ED2 : Lecture 25 March 2013

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Hopf-Lax formula for the Cauchy problem

$$(CP)\quad \begin{cases} u_t + H(Du) = 0 & t>0 \\ u(x,0) = g(x) \end{cases}$$

$$u(x,t) = \min_y \left\{ tH^*\left(\frac{x-y}{t}\right) + g(y) \right\} \quad , \quad t>0$$

Thm. u differentiable at (x,t), $t>0 \Rightarrow$

$$u_t + H(Du)\Big|_{(x,t)} = 0$$

Pf. "$\le$" Fix $q \in \mathbb{R}^n$, $h>0$

$$u(x+hq, t+h) \overset{(D.P.F.)}{=} \min_y \left\{ hL\left(\frac{x+hq-y}{h}\right) + u(y,t) \right\}$$

$$\underset{x=y}{\le} hL\left(\frac{\cancel{h}q}{\cancel{h}}\right) + u(x,t)$$

$$\Rightarrow \frac{u(x+hq, t+h) - u(x,t)}{h} \le L(q)$$

$h \to 0^+$

$$Du \cdot q + u_t \Big|_{(x,t)} \le L(q) \quad \forall q \in \mathbb{R}^n$$

$$Du \cdot q + u_t \Big|_{(x,t)}$$

$$u_t + \underbrace{\sup_q \left\{ Du \cdot q - L(q) \right\}}_{= H(Du)} \leq 0 \Big|_{(x,t)} \quad // \tfrac{1}{2}$$

" $\geq$ " $\exists z$: $u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$

$$u(x,t) - u(y,s) \geq tL\left(\frac{x-z}{t}\right) + \cancel{g(z)} - \left\{ sL\left(\frac{y-z}{s}\right) + \cancel{g(z)} \right\}$$

$$= tL\left(\frac{x-z}{t}\right) - sL\left(\frac{y-z}{s}\right) \quad \forall y, s$$

Choose y : $\frac{x-z}{t} = \frac{y-z}{s}$, i.e., $y = \frac{s}{t}(x-z) + z$

$$= \frac{s}{t} x + \left(1 - \frac{s}{t}\right) z$$

For such y

$$u(x,t) - u(y,s) \geq (t-s) L\left(\frac{x-z}{t}\right)$$

Want $y \approx x$ s.t. I can compute a directional derivative of u at (x,t) : $t - s = h > 0 \Rightarrow s = t - h$

$$y = \left(1 - \frac{h}{t}\right) x + \frac{h}{t} z = x - h \frac{x-z}{t}$$

$$\frac{u(x,t) - u\left(x - h\frac{x-z}{t}, t-h\right)}{h} \geq L\left(\frac{x-z}{t}\right)$$

$h \to 0+$

$$\frac{\partial u}{\ } \quad Du \cdot \frac{x-z}{t}\Big| \quad \geq L\left(\frac{x-z}{t}\right)$$

$h \to 0+$

$$\frac{\partial u}{\partial\left(\frac{x-z}{t}, 1\right)} = u_t + Du \cdot \frac{x-z}{t}\Big|_{(x,t)} \geq L\left(\frac{x-z}{t}\right)$$

$$u_t + H(Du)\Big|_{(x,t)} = u_t + \max_q \left\{ q \cdot Du - L(q) \right\}$$

$$\geq u_t + Du \cdot \frac{x-z}{t} - L\left(\frac{x-z}{t}\right) \geq 0 .$$

$q = \frac{x-z}{t}$ ∎

By combining Thm. with Rademacher Thm.

we get

<u>Corollary</u>. $u(x,t) = \begin{cases} \min_y \left\{ t\, H^*\left(\frac{x-y}{t}\right) + g(y) \right\}, & t > 0 \\ g(x) & t = 0 \end{cases}$

is Lip & solves (CP) $\begin{cases} u_t + H(Du) = 0 & \text{a.e. } (x,t) \\ & t > 0, \\ u(x,0) = g(x) . \end{cases}$

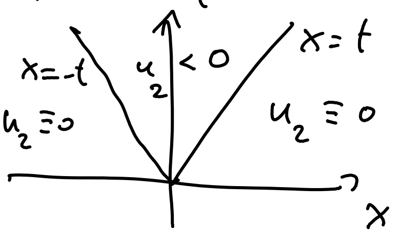
<u>Important RMK.</u> The H-L is NOT the unique Lip fn. satisfying (CP) A.E.

