Lect April 15, 2013

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 $\bar t, \bar s > 0$. The test fn (C^1)

$$\varphi(x,t) \doteq \frac{|x-\bar y|^2}{2\varepsilon} + \frac{|t-\bar s|^2}{2\varepsilon} + \beta f(x) + \eta t$$

is such that $u-\varphi$ has a max at $(\bar x, \bar t)$.

$$D_x\varphi(\bar x,\bar t) = \underbrace{\frac{\bar x-\bar y}{\varepsilon}}_{=: p_\varepsilon} + \beta Df(\bar x), \quad \varphi_t(\bar x,\bar t) = \frac{\bar t-\bar s}{\varepsilon}+\eta$$

Def. also the C^1 fn

$$\psi(y,s) \doteq -\frac{|\bar x-y|^2}{2\varepsilon} - \frac{|\bar t-s|^2}{2\varepsilon} - \beta f(y) - \eta s$$

is such that $v-\psi$ has a min at $(\bar y,\bar s)$

$$D_y\psi(\bar y,\bar s) = \underbrace{\frac{\bar x-\bar y}{\varepsilon}}_{=: p_\varepsilon} - \beta Df(\bar y), \quad \psi_s(\bar y,\bar s) = \frac{\bar t-\bar s}{\varepsilon}-\eta$$

Step 7 Use the def. of visc. sub- & supersol.

$$\varphi_t + H(D\varphi)\Big|_{(\bar x,\bar t)} \le 0 \le \psi_t + H(D\psi)_{(\bar y,\bar s)}$$

$$\eta + \cancel{\frac{\bar t-\bar s}{\varepsilon}} + H\left(p_\varepsilon+\beta Df(\bar x)\right) \le 0 \le \cancel{\frac{\bar t-\bar s}{\varepsilon}} - \eta + H\left(p_\varepsilon - \beta Df(\bar y)\right)$$

$$\Rightarrow 0 < 2\eta \le -H(p_\varepsilon+\beta P_1) + H(p_\varepsilon+\beta P_2) = (\star)$$

$$\Rightarrow 0 < 2\eta \leq -H(p_\varepsilon + \beta P_1) + H(p_\varepsilon + \beta P_2) = (\star)$$

$$P_1 := Df(\bar{x}), \quad P_2 := -Df(\bar{y}) \qquad |P_i| = 2 \quad i=1,2$$

<u>Def</u>. ω_H the modulus of continuity of H on $\overline{B}(0, 4L+2) \Rightarrow$

$$(\star) \leq \omega_H(\beta |P_1 - P_2|) \qquad \forall\ 0<\beta\leq 1$$

$$\leq \omega_H(4\beta) < \eta \quad \text{by choosing } \beta \text{ small enough}$$

Contradiction with $0 < 2\eta \leq \star$ ∎

<u>Rmk</u>. This proof has several variants and generalizations. For example, the Comparison Principle still holds for $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (depending on x !)

if

$$(H) \quad \begin{cases} |H(p,x) - H(p,y)| \leq C|x-y|(1+|p|) \\ |H(p,x) - H(q,x)| \leq C_1 |p-q| \end{cases}$$

With (H) it's enough that u, v are bdd.

... don't need Lip.

With (H) ...

& continuous, don't need Lip.

Pf. : see Evans' book, or make suitable changes to the previous proof (Exercise).

Much more in [BCD].

INTRODUCTION TO OPTIMAL CONTROL

Control system

$$(S) \quad \begin{cases} \dot{y}(s) = f(\underbrace{y(s)}_{\text{state}}, \underbrace{\alpha(s)}_{\text{control}}) & t < s < T \\ y(t) = x \end{cases}$$

($y(\cdot)$ = RESPONSE of the SYST. to the control)

DATA: $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ CONT.

