Lect April 8th, 2013

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Recall Hopf-Lax formula for solution of

$$(CP) \begin{cases} u_t + H(Du) = 0 & t > 0 \\ u = g & t = 0 \end{cases}$$

is $u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\}$

It remains to prove: u is a visco-supersol.

$\phi \in C^1$ s.t. $u - \varphi$ has a min at (x,t).

Let z: $u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$

$y = x - \frac{h}{t}(x-z)$ $\quad$ $u(y,s) \leq s L\left(\frac{y-z}{s}\right) + g(z)$

$s = t-h$. As in a previous proof

$u(x,t) - u(y,s) \geq (t-s) L\left(\frac{x-z}{t}\right)$

$(u - \phi)(x,t) \leq (u - \phi)(y,s)$

$u(x,t) - u(y,s) \leq \phi(x,t) - \phi(y,s) \quad \Rightarrow$

$$\phi(x,t) - \phi(y,s) \geq (t-s)\, L\left(\frac{x-z}{t}\right)$$

As in the pf. that u solves HJ at diff.ty points divide by $t-s = h$

$$\phi_t + D\phi \cdot \frac{x-z}{t} - L\left(\frac{x-z}{t}\right) \geq 0$$

$$\Rightarrow \quad \phi_t + H(D\phi)\Big|_{(x,t)} \geq 0 \qquad \blacksquare$$

CONSISTENCY (COERENZA in It.)
of visco. sols. with CLASSICAL SOLS.

More general PDEs :

(E) $\quad F(Du, u, x) = 0 \quad$ in $\Omega \subseteq \mathbb{R}^N$ OPEN SET

$F : \mathbb{R}^n \times \mathbb{R} \times \bar{\Omega} \to$ is CONT.

<u>Def.</u> (i) $u \in USC(\Omega)$ is a VISCO-SUBSOL. of (E) if $\forall\, \varphi \in C^1(\Omega)$ s.t. $u - \varphi$ has a MAX at $\bar{x}$ $\quad F(D\phi(\bar{x}), u(\bar{x}), \bar{x}) \leq 0$

(ii) $u \in LSC(\Omega)$ is a VISCO-SUPERSOL. of (E) if $\forall \varphi \in C^1(\Omega)$ s.t. $u-\varphi$ has a MIN at $\tilde{x}$ $F(D\phi(\tilde{x}), u(\tilde{x}), \tilde{x}) \geq 0$.

(iii) u is VISCO. SOL. if it is V-SUB- & SUPERSOLUTION

Recall a) $u \in LSC(\Omega)$ if $\forall x_0 \in \Omega$

$$u(x_0) \leq \liminf_{x \to x_0} u(x)$$

b) $u \in USC(\Omega)$ if $u(x_0) \geq \limsup_{x \to x_0} u(x)$.

Prop. (CONSISTENCY 1) (i) $u \in C(\Omega)$ has partial derivatives in Ω and solves (E) (in classical sense) $\Longrightarrow$ u is visco-soln.

(ii) u is visco soln. of (E) & $u \in C^1(\Omega)$ $\Longrightarrow$ u solves (E) in classical sense.

Pf. (i) "$\leq$" $\phi \in C^1(\Omega)$ $\bar{x} \in \operatorname{argmax}(u-\phi)$

$$D(u-\phi)(\bar{x})=0 \qquad F(Du(\bar{x}), u(\bar{x}), \bar{x})=0$$
$$\qquad\qquad\qquad\qquad \overset{\shortparallel}{D\phi(\bar{x})} \qquad \Rightarrow \leq 0$$

"$\geq$" the same! $\blacksquare$

(ii) $u \in C^1$ $\quad \phi = u$ is adm. test fn,

$u-\phi$ has max and min at each point of Ω.