<u>Example</u> $\begin{cases} u_t + u_x^2 = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = 0 & \text{at } t = 0 \end{cases}$

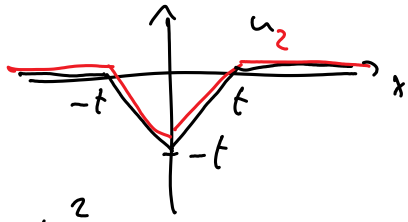
$u(x,t) = \min_y \left\{ t \left(\frac{x-y}{t}\right)^2 + 0 \right\} = 0 \quad \forall x, t.$

$$u_{HL}(x,t) = \min_{y\in\mathbb{R}} \left\{ \frac{t}{4}\left(\frac{x-y}{t}\right)^2 + 0 \right\} = 0 \qquad \forall x,t.$$

$$u_2(x,t) = \begin{cases} 0, & |x| \geq t \\ |x| - t, & t > |x| \end{cases}$$



$$\frac{\partial u_2}{\partial t} = -1 \qquad \frac{\partial u_2}{\partial x} = \operatorname{sign} x \qquad x \neq 0$$



$$\left|\frac{\partial u_2}{\partial x}\right| = 1 \qquad \Rightarrow \quad u_2 \text{ solves } \quad u_t + u_x^2 = 0$$

$$\forall\ |x| \neq t.$$

<u>Conclusion</u> the "a.e." soln. is a GOOD notion for $\exists$, but it's TOO WEAK for UNIQUENESS.

UNIQUENESS of CLASSICAL SOLUTIONS.

<u>Prop.</u> (COMPARISON PRINCIPLE) Ass. $H \in C(\mathbb{R}^n)$

$u, v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ with 1st. partial derivatives in $\mathbb{R}^n \times]0,T]$, Lip. & bounded,

$$\begin{cases} u_t + H(Du) \leq v_t + H(Dv) & \forall\ 0 < t \leq T \\ u(x,0) \leq v(x,0) \end{cases}$$

$$\Rightarrow \quad u(x,t) \leq v(x,t) \qquad \forall x, \forall t > 0.$$

<u>Cor.</u> If g is bounded and u_{HL} DIFFERENTIABLE everywhere in $\mathbb{R}^n \times]0, \bar{t}]$ for some $\bar{t} > 0$, then u_{HL} is the unique solution of (CP) $\begin{cases} u_t + H(Du) = 0 & 0 < t \le \bar{t} \\ u = g & t = 0 \end{cases}$

among Lip, bdd fns. with partial derivatives.

<u>Pf of Cor.</u> u_{HL} solves (CP) everywhere &

$$|u_{HL}(x,t) - g(x)| \le Ct \le C\bar{t} \quad \forall x, t, \; g \text{ bdd}$$

$\Rightarrow u_{HL}$ bdd. • If v is another "classical solution", $\begin{cases} u_t + H(Du) = 0 = v_t + H(Dv) \\ u(x,0) = g(x) = v(x,0) \end{cases}$

Comp. Princ. $\Rightarrow u \le v$ & $v \le u \Rightarrow v = u$ ∎

<u>Pf.</u> "Idea": Look at maxima of $u - v$ and show they are attained at $t = 0$.

$$\Phi(x,t) = \Phi_{\eta,\beta}(x,t) = u(x,t) - v(x,t) - \eta t - \beta \log(1 + |x|^2)$$

$\beta, \eta > 0$.

$$\Phi(x,0) \le u(x,0) - v(x,0) \le 0$$

1. $\forall \beta, \eta \qquad \Phi(x,0) \le u(x,0) - v(x,0) \le 0$

2. By contradiction ass. $\exists (x_0, t_0)$:

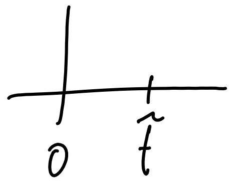
$(u - v)(x_0, t_0) > 0 \;.\; \Rightarrow \forall \eta, \beta \in]0, \gamma]$

$$\Phi(x_0, t_0) > 0$$

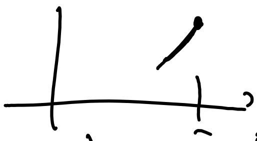
3. $\Phi \to -\infty$ as $|x| \to +\infty \Rightarrow$

$$\max_{\mathbb{R}^n \times [0,\hat{t}]} \Phi = \Phi(\tilde{x}, \tilde{t}) \quad , \quad \tilde{t} > 0$$

$$\Rightarrow \begin{cases} Du - Dv - \beta \dfrac{2x}{1+|x|^2} \Big|_{(\tilde{x},\tilde{t})} = 0 \\ u_t - v_t - \eta \Big|_{\tilde{x},\tilde{t}} \geq 0 \end{cases}$$



At $\tilde{x}, \tilde{t}$



$\hat{t} = \tilde{t}$

$$u_t + H(Du) \le v_t + H(Dv) \le u_t - \eta + H(Du - \beta p)$$

$$p = \frac{2\tilde{x}}{1+|\tilde{x}|^2} \qquad |p| \le 2$$

$u \in Lip \Rightarrow |Du(x,t)| \le C_1 \quad \forall x, t$

$|Du - \beta p| \le C_1 + 2\beta \le C' \quad$ if $\beta \le 1$

Use H uniformly cont. in $\overline{B}(0, 2C')$.