$A \subseteq \mathbb{R}^m$ compact, $T > 0$. Assume

$$(1) \quad \begin{cases} |f(x, a)| \le M \\ |f(x,a) - f(z,a)| \le L|x - z| \end{cases} \qquad \forall x, z \in \mathbb{R}^n \ \forall a \in A.$$

From ODE theory $\forall \alpha : [t, T] \to A$ cont.

$\exists !$ (unique) solution to (S) denoted by

$y(s) = y_x(s) = y_x(s; \alpha, t)$. Moreover :

(E1) $\quad |y_x(s;\alpha,t) - x| \leq M(t-s)$

(E2) $\quad |y_x(s;\alpha,t) - y_z(s,\alpha,t)| \leq e^{L(s-t)} |x-z|$

Reminder: (S), for $\alpha \in C$, is equivalent to

$$y(s) = x + \int_t^s f(y(\tau), \alpha(\tau))\, d\tau .$$

In control theory we need a larger class of controls:

$$\mathcal{A} := \{ \alpha : [0,T] \to A \text{ measurable} \}$$

Def. a sol. of (S) is an absolutely cont. fn. s.t. $\quad y(s) = x + \int_t^s f(y(\tau), \alpha(\tau))\, d\tau$

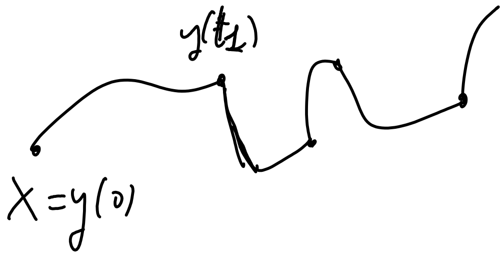
$$\forall s \in [t,T]$$

Thm. $\forall \alpha \in \mathcal{A}$ $\exists$ unique sol. $y(\cdot) = y_x(\cdot\,; \alpha, t)$ of (S) and it satisfies (E1), (E2).

Pf. Follows the same idea as for f cont. in time, see e.g. [BCD]. $\square$

RMK $\quad y(\cdot)$ is differentiable at points of

RMK $y(\cdot)$ is differentiable at points of continuity of $\alpha(\cdot)$, may not be diff.le at points of discontinuity.



← trajectory for piecewise continuous $\alpha(\cdot)$.

Exercise: prove the thm. if α is piecewise continuous. []

FINITE HORIZON COST FUNCTIONAL:

Given $\ell : \mathbb{R}^n \times A \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$

$$J(x,t,\alpha) = \underbrace{\int_t^T \ell(y(s), \alpha(s))\,ds}_{\text{RUNNING COST}} + \underbrace{g(y(T))}_{\text{TERMINAL COST}}$$

$$y(s) = y_x(s; \alpha, t)$$

Goal of the controller: MINIMIZE the COST.

Assume
$$\begin{cases} |\ell(x,a)|, |g(x)| \le C & \forall x, z \in \mathbb{R}^n \\ |g(x) - g(z)| \le C|x-z| & \forall a \in A \\ |\ell(x,a) - \ell(z,a)| \le C|x-z| \end{cases}$$

If $\ell, g \not\equiv 0$ this is called BOLZA problem

if $\ell \equiv 0$ → MAYER PB.

if $g \equiv 0$ → LAGRANGE PB.

if $g \equiv 0 \longrightarrow$ LAGRANGE PB.

Note the connection with Calc. of vars. :

$$\dot{y} = \alpha \quad , \quad \ell(x, a) = L(x, a) \quad , \quad g \equiv 0$$

$$J = \int_t^T L(y(s), \dot{y}(s))\, ds.$$

Note also that any Bolza pb. can be rewritten as a Mayer pb. by adding a state var. y_{n+1} :

$$\begin{cases} \dot{y}_{n+1} = \ell(y, \alpha) \\ y(t) = 0 \end{cases} \Rightarrow y_{n+1}(s) = \int_t^s \ell(y(\tau), \alpha(\tau))\, d\tau$$

We will make proofs only for Mayer pb.
(for simplicity).

PROBLEMS : FIND an OPTIMAL CONTROL and the MINIMAL COST.