"$\leq$" $\quad F(Du, u, x) \leq 0 \quad \forall x$

"$\geq$" $\quad F(Du, u, x) \geq 0 \quad \forall x$. $\boxed{}$

I want to prove that: u visco. soln. &

diff.le at $\bar{x} \in \Omega \Rightarrow F(Du(\bar{x}), u(\bar{x}), \bar{x}) = 0$

Recall: u diff.le at $\bar{x} \iff \exists p \in \mathbb{R}^N$:

$$u(x) = u(\bar{x}) + p\cdot(x-\bar{x}) + o(|x-\bar{x}|) \quad \text{as } x \to \bar{x}$$

<u>Def.</u> $u \in USC(\Omega)$, $p \in D^+u(x) \iff$

$$u(x) \leq u(\bar{x}) + p\cdot(x-\bar{x}) + o(|x-\bar{x}|) \quad \text{as } x \to \bar{x}$$

$u \in LSC(\Omega)$, $p \in D^-u(x) \iff$

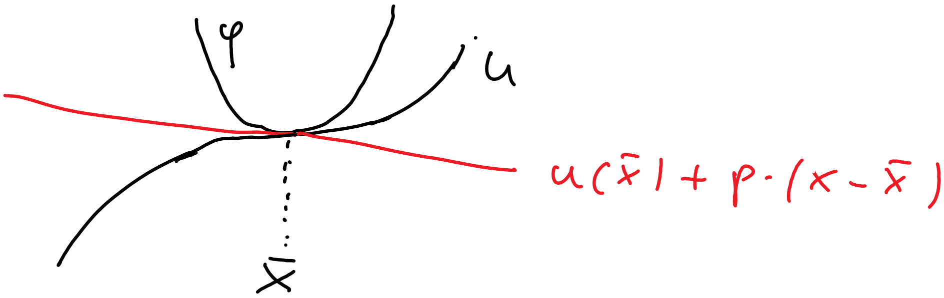
$u \in LSC(\Omega)$, $p \in D^- u(x) \iff$

$$u(x) \geq u(\bar{x}) + p\cdot(x-\bar{x}) + o(x-\bar{x}) \qquad x \to \bar{x}.$$

<u>RMK</u> u is diff.le at $\bar{x} \iff D^+u(\bar{x}) \cap D^-u(\bar{x}) \neq \emptyset$ and then $Du(\bar{x}) \in D^+u(\bar{x}) \cap D^-u(\bar{x})$.

<u>LEMMA</u> $p \in D^+u(\bar{x}) \iff \exists\, \varphi \in C^1(\Omega)$ s.t.

$u-\varphi$ has a max at $\bar{x}$, $u(\bar{x}) = \varphi(\bar{x})$, $p = D\varphi(\bar{x})$



Note that $(u-\phi)(x) \leq (u-\phi)(\bar{x}) = 0 \Rightarrow u(x) \leq \phi(x)$

<u>Pf.</u> "$\Rightarrow$" Elementary but not trivial, see Ch. II [BcD] or [Evans].

"$\Leftarrow$" $u-\varphi$ max at $\bar{x} \Rightarrow (u-\varphi)(x) \leq (u-\varphi)(\bar{x})$

$$u(x) - u(\bar{x}) \leq \varphi(x) - \varphi(\bar{x}) \underset{\text{Taylor}}{=} D\varphi(\bar{x})\cdot(x-\bar{x}) + o(|x-\bar{x}|) \quad \text{as } x \to \bar{x}$$

$\Rightarrow D\varphi(\bar{x}) \in D^+u(\bar{x})$. ∎

<u>Cor.</u> (CONSISTENCY 2) If u visco sol. of (E) & u diff.le at $\bar{x} \Rightarrow F(Du(\bar{x}), u(\bar{x}), \bar{x}) = 0$.

(E) & u diff.ble at $\bar{x}$ $\Rightarrow$ $F(Du(\bar{x}), u(\bar{x}), \bar{x}) = 0$.

Rmk. u visco sol. of (E) & $u \in Lip_{loc}(\Omega)$

$\Rightarrow$ u solves (E) A.E. in Ω

(by Rademacher + Cor.)

Pf of Cor. $Du(\bar{x}) \in D^+u(\bar{x}) \cap D^-u(\bar{x})$.

The LEMMA applied to $-u$.. gives:

$p \in D^-u(\bar{x}) \iff \exists \varphi \in C^1 : (u-\varphi)$ has a MIN at $\bar{x}$
$u = \varphi$, $p = D\varphi(\bar{x})$.