<u>Little detour.</u> f unif cont in $\bar{X}$

$\forall \varepsilon \; \exists \delta : \; |x-y| < \delta \Rightarrow |f(x) - f(y)| \le \varepsilon$.

<u>FACT</u> $f \in UC(\bar{X}) \; \exists \; \omega :]0, +\infty[\to]0, +\infty[$:

$\lim_{r \to 0+} \omega(r) = 0 \qquad |f(x) - f(y)| \le \omega(|x-y|)$

Then ω is called a MODULUS of CONTINUITY of f.

QED.

$$H(Du + \beta p) \le H(Du) + \omega(2\beta)$$

Now choose $\eta > \omega(2\beta) \Rightarrow$

$$u_t + H(Du) \le u_t + H(Du) - \eta + \omega(2\beta)$$
$$< u_t + H(Du) \qquad \blacksquare$$

This gives some hope for $\exists$ & UNIQUEN. of SOLN. of (CP) for some notion of weak solution

WEAKER THAN CLASSICAL
STRONGER THAN A.E.

I'll give 2 answers to this.

- SEMICONCAVE SOLUTIONS.
  (S.N. KRUŽKOV ~1960-1967)

VISCOSITY SOLUTIONS

M. CRANDALL, P.L. LIONS, L.C. EVANS ~ 1982

SEMICONCAVE FUNCTIONS.

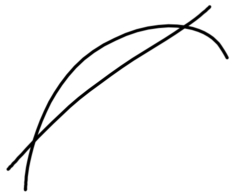
<u>Def.</u> $f \in C(\mathbb{R}^n)$ is SEMICONCAVE if $\exists C \in \mathbb{R}$

$$f(x+z) - 2f(x) + f(x-z) \leq C|z|^2$$

$$\forall x, z \in \mathbb{R}^n$$

<u>Ex. 1</u> f CONCAVE $\Rightarrow$ f SEMICONC. with $C = 0$

$x = \frac{x+z}{2} + \frac{x-z}{2}$ f conc. $\Rightarrow$



$$f(x) \geq \frac{1}{2} f(x+z) + \frac{1}{2} f(x-z)$$

<u>Lecture March 26, 2013.</u>

<u>Def.</u> φ is semiconvex if $-\varphi$ is SEMICONC.

$$\varphi(x+z) - 2\varphi(x) + \varphi(x-z) \geq -C|z|^2 .$$

Properties and examples of SEMICONCAVE FNS.

$$\frac{f(x+z) - 2f(x) + f(x-z)}{|z|^2} =$$

$$\left(f(x+z) - f(x) - f(x-z) + f(x) \right) \leq C$$

$$= \left(\frac{f(x+z) - f(x)}{|z|} - \frac{-f(x-z) + f(x)}{|z|} \right) \frac{1}{|z|} \le C$$

So semiconcavity means boundedness from above of 2^{nd} order difference quotients, a weak form of bdness. from above of $D^2 f$.

<u>Example</u>. $f \in C^2$, $\exists\, x_1 \in [x, x+z]$, $x_2 \in [x, x-z]$

$$f(x+z) = f(x) + \nabla f(x) \cdot z + \frac{1}{2} z^T D^2 f(x_1) z$$

$$f(x-z) = f(x) - \nabla f(x) \cdot z + \frac{1}{2} z^T D^2 f(x_2) z$$

$$f(x+z) - 2f(x) + f(x-z) = \frac{1}{2} z^T \left(D^2 f(x_1) + D^2 f(x_2) \right) z = \star$$

If $\exists C \in \mathbb{R}$ $D^2 f(y) \le C I_{n \times n}$ $\forall y$. (i.e. $D^2 f(y) - C I$ negative semidef.)

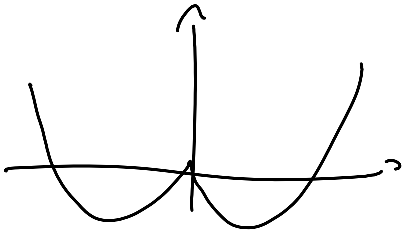
Then $\star \le \frac{1}{2} z^T (C I + C I) z = C |z|^2$.