Methods : • $\exists$ of opt. controls
• necessary conds. of optimality

We'll see DYNAMIC PROGRAMMING METHOD :
Sufficient conditions + strategy for constructing optimal sols.

CONSTRUCTING optimal SOLS.

Starting point of DPM is the def. of VALUE FUNCTION!

$$V(x,t) := \inf_{\alpha \in \mathcal{A}} J(x,t,\alpha)$$

= "minimal" cost as a function of initial time and position.

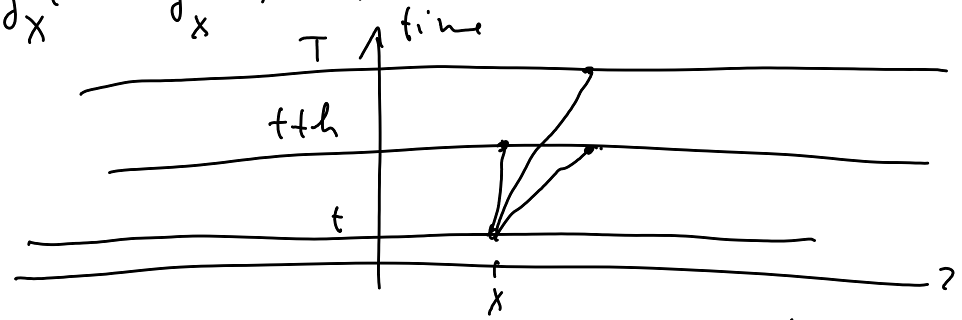
GOALS:
- show that v solves a PDE
- solving the PDE gives a way to find optimal controls.

Thm (Dynamic Programming Principle)

$\forall x \in \mathbb{R}^n$, $0 \le t < T$, $h > 0$: $t+h < T$

$$V(x,t) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_t^{t+h} \ell(y_x(s), \alpha(s))\, ds + V(y_x(t+h), t+h) \right\} \quad \text{(DPP)}$$

$y_x(s) = y_x(s; \alpha, t)$



Note that $h = T-t$ (DPP) is just the def. of $V(x,t)$

because $v(x,T) = \inf \int_T^T \ell + g(y_x(T)) = g(x)$

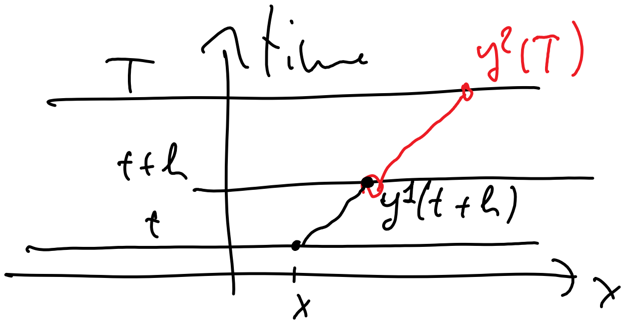
because $v(x,T) = \inf \int_T \ell + g(y_x(T)) = g(x)$

Lect. 16.4.13

Proof of DPP for Mayer pb. $\ell \equiv 0$.

"$\le$" Fix $\alpha_1 \in \mathcal{A}$, solve $\begin{cases} \dot y = f(y,\alpha_1) & t<s<t+h \\ y(t)=x \end{cases}$

Call the sol $y^1(\cdot)$



Fix $\varepsilon>0$ & choose $\alpha_2 \in \mathcal{A}$:

$$V(y^1(t+h), t+h) + \varepsilon \ge g(y^2(T)) \qquad (1)$$

$y^2(\cdot)$ solves $\begin{cases} \dot y = f(y,\alpha^2) & t+h<s<T \\ y(t+h) = y^1(t+h) \end{cases}$

Def. $\alpha^3(s) = \begin{cases} \alpha^1(s) & t<s<t+h \\ \alpha^2(s) & t+h \le s < T \end{cases}$ $\quad \alpha^3 \in \mathcal{A}$.