$p \in D^+u(\bar{x}) \iff$
$\exists \varphi \in C^1$ $(u-\varphi)$ MAX at $\bar{x}$ & $Du(\bar{x}) = D\varphi(\bar{x})$

$$\left.\begin{array}{ll} \text{"}\le\text{"} & F(\underset{\underset{Du(\bar{x})}{\parallel}}{D\varphi(\bar{x})}, u(\bar{x}), \bar{x}) \le 0 \\ \text{"}\ge\text{"} & F(\underset{\underset{Du(\bar{x})}{\parallel}}{D\varphi(\bar{x})}, u(\bar{x}), \bar{x}) \ge 0 \end{array}\right\} \Rightarrow F(Du(\bar{x}), u(\bar{x}), \bar{x}) = 0$$

∎

Equivalent DEF.: By the Lemma:

u visco sub.sol. of (E) $\iff$

$F(p, u(x), x) \le 0 \quad \forall p \in D^+u(x)$

$$F(p, u(x), x) \le 0 \quad \forall p \in Du(x)$$

u VISCO SUPER SOL. of (E) $\iff$

$$F(p, u(x), x) \ge 0 \quad \forall p \in D^- u(x).$$

∎

STABILITY of VISCO. SOLS. W.R.T. UNIFORM CONVERGENCE

Prop. $u_n \in C(\Omega)$ visco. sol. of

$$F_n(Du_n, u_n, x) = 0 \quad \text{in } \Omega$$

$$F_n \in C(\mathbb{R}^N \times \mathbb{R} \times \bar{\Omega}).$$

$u_n \to u$ loc. uniformly in Ω.

$F_n \to F$ loc unif in $\mathbb{R}^N \times \mathbb{R} \times \bar{\Omega}$

$\Longrightarrow$ u is a visco sol. of $F(Du, u, x) = 0$

RMK Even if $F_n = F$ $\forall n$ this property is not true for classical solutions.

Proof "$\le$". Goal: $\forall \varphi$ $\forall \bar{x} \in \operatorname{argmax}(u - \varphi)$

Proof "⊆". Goal: $\forall \varphi$ $\forall \bar{x}$ ... (cut off)

STRICT MAX $\quad F(D\varphi(\bar{x}), u(\bar{x}), \bar{x}) \leq 0$

Look at $u_n - \varphi$: we know that $u_{n_k} - \varphi$ has a max at x_k : $x_k \to \bar{x}$. By def. of u_{k_n} SUB sol

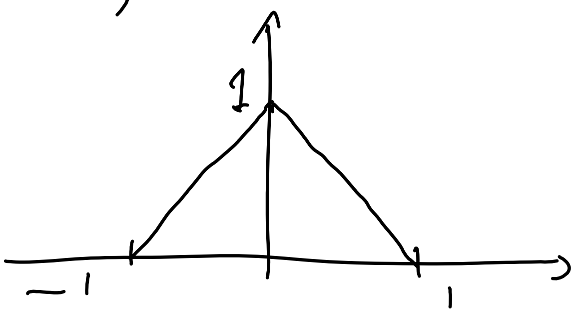
$$F_{h_n}(D\varphi(x_k), u_{n_k}(x_k), x_k) \leq 0$$
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$
$$F(D\varphi(\bar{x}), u(\bar{x}), \bar{x}) \leq 0 \qquad \blacksquare$$

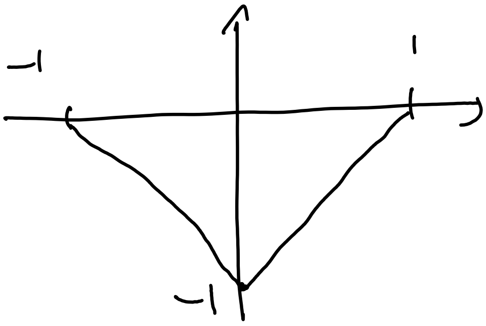
IMPORTANT EXAMPLE :

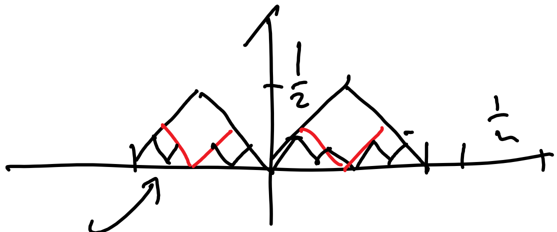
$$\begin{cases} |u'| - 1 = 0 & \text{in }]-1, 1[\\ u(-1) = 0 = u(1) \end{cases}$$

EIK. Eq. in $N = 1$

" $u' = \pm 1$ "







u_n solve a.e. $|u_n'| = 1$

$u_n \to 0$ uniformly in $[-1, 1]$
$\quad \overset{\shortparallel}{u}$

$u_n \to \underset{u}{\underset{\parallel}{U}}$ uniformly in [ ]

BUT $u \equiv 0$ doesn't solve $|u'|=1$ at ANY point.