<u>Exercise</u>. $f \in C^{1,1}(\mathbb{R}^n)$ (i.e., $f \in C^1$ with $\nabla f \in Lip$) $\Rightarrow$ f semiconc.

• f semiconc. with constant. C $\iff$

$$f(x) - C|x|^2 \text{ is concave}$$

<u>Examples.</u>

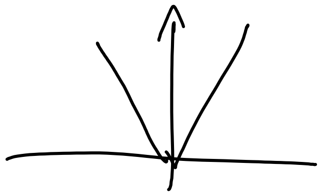


$|x|^2 - \alpha|x|$, $\alpha > 0$

non diff.ble in 0

but it is S.CONC.

• $|x|$

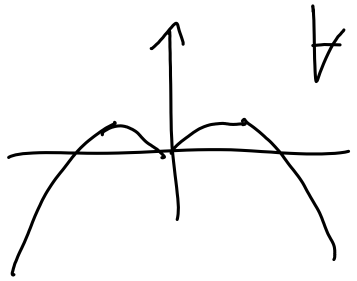


is <u>NOT</u> SEMICONCAVE

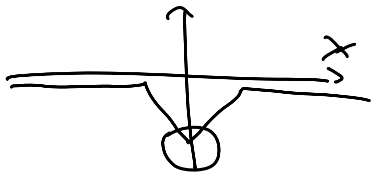
$\forall C$

because $|x| - C|x|^2$

is <u>not</u> concave for any C



• Counterexample seen yesterday (to uniqueness for CP for HJ)



$u_2(x,t)$

not S.conc. in x

• SEMICONC. FNS. are LOC. LIPSCHITZ

<u>Ref.</u> [BCD] II. 4.2

Recent reference Cannarsa-Sinestrari book 2004

<u>Rmk</u> S.CONCAV. is "intermediate regularity" between Lip & C^1.

between Lip & C .

Semiconcavity (in x) of Hopf-Lax formula.

<u>Lemma 1</u> g semiconcave with constant C

$$\Rightarrow \quad u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t H^*\left(\frac{x-y}{t}\right) + g(y) \right\}$$

satisfies $\forall t>0, \ \forall x, z \in \mathbb{R}^n$

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C|z|^2 .$$

<u>Pf</u> $\exists y : u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$

Use $y+z$ in the def. of $u(x+z,t)$ & $y-z$ for $u(x-z,t)$:

$$u(x+z,t) - 2u(x) + u(x-z,t) \leq \cancel{tL\left(\frac{x-y}{t}\right)} + g(y+z)$$
$$\cancel{-2tL\left(\frac{x-y}{t}\right)} - 2g(y) + \cancel{tL\left(\frac{x-y}{t}\right)} + g(y-z) \leq C|z|^2$$

semiconc. of g ∎

<u>Def.</u> $H \in C^2$ is UNIFORMLY CONVEX if $\exists \theta > 0$:

$$\sum_{i,j=1}^n \frac{\partial^2 H(p)}{\partial p_i \partial p_j} \xi_i \xi_j \geq \theta |\xi|^2 \quad \forall p, \xi \in \mathbb{R}^n$$

i.e. $D^2 H(p) - \theta I_{n\times n} \geq 0 \quad \forall p \in \mathbb{R}^n$.

Rmk. H unif. convex $\Rightarrow$

$$H(p+z) - 2H(p) + H(p-z) \geq \theta |z|^2 \quad (UC)$$

(Pf.! as before by Taylor expansion.)

FACT (UC) $\Rightarrow$ $L = H^*$ is semiconcave with $C = \frac{1}{\theta}$. Pf. Exercise!

Lemma 2 H unif. convex, $g \in Lip$, then $u_{HL}(x,t)$ is semiconc. in x & $\forall x \in \mathbb{R}^n$ $\forall t > 0$

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq \frac{1}{\theta t} |z|^2$$

Pf. $\exists y$: $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$

use y for $x+z$ & y for $x-z$:

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq tL\left(\frac{x+z-y}{t}\right) + \cancel{g(y)}$$

$$-2tL\left(\frac{x-y}{t}\right) - \cancel{2g(y)} + tL\left(\frac{x-z-y}{t}\right) + \cancel{g(y)}$$

$$\underset{\text{s.conc. of } L}{\leq} t \frac{1}{\theta} \left|\frac{z}{t}\right|^2 = \frac{1}{\theta t} |z|^2 .$$

■

Uniqueness of semiconcave solutions.