$\begin{cases} \dot y = f(y,\alpha^3) & t<s<T \\ y(t)=x \end{cases}$ call the sol. $y^3(\cdot)$.

$\Rightarrow y^3(s) = \begin{cases} y^1(s) & t<s<t+h \\ y^2(s) & t+h<s<T \end{cases}$ by UNIQ. of sol. to (S)

In partic $y^3(T) = y^2(T)$

In partic $y^3(T) = y^2(T)$

$$V(x,t) \le J(x,t,\alpha^3) = g(y^3(T)) = g(y^2(T))$$

$$\overset{(1)}{\le} \varepsilon + V(y^1(t+h), t+h) \qquad \forall\, \alpha_1 \in \mathcal{A}$$

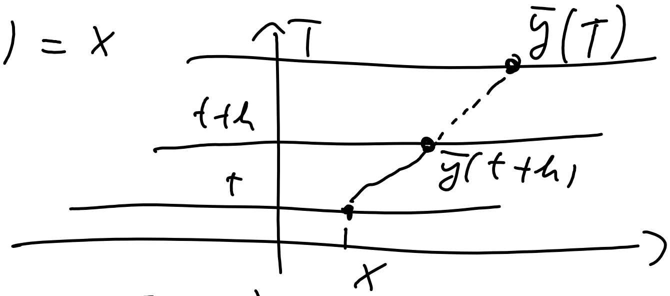
$$\Rightarrow \; V(x,t) \le \varepsilon + \inf_{\alpha} V(y_x(t+h;\alpha,t), t+h)$$

$\varepsilon \searrow 0$ & get "$\le$" in (DPP). $\square_{1/2}$

"$\ge$" $\forall\, \varepsilon > 0 \;\exists\, \bar{\alpha} \in \mathcal{A}$:

$$V(x,t) + \varepsilon \ge J(x,t,\bar{\alpha}) = g(\bar{y}(T))$$

$\bar{y}(\cdot)$ solves $\begin{cases} \dot{y} = f(y,\bar{\alpha}) & t < s < T \\ y(t) = x \end{cases}$



Def. of V

$$V(\bar{y}(t+h), t+h) \le g(\bar{y}(T))$$

$$\Rightarrow \; V(x,t) + \varepsilon \ge V(\underset{\substack{\parallel \\ y_x(t+h;\,\bar{\alpha},t)}}{\bar{y}(t+h)}, t+h) \ge$$

$$\ge \inf_{\alpha} V(y_x(t+h;\alpha,t), t+h)$$

Let $\varepsilon \searrow 0$ & get "$\ge$" in (DPP). $\blacksquare$

<u>Prop.</u> Under the standing assumptions on

Prop. Under the standing assumptions on f, ℓ, g, A, the value fn. $v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is bounded & Lipschitz.

Pf S.1 $|v(x,t)| \le T \sup|\ell| + \sup|g| \quad \forall x, t.$ ∎

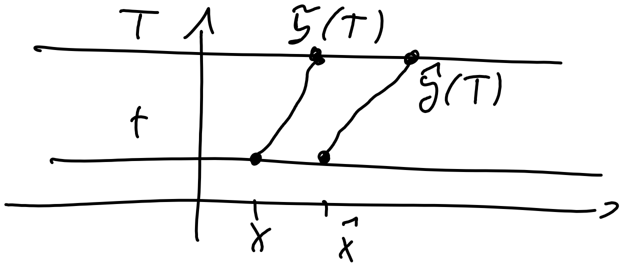
For simpl. $\ell \equiv 0$.