<u>Exe: HW</u> Are some of these visco. sols.?
WHICH ONE?

<u>Lect. April 9. 2013</u>

Is $u(x) = 1-|x|$ sol. of $|u'|-1=0$?

No problem $\forall x \neq 0$.

$D^+u(0) = [-1,1] \qquad D^-u(0) = \emptyset$

Then u sats. the supersol. cond. at 0.

Subsol.? $\quad |p|-1 \leq 0 \quad \forall p \in D^+u(0)$

YES $\Rightarrow$ u is a viscosol.

<u>Next</u>: $v(x) = |x|-1$

$D^+v(0) = \emptyset \qquad D^-v(0) = [-1,1]$.

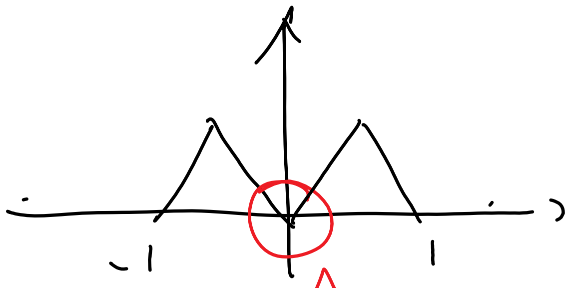
$\Rightarrow$ v is subsol.

Super? $\quad |p|-1 \overset{?}{\geq} 0 \quad \forall p \in D^-v(0)$

SUPERSOL.? $\quad |p|-1 \overset{?}{\geq} 0 \quad \forall p \in D^- v(0)$

No, because $0 \in D^- v(0)$.

What about



$D^- u(0) = [-1, 1]$

the SUPERSOL COND. IS FALSE.

Conclusion: Among these explicit a.e. solutions only one is a visco. sol.

We'll see that in fact (DP) has a unique visco sol.

RMK 1. $\quad u(x) = 1 - |x| = \mathrm{dist}(x, \partial\Omega)$

$\Omega =]-1, 1[$

RMK 2. Note that among all the functions above $v(x) = |x| - 1$ is the only visco. sol. of $\quad 1 - |v'| = 0$

Subsol. because $D^+ v(0) = \emptyset$

Subsol. because $D^+v(0) = \emptyset$

Supersol because $D^+v(0) = [-1, 1]$

$$1 - |p| \geq 0 \quad \forall p \in D^+(0).$$

Note that $|u'| - 1 = 0$ & $1 - |u'| = 0$

do NOT have the same solutions !!

<u>RMK 3</u> u subsol. of $F(Du, u, x) = 0$

$\forall \varphi$: $u - \varphi$ has a max at x $F(D\varphi, u, x)|_x \leq 0$

$V = -u$ $\quad V - (\underbrace{-\varphi}_{\psi}) = -(u - \varphi)$ has a MIN at x

$\Rightarrow \forall x \in \mathrm{argmin}(V - \psi)$ $\quad F(-D(-\psi)(x), -V(x), x) \leq 0$

$\Rightarrow \quad$ '' $\quad -F(-D(-\psi)(x), -V(x), x) \geq 0$

$\Rightarrow V = -u$ is a SUPERSOL of

$$-F(-DV, -V, x) = 0.$$

<u>RMK.</u> This is due to the fact that the inequalities in def of viscosol. "ORIGINATE" from the vanishing viscosity method

from the vanishing viscosity method.

Then visco sol. of $F(Du, u, x) = 0$

is related to

$$F(Du^\varepsilon, u^\varepsilon, x) = \varepsilon \Delta u^\varepsilon \qquad (E^1_\varepsilon)$$

Instead viscosols. of $-F(\quad) = 0$

are related to

$$-F(Dv^\varepsilon, v^\varepsilon, x) = \varepsilon \Delta v^\varepsilon \qquad (E^2_\varepsilon)$$

and (E^1_ε) is NOT equivalent to (E^2_ε)!