Def. $u : \mathbb{R}^n \times [0,+\infty[\to \mathbb{R}$ Lip. is a <u>weak</u> <u>soln</u> of (CP) $\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(x,0) = g(x) \end{cases}$

if

(a) $u(x,0) = g(x)$

(b) $u_t + H(Du) = 0$ a.e. in $\mathbb{R}^n \times (0,+\infty)$

(c) $\exists C_1 \geq 0 : \forall x, z \in \mathbb{R}^n \ \forall t > 0$

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C_1\left(1+\frac{1}{t}\right)|z|^2$$

Thm. (Kružkov '67) $H \in C^2$ sats. (C) & (S), $g \in Lip \Rightarrow \exists$ at most ONE weak soln. of (CP).

Pf. No, (see [E])

Corollary $H \in C^2$, (C), (S), $g \in Lip$, & either g semiconc. OR H uniformly convex.

$\Rightarrow u(x,t) = \min_y \left\{ tH^*\left(\frac{x-y}{t}\right) + g(y) \right\}$ is the unique weak sol. of (CP)

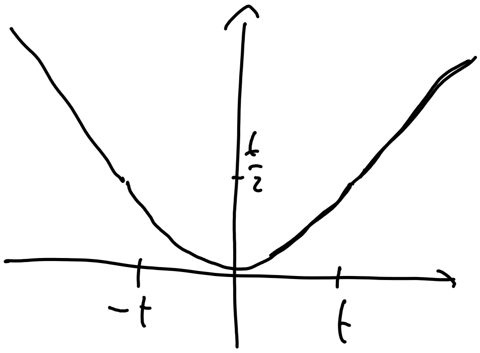
Pf. Combine Lemmata 1,2 with uniqueness Thm. ∎

Exa 1. $\begin{cases} u_t + \frac{|Du|^2}{2} = 0 \end{cases}$ $\quad u(x,t) = \min\left\{ \frac{|x-y|^2}{\ldots} + |y| \right\}$

Exa 1. $\begin{cases} u_t + \frac{|Du|^2}{2} = 0 \\ u(x,0) = |x| \end{cases}$ $u(x,t) = \min_y \left\{ \frac{|x-y|^2}{2t} \cancel{t} + |y| \right\}$

By standard calculations

$$u(x,t) = \begin{cases} |x| - \frac{t}{2}, & |x| \geq t \\ \frac{|x|^2}{2t} & |x| < t \end{cases}$$



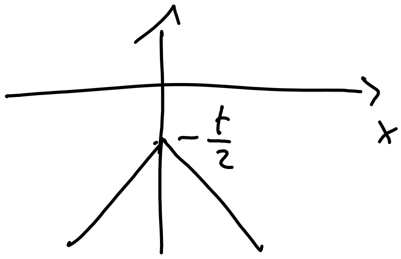
Conv. in x, semiconc. in x $\forall t > 0$

Exerc.: Is it C^1 ?

Exa 2 $\begin{cases} u_t - \frac{|Du|^2}{2} = 0 & t > 0 \\ u(x,0) = -|x| \end{cases}$ $u(x,t) = \min_y \left\{ \frac{|x-y|^2}{2t} - |y| \right\}$

After elementary calculations

$$u(x,t) = -|x| - \frac{t}{2}$$



concave & not diff.le at $x = 0$.

Exa 3 $\begin{cases} u_t + \frac{|Du|^2}{2} = 0 \\ u(x,0) = a|x|^2 \end{cases}$ $a \in \mathbb{R}$

HW: Compute u_{HL}.

Hystorical remarks. $\left(\frac{v^2}{2}\right)_x$

E. Hopf 1950 $\quad v_t + v v_x = \varepsilon v_{xx} \quad \varepsilon > 0$

$n=1 \quad v = u_x \quad \updownarrow \quad u_t + \frac{(u_x)^2}{2} = \varepsilon u_{xx} \quad (HJ)$

$u = \int_{-\infty}^{x} v$

uses a log transformation for u (Hopf – Cole transformation)

which converts (HJ) into a linear heat equation.

P. Lax
O. Oleinik } 1957 $\quad v_t + f(v)_x = 0$

scalar cons. law with f convex

$n=1$

write formulas for v related to formulas

for u : $\quad u_t + f(u_x) = 0$

That's why H-L is sometimes called Lax-Oleinik formula.