S.2 : Goal $|v(t,x) - v(t,\hat{x})| \le C|x - \hat{x}| \quad (T1)$

Start by $v(t,x) - v(t,\hat{x}) \le \dots$

Fix $\varepsilon > 0$ choose $\hat{\alpha} \in \mathcal{A}$. $v(t,\hat{x}) + \varepsilon \ge g(\hat{y}(T))$

$\hat{y}(s) := y_{\hat{x}}(s; \hat{\alpha}, t)$



$\tilde{y}(s) := y_x(s; \hat{\alpha}, t) \implies |\hat{y}(T) - \tilde{y}(T)| \le e^{4(T-t)} |\hat{x} - x|$

$$v(t,x) - v(t,\hat{x}) \le g(\tilde{y}(T)) - g(\hat{y}(T)) + \varepsilon$$

$$\le L_g |\tilde{y}(T) - \hat{y}(T)| + \varepsilon$$

$$\le C|\hat{x} - x| + \varepsilon$$

Let $\varepsilon \searrow 0$, exchange x & $\hat{x}$ $\implies |v(t,x) - v(t,\hat{x})| \le C|x - \hat{x}|$

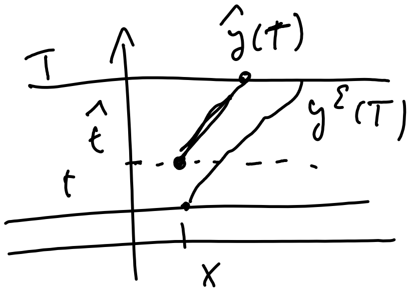
St.2 ∎

Step 3 $v(x,\hat{t}) - v(x,t) \le \dots$

$0 \le t < \hat{t} \le T$. Fix $\varepsilon > 0$, take $\alpha_\varepsilon \in \mathcal{A}$.

$$v(x,t) + \varepsilon \ge g(y^\varepsilon(T)), \quad y^\varepsilon(\cdot) = y(\cdot\,; \alpha^\varepsilon, t)$$

$V(x,t) + \varepsilon \geq g(y^\varepsilon(T))$, $y^\varepsilon(\cdot) = y_x(\cdot\,; \alpha^\varepsilon, t)$



$\hat{\alpha}(s) := \alpha^\varepsilon(s + t - \hat{t})$

$\hat{t} < s < T$

$\hat{y}(\cdot)$ sol of $\begin{cases} \dot{y} = f(y, \hat{\alpha}) & \hat{t} < s < T \\ y(\hat{t}) = x \end{cases}$

Claim $\hat{y}(s) = y^\varepsilon(s + t - \hat{t})$, by uniqueness of

sol. to (S) $\hat{y}(\hat{t}) = y^\varepsilon(t) = x$

$$\dot{\hat{y}}(s) = \dot{y}^\varepsilon(s + t - \hat{t}) = f(y^\varepsilon(s+t-\hat{t}), \alpha^\varepsilon(s+t-\hat{t}))$$

$$= f(\hat{y}(s), \hat{\alpha}(s)) .$$

$$V(x,\hat{t}) - V(x,t) \leq g(\hat{y}(T)) - g(y^\varepsilon(T)) + \varepsilon$$

$$\leq L_g |\hat{y}(T) - y^\varepsilon(T)| + \varepsilon$$

$$= L_g |y^\varepsilon(T + t - \hat{t}) - y^\varepsilon(T)| + \varepsilon$$

$$\leq L_g M |t - \hat{t}| + \varepsilon$$

Let $\varepsilon \searrow 0$ & get ½ of desired $\leq$

$$V(x,\hat{t}) - V(x,t) \leq C|t - \hat{t}| \quad \text{if } t < \hat{t}$$

Step 4 $V(x,t) - V(x,\hat{t}) \leq \ldots$

$\varepsilon > 0 \quad \exists \hat{\alpha} : \quad V(x,\hat{t}) + \varepsilon \geq g(\hat{y}(T))$

$$\bar{\alpha}(s) := \begin{cases} \hat{\alpha}(s + \hat{t} - t) & t \leq s < T + t - \hat{t} \\ \hat{\alpha}(T) & T + t - \hat{t} \leq s \leq T \end{cases}$$

Use $\bar{\alpha}$: $V(x,t) \leq g(\bar{y}(T))$ ....