Exer. Solve
$$\begin{cases} |u_\varepsilon'| - 1 = \varepsilon u_\varepsilon'' & \text{in }]-1,1[\\ u_\varepsilon(-1) = 0 = u_\varepsilon(1) \end{cases}$$

and show that $u_\varepsilon \to u(x) = 1 - |x|$.

Instead
$$\begin{cases} 1 - |v_\varepsilon'| = \varepsilon v_\varepsilon'' \\ v_\varepsilon(-1) = 0 = v_\varepsilon(1) \end{cases}$$

$$v_\varepsilon \to v(x) = |x| - 1 = -u(x).$$

COMPARISON PRINCIPLES

COMPARISON PRINCIPLES

FIRST PROBLEM.

(SE) $\quad u + H(Du, x) = 0 \quad$ in $\Omega \subseteq \mathbb{R}^n$ BOUNDED

<u>Want</u> u SUBSOL, v SUPERSOL. of (SE)

& $u \le v$ on $\partial\Omega \Rightarrow u \le v$ in $\bar{\Omega}$

If sub & supersol. are C^1 this follows from

$\Phi(x) = u(x) - v(x)$ has a max at $\bar{x}$

<u>Case 1</u> $\bar{x} \in \partial\Omega \Rightarrow \Phi(\bar{x}) = \max \Phi \le 0$

<u>Case 2</u> $\bar{x} \in \Omega \Rightarrow Du(\bar{x}) = Dv(\bar{x}) = \bar{p}$

$$u(\bar{x}) + H(\bar{p}, \bar{x}) \le 0 \le v(\bar{x}) + H(\bar{p}, \bar{x})$$

$\Rightarrow u(\bar{x}) - v(\bar{x}) \le 0 \quad \Rightarrow \Phi(\bar{x}) \le 0$

$\Rightarrow u \le v$ in $\bar{\Omega}$.

<u>Thm</u> (COMPARISON PRINC. for (SE)).

$\Omega \subseteq \mathbb{R}^n$ OPEN & BOUNDED, $u, v \in C(\bar{\Omega})$,

u visco. subsol. of (SE), v visco supers. of (SE),

u visco. subsol. of (SE), v visco supers. of (SE), $u \leq v$ on $\partial\Omega$. $H: \mathbb{R}^N \times \bar{\Omega} \to \mathbb{R}$ cont. &

$$|H(p,x) - H(p,y)| \leq \omega(|x-y|(1+|p|)) \quad (RH)$$

where $\omega: [0,+\infty[\to [0,+\infty[$, cont. at 0, $\omega(0)=0$

THEN $u \leq v$ in $\bar{\Omega}$.

<u>RMK 1</u>. (RH) is condition of UNIFORM CONT.TY of H in x, unif. in p. Examples:

$$H(p,x) = H_1(p) + f(x), \quad H_1, f \in C$$

$$H(p,x) = g(x)|p| + f(x), \quad g \in Lip, \ f \in C$$

$$|H(p,x) - H(p,y)| \leq |p||g(x) - g(y)| + |f(x) - f(y)|$$
$$\leq L_g |x-y||p| + \omega_f(|x-y|) \qquad \square$$

<u>COR.</u> There is at most one visco sol. in $C(\bar{\Omega})$ of (DP) $\begin{cases} u + H(Du,x) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

<u>Pf</u>. u, v sols. $u = v$ on $\partial\Omega$

Comp Princ $\Rightarrow \begin{cases} u \leq v \\ v \leq u \end{cases} \Rightarrow u = v. \qquad \blacksquare$

comp. princ. $\Rightarrow \begin{cases} u \le v \\ v \le u \end{cases} \Rightarrow u = v$. $\blacksquare$

<u>Proof of Thm.</u> Idea (Kružkov): doubling the variables: $\varepsilon > 0$

$$\Phi_\varepsilon(x,y) \doteq u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon}$$

has a max at $(x_\varepsilon, y_\varepsilon) \in \bar\Omega \times \bar\Omega$.

