in U$$

$$\Rightarrow \quad -\frac{\partial V}{\partial t} + \sup_{u \in U} \{ \underbrace{-A^u}_{} V - L(x,u) \} \leq 0 \quad \text{in }]0,T[\times \mathbb{R}^n$$

$$-\mathrm{tr}\left(\frac{\sigma\sigma^T}{2}(x,u) D^2 V\right) - f(x,u) \cdot DV$$

<u>Notat.</u> $\mathcal{H}(x, p, M) := \sup_{u \in U} \left\{ -\mathrm{tr}\left(\frac{\sigma\sigma^T}{2}(x,u) M\right) - f(x,u) \cdot p - L(x,u) \right\}$

$x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $M \in \mathrm{Sym}(n)$

The HJB PDE is

$$(HJB) \quad -\frac{\partial W}{\partial t} + \mathcal{H}(x, D_x W, D_x^2 W) = 0 \quad]0,T[\times \mathbb{R}^n = Q_0$$

We proved : $V \in C_p^{1,2}(Q_0) \Rightarrow -\frac{\partial V}{\partial t} + \mathcal{H}(x, D_x V, D_x^2 V) \leq 0$

<u>Q</u> : Is V a SOLUTION of HJB ?

<u>Lect May 14, 2013</u>

Assume $\exists$ opt. control for (t,x)

$u^*_\cdot \in \Theta_{tx}$ & $\forall h > 0$, $t+h < T$,

$$(DPP^*) \quad V(t,x) = \mathbb{E}_{tx}\left[\int_t^{t+h} L(X_s^*, u_s^*)\, ds + V(t+h, X_{t+h}^*) \right]$$

Use Dynkin formula (as last time)

$$-\mathbb{E}_{tx} \frac{1}{h} \int_t^{t+h} L(X_s^*, u_s^*)\, ds = \mathbb{E}_{tx} \frac{1}{h} \int_t^{t+h} \left(\frac{\partial V}{\partial t} + A^{u_s^*} V \right)(s, X_s^*)\, ds$$

$$-\mathbb{E}_{tx}\frac{1}{h}\int_t^{t+h} L(X_s^*, u_s^*)\,ds = E_{tx}\frac{1}{h}\int_t^{t+h}\left(\frac{\partial V}{\partial t} + A^{u_s^*}V\right)(s, X_s^*)\,ds$$

$h>0$, Under suitable ass., including the continuity of u_s^* at $s=t$

$$\left(-\frac{\partial V}{\partial t} - A^{u_t^*}V\right)(t,x) - L(x, u_t^*) = 0$$

$$\Rightarrow \quad -\frac{\partial V}{\partial t} + \sup_u\left(-A^u V - L(x,u)\right) \geq 0$$

$\blacksquare$

<u>Conclusion</u> We hope that V be, under ass. ..., a sol. of the <u>terminal</u> <u>value</u> pb.

$$(CP)\quad \begin{cases} -\dfrac{\partial V}{\partial t} + \mathcal{H}(x, D_x V, D_x^2 V) = 0 &]0,T[\times\mathbb{R}^n \\ V(T,x) = g(x) & \forall x\in\mathbb{R}^n \end{cases}$$

<u>Rmk 1</u>. $Z(t,x) \doteq V(T-t,x) =$

$$= \inf_u \int_{T-t}^T L(\quad)\,ds + g(X_T)$$

$$\frac{\partial Z}{\partial t} = -\frac{\partial V}{\partial t} \qquad Z(0,x) = V(T,x) = g(x)$$

Z "solves" the <u>initial</u> value pb.

$$\begin{cases} \dfrac{\partial Z}{\partial t} + \mathcal{H}(x, D_x Z, D_x^2 Z) = 0 \\ Z(0,x) = g(x) \end{cases}$$

This is an I.V.P. for a DEGENERATE PARABOLIC EQ., i.e.,

$$\mathcal{H}(x, p, M + \underline{P}) \le \mathcal{H}(x, p, M) \qquad \forall M \in Sym(n), \ \forall \underline{P} \in \text{''}, \ \underline{P} \ge 0$$

because $a_{ij} = \dfrac{\sigma\sigma^T}{2} \ge 0$

$$tr(a(M+\underline{P})) = tr(aM) + \underbrace{tr(a\underline{P})}_{\ge 0} \ge tr(aM)$$

<u>RMK 2</u> Ass. min. EIGENvalue of $a(x,u)$ is $\ge \lambda > 0$, i.e.