E. Hopf: 1965 $\quad n \geq 1 \quad u_t + H(D_x u) = 0$

His "derivation": Complete integral for

$u_t + H(Du) = 0$:

$$\varphi(x,t;a,b) = a \cdot x - tH(a) + b$$

$$\varphi(x,t;a, g(z) - a\cdot z) = a\cdot(x-z) - tH(a) + g(z)$$

<u>Note</u> $\varphi(z, 0; a, g(z) - a\cdot z) = g(z) \quad \forall a.$

Instead of, but related to, "stationarizing" w.r.t. a & z,

$$\sup_a \left[a\cdot(x-z) - tH(a) + g(z)\right] =$$

$$= g(z) + t\sup_a\left[a\cdot\frac{x-z}{t} - H(a)\right]$$

$$= g(z) + tH^*\left(\frac{x-z}{t}\right)$$

<u>Next</u>. $\displaystyle\inf_{z\in\mathbb{R}^n} \underbrace{\qquad\qquad} = u_{HL}(x,t)$

What happens if, instead, I compute $\sup_a \inf_z$ instead of $\inf_z \sup_a$

$$\inf_z\left[a\cdot(x-z) - tH(a) + g(z)\right] = a\cdot x - tH(a) + \sup_z\left[a\cdot z - g(z)\right]$$

$$= a\cdot x - tH(a) - g^*(a)$$

$$\sup_a \inf_z [\quad] = \sup_a\left[a\cdot x - tH(a) - g^*(a)\right].$$

2^{nd} Hopf formula

Hopf. proves this is a solution for g convex & $H \in C(\mathbb{R}^n)$.

Lect. March 27 - 2013

Inf & Sup - convolution

$g : \mathbb{R}^n \to \mathbb{R}$ bounded

$$g_\varepsilon(x) := \inf_{y \in \mathbb{R}^n} \left\{ g(y) + \frac{1}{2\varepsilon} |x-y|^2 \right\}$$ INF-CONV. of g

$$g^\varepsilon(x) := \sup_{y \in \mathbb{R}^n} \left\{ g(y) - \frac{1}{2\varepsilon} |x-y|^2 \right\}$$ SUP-CONV.

N.B. $g_\varepsilon(x) = u_{HL}(x, \varepsilon)$ Hopf-Lax formula at $t = \varepsilon$ for

$$\begin{cases} u_t + \frac{|Du|^2}{2} = 0 & t > 0 \\ u = g & t = 0 \end{cases}$$

We know: $g \in Lip \Rightarrow -C\varepsilon \le g_\varepsilon(x) - g(x) \le 0$

(because $H^*(0) = 0$) so $g_\varepsilon \to g$ UNIF. in $\mathbb{R}^n$ as $\varepsilon \to 0$.

Exercise. $\forall g$ bdd.

$g_\varepsilon \leq g$, $g^\varepsilon \geq g$, $\begin{cases} g_\varepsilon \leq g_{\varepsilon'} \\ g^\varepsilon \geq g^{\varepsilon'} \end{cases}$ if $\varepsilon' < \varepsilon$

& g_ε and $-g^\varepsilon$ SEMICONC.

- g CONTINUOUS $\Rightarrow$ $g_\varepsilon, g^\varepsilon \xrightarrow[\varepsilon \searrow 0]{} g$ LOCALLY UNIFORMLY
- g lower semicont. $\Rightarrow$ $g_\varepsilon \nearrow g$ pointwise

  g upper " $\Rightarrow$ $g^\varepsilon \searrow g$ " .

Asymptotics as $t \to +\infty$

(CP) $\begin{cases} u_t + H(Du) = 0 & t > 0 \\ u(x,0) = g(x) \end{cases}$ H convex & superlinear, $g \in Lip$ & bounded

$$u(x,t) = \min_y \left\{ t\, H\left(\frac{x-y}{t}\right) + g(y) \right\}$$

Exercise Prove the

Thm. Ass. also $H(p) \geq H(0) = \alpha$ $\forall p \in \mathbb{R}^n$

Then $\lim_{t \to +\infty} \big(u(x,t) + t\alpha\big) = u_\infty(x)$ loc. uniformly with $u_\infty(x) = \inf_{y\in\mathbb{R}^n} \left\{ g(y) + \max_{\{p : H(p)=\alpha\}} p\cdot(x-y) \right\}$

Rmk. In partic., if $H(p) > \alpha$ $\forall p \neq 0$ $u_\infty(x) \equiv \inf g$

In partic.: $H = |p|^2$, $\alpha = 0$ $u(x,t) \longrightarrow \inf g$ loc. unif. as $t\to\infty$.

Hint 1 First treat the case $\alpha = 0$

Hint 2 Prove & use ($\alpha = 0$)

$$\inf_{t>0} t H^*\left(\frac{\xi}{t}\right) = \max_{\{p : H(p)=0\}} \xi\cdot p \quad \forall \xi\in\mathbb{R}^n$$

Introduction to "VISCOSITY SOLUTIONS"

Goal Find a notion of "generalized solution" weaker than classical such that Cauchy problems (& Dirichlet pbs.) have a unique solution.