$\bar{y}$ = traj. corr. to $\bar{\alpha}$ ... show that

$\bar{y}(s) = \hat{y}(s + \hat{t} - t)$ and conclude as before.

Conclusion: Exercise (or check Ch. 10 of [Evans]).

∎

The Hamilton - Jacobi - Bellman PDE

$$H(p,x) := \max_{a \in A} \{ -f(x,a) \cdot p - \ell(x,a) \}$$

$$= - \min_{a \in A} \{ f(x,a) \cdot p + \ell(x,a) \} , \quad p, x \in \mathbb{R}^n$$

Theorem Under the standing ass., the value function V of the opt. contr. problem is the UNIQUE VISCOSITY SOLUTION of the TERMINAL VALUE problem for the H-J-B eq

$$\begin{cases} -u_t + H(D_x u, x) = 0 & \text{in } \mathbb{R}^n \times]0,T[\\ u(x,T) = g(x) . \end{cases}$$

Rmk. $w(x,t) := u(x, T-t)$ $\quad w_t = -u_t$

& $w(x,0) = u(x,T) = g(x)$, w solves

the initial v. pb. $\begin{cases} w_t + H(D_x w, x) = 0 & 0<t<T \\ w(x,0) = g(x) \end{cases}$

Check it for visco. sols.!

Proof Step. 1. By Prop. v is bdd. & Lip.

& $v(x,T) = g(x)$.

Let's prove that v is visco. sol., in case $l \equiv 0$.

Step 2 "SUBSOL." $\phi \in C^1$: $v - \phi$ max at (x,t)

$0<t<T$. GOAL: $-\phi_t + \max_{a \in A} \{ -f(x,a) \cdot D\phi \} \Big|_{(x,t)} \leq 0$ (T1)

Fix $\bar{a} \in A$, take $\bar{\alpha}(s) \equiv \bar{a}$ $\forall s$

$\bar{y}(\cdot) = y_x(\cdot; \bar{\alpha}, t)$. Note that $\dot{\bar{y}}(s) \; \exists \; \forall s$

Use "$\leq$" in (DPP):

$$v(x,t) \leq v(\bar{y}(t+h), t+h) \quad \forall \; 0<h<T-t$$

know that $(v-\phi)(x,t) \geq (v-\phi)(z,s)$ $\quad \forall z, s$

$$\phi(x,t) - \phi(\bar{y}(t+h), t+h) \leq v(x,t) - v(\bar{y}(t+h), t+h) \leq 0$$

Divide by $h>0$, let $h \to 0+$

$$-\phi_t(x,t) - D_x\phi(x,t)\cdot \underbrace{\dot{\bar y}(t)}_{= f(\bar y(t),\bar a)} \le 0 \qquad \forall \bar a \in A$$

$$\Rightarrow \quad -\phi_t(x,t) + \max_{a\in A}\{-D\phi(x,t)\cdot f(x,a)\} \le 0$$

which was (T1).

<u>Step 3</u>. "SUPERSOL." $\phi\in C^1$, $V-\phi$ MIN at (x,t)

$0<t<T$. Goal: $\quad -\phi_t + H(D_x\phi, x)\Big|_{(x,t)} \ge 0 \quad$ (T2)

By "$\ge$" in DPP $\forall \varepsilon>0\ \exists \bar\alpha\in Q$:

$$V(x,t) \ge V(\bar y(t+h), t+h) - \varepsilon h$$

$\bar y(s) = y_x(s;\bar\alpha, t)$. $\qquad (V-\phi)(x,t) \le (V-\phi)(z,s)$

$$\Rightarrow \quad \phi(x,t) - \phi(\bar y(t+h), t+h) \ge V(x,t) - V(\bar y(t+h), t+h)$$

$(\ast)$ $\qquad\qquad\qquad\qquad\qquad \ge -\varepsilon h$

Want to divide by $h>0$ & let $h\to 0$,

but now $\dot{\bar y}(t)$ MAY NOT EXIST.