$$\sum_{i,j} a_{ij}(x,u)\xi_i\xi_j \ge \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n$$

then $P \ge 0$ $\quad tr(aP) \ge \lambda \, tr P \quad \Rightarrow$

$$\mathcal{H}(x, p, M + \underline{P}) \le \mathcal{H}(x, p, M) - \lambda \, tr P$$

this is called UNIFORM PARABOLICITY of (HJB)

<u>RMK 3</u>. In general (HJB) is FULLY NONLINEAR (nonl. in D^2V), if $\sigma(x,u)$ is <u>NOT</u> <u>cst</u>. in u, and $\mathcal{H}$ is CONVEX in $(D_x u, D_x^2 u)$.

If $\sigma = \sigma(x)$ is indep of the control.

(HJB) becomes $\qquad H(x, D_x V)$

(HJB) becomes

$$-\frac{\partial V}{\partial t} - \operatorname{tr}\frac{\sigma\sigma^T}{2} D_x^2 V + \overbrace{\sup_u\{f(x,u)\cdot D_x V - L(x,u)\}}^{H(x,D_x V)} = 0$$

is SEMILINEAR. In particular

$d = n \qquad dX_s = f(X_s, u_s)\,ds + dB_s$

$\sigma = I_n \Rightarrow$ (HJB) becomes

$$-\frac{\partial V}{\partial t} - \Delta_x V + H(x, D_x V) = 0$$

VISCOUS H-J eq.

Exerc. $H(x,p) = a(x)|p|^2 + V(x)$

find $f(\ )$ & $L(\ )$ such that get this H.
(if $a(x) \geq 0$).

Rmk 4. If $\sigma \equiv 0$

(HJB) becomes $\quad -\frac{\partial V}{\partial t} + H(x, D_x V) = 0$

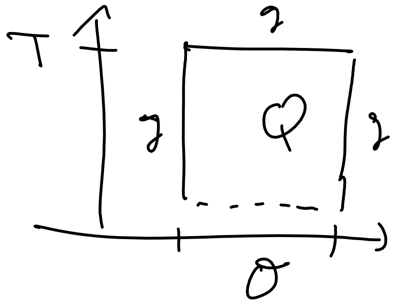
1st order eq. fully nonlinear, H-J eq.

Rmk. 5 $\quad \mathcal{O} \subseteq \mathbb{R}^n \quad Q =]0,T[\times\mathcal{O}$

$\tau = \tau(t,x,u_\cdot) =$ 1st exit time of (s, X_s)
from Q

from Q

$$J_Q = E\left(\int_0^\tau L(\ ) + g(\tau, X_\tau)\right)$$



The Value fn. is expected to solve an TERM.-BDRY-Value pb.

$$\begin{cases} HJB & \text{in }]0,T[\times \mathcal{O} = Q \\ V(T,x) = g(T,x) & \forall x \\ V(t,x) = g(t,x) & \forall x \in \partial\mathcal{O} \\ & \forall t \in]0,T[. \end{cases}$$

<u>Verification Thm.</u>

Now we ASSUME $\exists$ solution to (CP) and want infos. on the control pb.

<u>Thm</u> Ass. $W \in C^{1,2}(Q_0) \cap C_p(\overline{Q_0})$ solves (CP). Then

(i) $W(t,x) \le J(t,x;u_\cdot) \quad \forall u_\cdot \in \mathcal{A}_{t\nu}$

(ii) if for (t,x) $\exists$ $u^*_\cdot \in \mathcal{A}_{t\nu}$, with trajectory X^*_s, $X^*_t = x$ such that

$$u^*_s \in \operatorname*{argmin}_u \left\{ A^u W(s, X^*_s) + L(X^*_s, u) \right\}$$

for Leb.$\times P$-a.e. (s,ω). Then ...

$$W(t,x) = J(t,x,u^*_\cdot)$$

and then $u^*_\cdot$ is optimal for (t,x) &

$$W(t,x) = V(t,x).$$

Pf. For simplicity ass. $W \in C^{1,2}_p(Q_0)$ (in general use cutoff)

Dynkin Form. $\Rightarrow$ $\forall$ $0<t<s<T$

$$\mathbb{E}_{tx} W(s,X_s) = W(t,x) + \mathbb{E}_{tx}\int_t^s \underbrace{\left(\frac{\partial W}{\partial t} + A^{u_s} W\right)}(r,X_r)\,dr$$