$$\max_{\bar\Omega}(u-v) = \max \Phi_\varepsilon(x,x) \le \max_{\bar\Omega\times\bar\Omega} \Phi_\varepsilon =$$

$$= \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \le u(x_\varepsilon) - v(y_\varepsilon)$$

<u>GOAL</u> $\liminf_{\varepsilon \to 0+} \left(u(x_\varepsilon) - v(y_\varepsilon)\right) \le 0$.

$$\Phi(x_\varepsilon, x_\varepsilon) \le \Phi(x_\varepsilon, y_\varepsilon) \Rightarrow$$

$$\cancel{u(x_\varepsilon)} - v(x_\varepsilon) \le \cancel{u(x_\varepsilon)} - v(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon}$$

$$\Rightarrow \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \le v(x_\varepsilon) - v(y_\varepsilon) \qquad (I1)$$

$\therefore \; | \quad |^2 \le 2\varepsilon \; 2 \sup |v|$

$\Rightarrow \quad |x_\varepsilon - y_\varepsilon|^2 \leq 2\varepsilon \; 2 \sup_{\bar\Omega} |V|$

$\Rightarrow \quad |x_\varepsilon - y_\varepsilon| \to 0 \quad$ as $\varepsilon \to 0 \quad (I2)$

By $(I1)$ & $V \in C(\bar\Omega) \Rightarrow$

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0. \qquad (I3)$$

<u>Case 1</u>. $\exists\, (x_{\varepsilon_n}, y_{\varepsilon_n}) \in \partial(\bar\Omega \times \bar\Omega)$, with $\varepsilon_n \to 0$

<u>Case 2</u> $\quad (x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega \quad \forall \varepsilon \in]0, \bar\varepsilon]$.

Case 1: either $x_{\varepsilon_n} \in \partial\Omega$ or $y_{\varepsilon_n} \in \partial\Omega$

If $x_{\varepsilon_n} \in \partial\Omega$

$$u(x_{\varepsilon_n}) - V(y_{\varepsilon_n}) \overset{\text{B.C.}}{\leq} V(x_{\varepsilon_n}) - V(y_{\varepsilon_n}) \to 0$$

by $(I2)$ & cont.ty of V.

If $y_{\varepsilon_n} \in \partial\Omega$

$$u(x_{\varepsilon_n}) - V(y_{\varepsilon_n}) \overset{\text{Bound. cond.}}{\leq} u(x_{\varepsilon_n}) - u(y_{\varepsilon_n}) \to 0$$

by cont.ty of u.

Case 2: $\quad \varphi(x) := V(y_\varepsilon) + \dfrac{|x - y_\varepsilon|^2}{2\varepsilon} \quad \in C^\infty$

$u - \varphi$ has a max at $x = x_\varepsilon$

$$\psi(y) \doteq u(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon} \in C^\infty$$

$-v + \psi$ has a max at $y = y_\varepsilon$, i.e. $v - \psi$ has a MIN at $y = y_\varepsilon$

$$D\varphi(x_\varepsilon) = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} = D\psi(y_\varepsilon)$$

u V-SUBSOL., v V-SUPERSOL. $\Rightarrow$

$$u(x_\varepsilon) + H\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, x_\varepsilon\right) \leq 0 \leq v(y_\varepsilon) + H\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, y_\varepsilon\right)$$

$$\Rightarrow u(x_\varepsilon) - v(y_\varepsilon) \underset{(RH)}{\leq} \omega\left(|x_\varepsilon - y_\varepsilon|\left(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\right)\right)$$

$$= \omega\Big(\underbrace{|x_\varepsilon - y_\varepsilon|}_{(I2)\ \downarrow} + \underbrace{\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}}_{(I3)\ \searrow 0}\Big) \xrightarrow[\varepsilon \to 0]{} 0$$

$\blacksquare$

APPLICATION TO THE EIKONAL EQ.

Thm. $\Omega \subseteq \mathbb{R}^N$ open bounded. Then

<u>Thm.</u> $\Omega \subseteq \mathbb{R}^N$ open bounded. Then

$u(x) := \text{dist}(x, \partial\Omega)$ is the unique visco. sol.

of (DP) $\begin{cases} |Du| - 1 = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$

<u>Pf</u>: tomorrow. Need

<u>Prop.</u> $u \in C(\Omega)$ v. sol. of $F(Du, u, x) = 0$ in Ω.