Classical notion of weak soln. for divergence form equations:

(VE) $\quad \operatorname{div} F(x,u) = 0$ in Ω open set

$F = (F_1, \dots, F_n)$ vector field

Take $\varphi \in C^1_c(\Omega)$ "test functions"

$$\int_\Omega \varphi \, \mathrm{div}\, F(x,u)\, dx = 0 \qquad \forall \varphi$$

$$\shortparallel$$

$$\int_\Omega \left[\mathrm{div}(\varphi F) - F \cdot \nabla \varphi \right] dx =$$

$$= \underbrace{\int_{\partial\Omega} \varphi F \cdot \nu \, d\sigma}_{=0} - \int_\Omega F(x,u) \cdot \nabla \varphi \, dx$$

ν unit normal to $\partial\Omega$

If u sol. of (VE) $\exists \Rightarrow$

$$\int_\Omega F(x,u) \cdot \nabla \varphi \, dx = 0 \quad (*)$$

$$\forall \varphi \in C^1_c(\Omega)$$

For (VE) one defines a weak solution as $u \in L^1_{loc}(\Omega)$ s.t. $(*)$ holds $\forall \varphi \in C^1_c(\Omega)$.

<u>BUT</u> $u_t + H(Du, x) = 0$ is <u>NOT</u> in div form, so that method doesn't work.

IDEA: MAXIMUM PRINCIPLE & VANISHING VISCOSITY METHOD.

Consider

$$(HJ_\varepsilon) \qquad u^\varepsilon_t + H(D_x u^\varepsilon, x) = \underbrace{\varepsilon \Delta_x u^\varepsilon}_{} \quad , \ \varepsilon > 0$$

$$\Delta_x u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$$

artificial or math viscosity.

These equations (+ Init. cond.) have solutions $u^\varepsilon \in C^2$, and I expect that the "physically correct" weak soln of

$$u_t + H(Du, x) = 0$$

is the limit as $\varepsilon \searrow 0$ of u^ε.

But the best I can show are estimates on u^ε of EQUIBOUNDEDNESS & EQUICONTINUITY

Then Ascoli-Arzelà $\Rightarrow \exists\, \varepsilon_j \to 0$:

$u^{\varepsilon_j} \to u$ locally uniformly

Let's assume $u^\varepsilon \in C^2$ $\exists$ & $u^{\varepsilon_j} \to u$ loc. unif.

Important Rmk. Let $\varphi \in C^2$, $u^\varepsilon - \varphi$ has

Important Rmk. Let $\varphi \in C^2$, $u^\varepsilon - \varphi$ has a max at $(\bar{x},\bar{t})$, $\bar{t}>0$
(MIN)

$$u^\varepsilon_t = \varphi_t \quad , \quad D_x u^\varepsilon = \nabla\varphi \quad , \quad D^2(u^\varepsilon - \varphi) \leq 0 \quad (\geq)$$

at $(\bar{x},\bar{t})$

$$\Delta u^\varepsilon \leq \Delta\varphi$$

$$u^\varepsilon_t + H(D_x u^\varepsilon, x) = \varepsilon \Delta_x u^\varepsilon$$

$$\varphi_t + H(D\varphi, x) \leq \varepsilon\Delta\varphi \quad \text{at } \bar{x},\bar{t}$$
$(\geq)$

Motivated by this I give

Def. $u \in C(\mathbb{R}^n \times]0,\infty[)$ is a viscosity soln. of (HJ) $u_t + H(Du,x) = 0$ in $\mathbb{R}^n \times]0,\infty[$ if

$\forall \varphi \in C^2(\mathbb{R}^n \times]0,\infty[)$ $\quad \varphi_t + H(D\varphi, x)\Big|_{(\bar{x},\bar{t})} \leq 0$

if $u-\varphi$ has a max at $(\bar{x},\bar{t})$

& $\quad \varphi_t + H(D\varphi, x)\Big|_{(\bar{x},\bar{t})} \geq 0$ if $u-\varphi$ has a MIN at $(\bar{x},\bar{t})$

Prop. $H \in C(\mathbb{R}^n)$, $u^\varepsilon \in C^2$ solve (HJ_ε), $u^{\varepsilon_j} \to u$ loc. unif. $\Rightarrow$ u is a visco sol. of (HJ)

Proof. Only "$\leq$", the other is analogous.