<u>Lect</u>. 4/<u>17</u>/13

$$\phi(x,t) - \phi(\bar y(t+h), t+h) = -\int_t^{t+h} \frac{d}{ds}\phi(\bar y(s), s)\, ds$$

$$= -\int^{t+h} [\phi_t(\bar y(s), s) + D_x\phi(\bar y(s), s)\cdot f(\bar y(s), \bar\alpha(s))]\, ds$$

$$= -\int_t^{t+h} \Big[\underbrace{\phi_t(\bar y(s),s)}_{= \ \phi_t(x,t)+o(1)} + \underbrace{D_x\phi(\bar y(s),s)}_{= \ D\phi(x,t)+o(1)} \cdot \underbrace{f(\bar y(s),\bar\alpha(s))}_{= \ f(x,\bar\alpha(s))+O(h)} \Big]\, ds$$

as $h \to 0+$

$$\boxed{\bar y(s) - x = O(h)}$$

$$= o(h) - h\,\phi_t(x,t) + \int_t^{t+h} \underbrace{-D\phi(x,t)\cdot f(x,\bar\alpha(s))}_{\le \max_{a\in A}[-D\phi(x,t)\cdot f(x,a)] = H(D\phi(x,t),x)}\, ds$$

Then

$$-\varepsilon \frac{\cancel{h}}{\cancel{h}} \le \frac{o(h)}{h} - \frac{\cancel{h}}{\cancel{h}}\phi_t(x,t) + \frac{\cancel{h}}{\cancel{h}} H(D\phi(x,t),x)$$

$h \searrow 0 \qquad -\varepsilon \le -\phi_t(x,t) + H(D\phi(x,t),x)$

$\varepsilon \searrow 0 \Rightarrow 0 \le \ldots\ldots \qquad$ which is (T2)

<u>Step 4</u> By the Rmk. we get uniqueness from the Comp. Princ. for Cauchy pb. if H is "good enough". Check the structure conditions

$$(SH)\quad \begin{cases} |H(x,p) - H(y,p)| \le C|x-y|(1+|p|) \\ |H(x,p) - H(x,q)| \le C|p-q| \end{cases}$$

Take $\bar a$: $H(x,p) = -f(x,\bar a)\cdot p$

$$H(x,p) - H(y,p) \le -f(x,\bar a)\cdot p + f(y,\bar a)\cdot p$$

$$\le |p|\,|f(x,\bar a) - f(y,\bar a)| \le |p| L |x-y|$$

$$H(x,p) - H(x,q) \le -f(x,\bar a)\cdot p + f(x,\bar a)\cdot q$$
$$\le |f(x,\bar a)|\,|p-q| \le M|p-q|.$$

By exchanging x & y, p & q we get (SH). ∎

Verification Theorem & Synthesis of an optimal FEEDBACK

Assume $\exists\ w \in C(\mathbb{R}^n \times [0,T]) \cap C^1(\mathbb{R}^n \times]0,T[)$

solving

$$\begin{cases} -w_t + H(D_x w, x) = 0 & 0<t<T \\ w(x,T) = g(x) \end{cases}$$

Then $w_t + \min_{a\in A} D_x w \cdot f(x,a) = 0$

Take $a^*(x,t) \in \arg\min D_x w(x,t)\cdot f(x,\cdot)$

$a^* : \mathbb{R}^n \times]0,T[\to A$, & consider

$$(CP)\quad \begin{cases} \dot y(s) = f(y(s), a^*(y(s),s)) & 0<s<T \\ y(0) = x \end{cases}$$

and assume it has a sol. $y^*(\cdot)$, and

call $\alpha^*(s) \doteq a^*(y^*(s), s)$.