(HJB) $\frac{\partial W}{\partial t} + \inf_u \{A^u W + L\} = 0 \Rightarrow \quad \geq -L(X_r,u_r)$

$$W(t,x) \leq \mathbb{E}_{t,x}\left[\int_t^s L(X_r,u_r)\,dr + W(s,X_s)\right]$$

let $s \to T$ $\qquad\qquad W(T,X_T) = g(x)$

$$\Rightarrow W(t,x) \leq J(t,x,u_\cdot) \quad . \quad (i)$$

(ii) if $u^*_r \in \operatorname{argmin}\{\dots\}$ along X^*_r

$$\Rightarrow W(t,x) = \mathbb{E}_{t,x}\left[\int_t^s L(X^*_r,u^*_r)\,dr + W(s,X^*_r)\right]$$

$s \to T$ $\qquad\qquad g(x)$

$$\Rightarrow W(t,x) = J(t,x,u^*_\cdot). \quad \blacksquare$$

Rmk Define the FEEDBACK $]0,T[\times\mathbb{R}^n \to U$

$$\underline{u}^*(t,x) \in \operatorname{argmin}_{u\in U}\{A^u W(t,x) + L(t,x)\}.$$

$$\underline{u}^*(t,x) \in \operatorname*{argmin}_{u \in \mathcal{V}} \left\{ A^u W(t,x) + L(t,x) \right\}.$$

suppose that

$$\begin{cases} d\underline{X}_s = f(X_s, \underline{u}^*(s,X_s))\,ds + \sigma(X_s, \underline{u}^*(s,X_s))\,dB_s \\ X_t = x \end{cases}$$

has a UNIQUE SOLN. Then $u^*_s = \underline{u}^*(s, X^*_s)$

is optimal by Verif. Thm.

This procedure is called SYNTHESIS

of OPTIMAL FEEDBACK.

<u>Examples</u> where this works.

- Linear system & Quadratic cost
- Merton portfolio optimization

see Vargiolu's lect. tomorrow.

<u>Rmk</u>. $\exists !\, W \in C^{1,2}(Q_0) \cap C_p(\bar{Q}_0)$ sol. of (HJB)

is known if (HJB) is UNIFORMLY PARABOLIC

+ technical conds. (L.C. Evans, N.S. KRYLOV)

What can we say if V is not $C^{1,2}$

$\nexists$ (CP) does not have smooth solutions?

Ex. 1 DETERMINISTIC CONTROL $\sigma \equiv 0$

Ex. 2 Newton's law with control & noise

$$\ddot{X}_s = f(X_s, \dot{X}_s, u_s) + \frac{dB_s}{ds} \qquad Y_s = \dot{X}_s$$

$$\begin{cases} dX_s = Y_s\, ds \cancel{+} 0\, dB_s \\ dY_s = f(X_s, Y_s, u_s)\, ds + dB_s \end{cases}$$

$\sigma\sigma^T$ is degenerate in 3 dim / 6 $\square$

Need a notion of soln. to HJB weaker than classical.

VISCOSITY SOLUTIONS

$F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times Sym(N) \to \mathbb{R}$ cont.

$\mathcal{O} \subseteq \mathbb{R}^N$, look at

(PDE) $F(x, w, Dw, D^2 w) = 0$ in $\mathcal{O}$

Ex. • $-\Delta w + H(x, w, Dw) = 0 \qquad \mathcal{O} \subseteq \mathbb{R}^N$

• $w_t - \Delta_x w + H(\qquad) = 0 \qquad]0,T[\times \tilde{\mathcal{O}} = \mathcal{O}$

- $-w_t - \Delta_x w + H(\quad) = 0$
- $-w_t + H(x, Dw, D^2 w) = 0$

Ass. F is DEGEN. ELL.

$$F(x, r, p, M + P) \le F(x, r, p, M) \quad \forall P \ge 0$$

Def. • $w \in USC(\mathcal{O})$ is a visco. SUB sol. of (PDE) if $\forall \varphi \in C^{1,2}(\mathcal{O})$ $\forall$ $\bar{x} \in \operatorname{argmax}(w - \varphi)$

$$F(\bar{x}, w(\bar{x}), D\varphi(\bar{x}), D^2\varphi(\bar{x})) \le 0$$

- $w \in LSC(\mathcal{O})$ is a visco SUPER sol. of (PDE) if $\forall \varphi \in C^{1,2}(\mathcal{O})$ $\forall$ $\tilde{x} \in \operatorname{argmin}(w - \varphi)$

$$F(\tilde{x}, w(\tilde{x}), D\varphi(\tilde{x}), D^2\varphi(\tilde{x})) \ge 0.$$