Case 1 $u-\varphi$ has a STRICT MAX at $(\bar{x},\bar{t})$. (LOC)

Claim. For $0<\varepsilon_j\leq\bar{\varepsilon}$ $\exists(x_j,t_j)$:

$$\begin{cases} u^{\varepsilon_j}-\varphi \text{ has a loc. max at } (x_j,t_j) \\ (x_j,t_j)\to(\bar{x},\bar{t}) \end{cases}$$

Pf of the Claim. $0<r$: $B_r=B((\bar{x},\bar{t}),r)$ open.

$\exists\delta>0$: $\underbrace{(u-\varphi)(\bar{x},\bar{t})}_{d}-\delta=\max\limits_{\partial B_r}(u-\varphi)$

$u^{\varepsilon_j}\to u$ unif. $\quad \max\limits_{\partial B_r}(u^{\varepsilon_j}-\varphi)\leq d-\frac{\delta}{3}<$

j large $\quad d-\frac{2\delta}{3}\leq(u^{\varepsilon_j}-\varphi)(\bar{x},\bar{t})$

Then $u^{\varepsilon_j}-\varphi$ attains its max on $\bar{B}_r$ in some $(x_j,t_j)\in B_r$. Repeat the argument with $r_j\searrow 0$ & get $(x_j,t_j)\in B_{r_j}$ $\Rightarrow$ $(x_j,t_j)\to(\bar{x},\bar{t})$ ∎

Then at (x_j,t_j) $u_t^{\varepsilon_j}=\varphi_t$, $Du^{\varepsilon_j}=D\varphi$

$\Delta u^{\varepsilon_j}\leq\Delta\varphi$

$$0=u_t^{\varepsilon_j}+H(Du^{\varepsilon_j},x)-\varepsilon\Delta u^{\varepsilon_j}\Big|_{(x_j,t_j)}$$

$$\boxed{0 = u^{\varepsilon_j}_t + H(Du^{\varepsilon_j}, x) - \varepsilon \Delta u^{\varepsilon_j}\Big|_{(x_j, t_j)}}$$

$$\geq \varphi_t + H(D\varphi, x) - \varepsilon \Delta \varphi \Big|_{(x_j, t_j)}$$

$j \to \infty$ $\downarrow$ $\downarrow$ $\downarrow$

$$\boxed{\varphi_t(\bar{x}, \bar{t}) + H(D\varphi(\bar{x}, \bar{t}), \bar{x})} + 0$$

Case 2 $u - \varphi$ has a max at $(\bar{x}, \bar{t})$ not loc. strict.

$$\tilde{\varphi}(x, t) = \varphi(x, t) + |x - \bar{x}|^2 + (t - \bar{t})^2$$

$\Rightarrow$ $u - \tilde{\varphi}$ has a STRICT MAX at $(\bar{x}, \bar{t})$

Case 1 $\Rightarrow$ $\tilde{\varphi}_t + H(D\tilde{\varphi}, x)\Big|_{(\bar{x}, \bar{t})} \leq 0$

$\tilde{\varphi}_t = \varphi_t$, $D\tilde{\varphi} = D\varphi$

$\blacksquare$ "$\leq$"

N.B. Vanishing viscosity was only used to MOTIVATE the definition, but visco. sols. can be constructed in many other ways.

To further motivate the def we prove

Thm. H convex & superlinear, $g \in Lip$

$\Rightarrow u = u_{HL}(x,t) = \min_y \left\{ t H\left(\frac{x-y}{t}\right) + g(y) \right\}$

is a visco. soln. of $u_t + H(Du) = 0$

in $\mathbb{R}^n \times]0,+\infty[$.

Pf. $u(x,t) = \min_y \left\{ u(y,s) + (t-s) L\left(\frac{x-y}{t-s}\right) \right\}$

$\forall\ t > s$

"SUB-SOL" $y = x - hq \quad s = t-h \quad , h > 0$

$$u(x,t) - u(x-hq, t-h) \leq h L(q)$$

take $\varphi \in C^2$: $u - \varphi$ has a loc. max at x,t

$$(u-\varphi)(x,t) \geq (u-\varphi)(x-hq, t-h) \quad \forall h > 0 \text{ small}$$

$$\varphi(x,t) - \varphi(x-hq, t-h) \leq u(x,t) - u(x-hq, t-h)$$

$$\Rightarrow \frac{\varphi(x,t) - \varphi(x-hq,t-h)}{h} \leq L(q) \quad \forall q$$

$h \to 0$ $\quad D\varphi \cdot q + \varphi_t - L(q) \leq 0 \quad \forall q$ at (x,t)

$$\varphi_t + H(D\varphi)\Big|_{(x,t)} \leq 0 .$$

$\square$ "$\leq$"

"$\geq$" Next time!