Then α^* is optimal for the min. pb with $y(0)=x$, i.e., $J(x,0,\alpha^*) \le J(x,0,\alpha)$ $\forall \alpha \in \mathcal{A}$

<u>Pf.</u> Let $y(s) = y_x(s;\alpha,0)$

$$\frac{d}{ds} w(y(s),s) = w_t + Dw \cdot f(y(s),\alpha(s))$$
$$\ge w_t + \underbrace{\min_a Dw \cdot f(y(s),a)}_{H(Dw,\,y(s))} = 0$$

$$\Rightarrow \quad w(x,0) \le w(y(T),T) = g(y(T)) = J(x,0,\alpha)$$

By construction of α^* $\quad \frac{d}{ds} w(y^*(s),s) = 0$

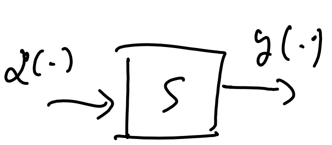
$$\Rightarrow \quad w(x,0) = g(y^*(T)) = J(x,0,\alpha^*)$$

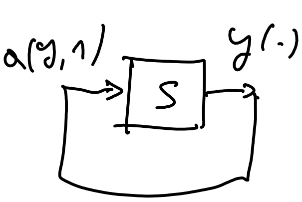
$$\Rightarrow \quad J(x,0,\alpha) \ge J(x,0,\alpha^*) \quad \forall \alpha \in \mathcal{A}. \ \blacksquare$$

<u>Comments</u>

- Verific. thm. because, if you have w, you can verify if $\bar\alpha$ is optimal by checking that $\bar\alpha(s) \in \arg\min Dw \cdot f(\bar y(s),\cdot)$ $\forall s$. $\bar y(s) = y_x(s;\bar\alpha,0)$.

- $\alpha^* : \mathbb{R}^n \times]0,T[\to A$ is called a FEEDBACK or CLOSED LOOP control





We built $\alpha^*(s) = a^*(y^*(s), s)$.
This is important for applications because feedbacks are more "ROBUST".

- DIFFICULTIES to apply the method
  - smooth sols of (CP) $\nexists$ in general
  - even if $\exists\, w \in C^1$ solving (CP) $\operatorname{argmin} Dw(x,t)\cdot f(x,\cdot)$ is not continuous in gen: have troubles to solve
    $$\dot{y} = f(y, a^*(y,s))$$

- The syntesis of optimal feedback via HJB eqs. works for LINEA-QUADRATIC problems, i.e.

$$\dot{y} = Ay + B\alpha \qquad \begin{array}{l} A, B \text{ matrices} \\ \alpha \in \mathbb{R}^m \end{array}$$

$$J(x,t,\alpha) = \int_t^T \Big[y(s)^T \underset{\substack{\uparrow \\ \mathrm{Sym}(N)}}{M}\, y(s) + \alpha(s)^T \underset{\substack{\uparrow \\ \mathrm{Sym}(M)}}{R}\, \alpha(s) \Big] ds + y(T)^T \underset{\substack{\uparrow \\ \mathrm{Sym}(N}}{Q}\, y(T)$$

Can prove that $w(x,t) = x^T K(t) x$
for $K(t) \in \mathrm{Sym}(N)$, K solves a matrix ODE

called RICCATI eq.

Here $a^*(x,t)$ is LINEAR in x ($\Rightarrow$ Lip)

Refs. • FLEMING-RISHEL book Springer 75

• notes of my course ED2 2011-12

• And for NON-LINEAR systems & NON-QUADRATIC COSTS?

By viscosity solutions theory can produce APPROXIMATE OPTIMAL FEEDBACKS

See e.g. Chpt. I & VI of BCD & Appendix (M.FALCONE).

History of viscosity sols:

• 1982 M. CRANDALL - P.L. LIONS
  1984 " - L.C. Evans - "

• Appl. to control I. CAPUZZO-DOLCETTA, Evans, P.L. Lions...

• STOCH. Control. P.L. Lions
  book by Fleming-Soner

• Many other applications of viscosity solutions to different fields, see, e.g.

Crandall, Ishii, Lions: Users' guide to visco. sols., Bull. A.M.S. 1992.