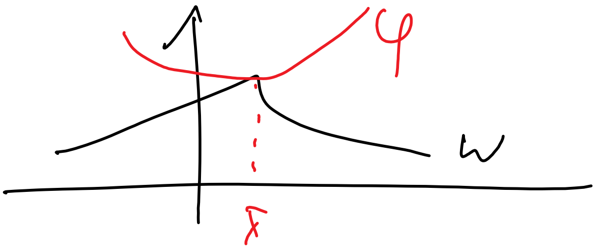
- $w \in C(\mathcal{O})$ is a visco. sol. if SUB & SUPER sol.

Rmks In the def it's equivalent to take

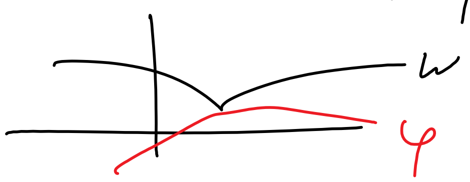
- $\varphi \in C^\infty$
- max or min merely LOCAL
- " " " are STRICT
- $w(\bar{x}) = \varphi(\bar{x})$

For subsol. $(w - \varphi)(x) \le 0 \quad \forall x$

$\Rightarrow$ $w \leq \varphi$ $\forall x$ $w = \varphi$ at $\bar{x}$, i.e. φ touches the graph of w at $\bar{x}$ from ABOVE



For supersol., instead, touch w by φ from BELOW



- CONSISTENCY: CLASSICAL $\Rightarrow$ VISCOSITY

VISCOSITY + TWICE DIFF.LE at $\bar{x}$

$$\Rightarrow F(\bar{x}, w(\bar{x}), Dw(\bar{x}), D^2 w(\bar{x})) = 0.$$

- MOTIVATION.

(1) 1st. order HJ $\quad u_t + H(x, D_x u) = 0$

VANISHING VISCO. METHOD $\quad u^\varepsilon_t + H(x, D_x u^\varepsilon) = \varepsilon \Delta_x u^\varepsilon$

(2) the value fn. V of our stoch. control pb. is a visco. soln. of (HJB) as soon as $V \in C$.

Thm. [Soner, p.148] In addition to our basic ass. of 1st lect. suppose

- $\mathcal{V} \subseteq C(Q_0)$
- either $\mathcal{U}$ is COMPACT or

  $V_m :=$ value with controls in $\mathcal{U} \cap B(0,m)$

  $\to V$ loc. unif. as $m \to \infty$

Then V is a visco. soln. of (HJB)

<u>Pf</u>. "$\geq$" is harder but doable, see [So] or [FSo]. (without assuming $\exists$ of opt. control u^* & (DPP*))

"$\leq$" $\varphi \in C^{1,2}(Q_0)$, $(x,t) \in \operatorname{argmax}(V - \varphi)$.

Not restrictive ass. $\varphi \in C^{1,2}_p(Q_0)$

Recall the beginning of the formal derivation of (HJB):

Fix $u_s \equiv u$ cost. DPP $\Rightarrow$

$$E_{tx} V(t+h, X_{t+h}) - V(t,x) \overset{\otimes}{\geq} -E_{tx} \int_t^{t+h} L(X_s, u)\, ds$$

$$(V - \varphi)(t,x) \geq (V - \varphi)(t+h, X_{t+h}) \quad \left(\text{because } (t,x) \in \operatorname{argmax}(V - \varphi)\right)$$

$$\Rightarrow \varphi(t+h, X_{t+h}) - \varphi(t,x) \geq V(t+h, X_{t+h}) - V(t,x)$$

$$\Rightarrow \mathbb{E}_{tx} \varphi(t+h, X_{t+h}) - \varphi(t,x) \geq -E_{tx} \int_t^{t+h} L(X_s, u)\, ds .$$

Use Dynkin form.,
$\forall$ divide by $h > 0$, let $h \searrow 0$, as last time,

& get

$$-\frac{\partial \varphi}{\partial t} + \sup_{u \in U} \left\{ -A^u \varphi - L(x,u) \right\} \leq 0$$

at (t, x).

$\Rightarrow$ V is subsol. $\square$

MAIN RESULT in VISCOSITY SOLS. THEORY:

COMPARISON PRINCIPLES between

SUB & SUPERSOLS. of I.V.P. & BVP.

$\Rightarrow$ UNIQUENESS of VISCO SOLS.

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