$\Phi \in C^1(\mathbb{R})$, $\Phi'(t) > 0$. Then $v = \Phi(u)$ is a

visco sol. of

$$F(\Psi'(v) Dv, \Psi(v), x) = 0 \quad \text{in } \Omega$$

where $\Psi = \Phi^{-1}$.

<u>Rmk</u> $u \in C^1$ $\begin{cases} u = \Psi(v), \\ Du = \Psi'(v) Dv \end{cases}$.

<u>LECTURE April 10th, 2013</u>

<u>Proof</u> of the Prop.: "$\leq$" Fix $x \in \Omega$,

$p \in D^+v(x)$. <u>GOAL</u>: $F(\Psi'(v)p, \Psi(v), x) \leq 0$

By def. of D^+v: $v(y) \leq v(x) + p\cdot(y-x) + o(|y-x|)$

By def. of D^+v: $\quad v(y) \le \underbrace{v(x) + p\cdot(y-x) + o(|y-x|)}$ as $y \to x$

$$\Psi(v(y)) \le \Psi\big(\qquad \big)$$

$$= \Psi(v(x)) + \Psi'(v(x))\big[p\cdot(y-x) + o(|y-x|)\big]$$

Taylor for Ψ

$$+\, o\big(|p\cdot(y-x)| + o(|y-x|)\big)$$

$$= \Psi(v(x)) + \Psi'(v(x))\, p\cdot(y-x) + o(|y-x|)$$

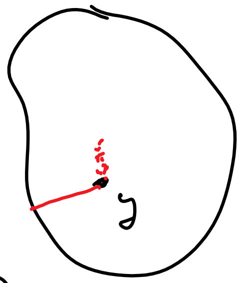
$$\Rightarrow \Psi'(v(x))\,p \in D^+(\Psi\circ v)(x) = D^+u(x)$$

u SUBSOL $\Rightarrow F\big(\Psi'(v(x))\,p,\ \underset{\Psi(v(x))}{\overset{\shortparallel}{u(x)}},\ x\big) \le 0$ $\blacksquare$

<u>Proof of Thm.</u> <u>Step 1</u> $u(x) \doteq dist(x, \partial\Omega) \doteq$

$= \overset{\min}{\cancel{\inf}}\{|y-x| : y \in \partial\Omega\}$ is a SUBSOL. of

$$|Du| - 1 = 0 .$$



$$u(x) \le |x-y| + u(y)$$

$\forall y \in \Omega$

$$|x-y| \ge u(x) - u(y) \ge \varphi(x) - \varphi(y) \quad =$$

$\varphi \in C^1$ $\quad \forall \varphi$: $u(x) - \varphi(x) \geq u(y) - \varphi(y)$, i.e.

$u - \varphi$ max at x

Taylor $= D\varphi(x) \cdot (x-y) + o(|x-y|)$ as $y \to x$

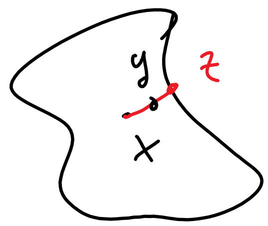
$\Rightarrow 1 \geq D\varphi(x) \cdot q + o(1)$ as $y \to x$ $\quad q = \dfrac{x-y}{|x-y|}$

$\Rightarrow 1 \geq D\varphi(x) \cdot q \quad \forall q, |q| = 1$

$\Rightarrow 1 \geq |D\varphi(x)| \Rightarrow u$ visco. sub sol. $\blacksquare$

<u>Step. 2</u> u is SUPERSOL. Fix $x \in \Omega$, $z \in \partial\Omega$ such that

$u(x) = |x - z|$



z = a point of min. distance

Choose $y \in [x, z]$

i.e. $y = x + t \dfrac{z-x}{|z-x|} \quad 0 \leq t \leq |z-x|$

$u(x) = |x-y| + |y-z| \geq |x-y| + u(y)$

$|x-y| \leq u(x) - u(y) \leq \varphi(x) - \varphi(y) =$

$\varphi \in C^1$ $\quad$ if $(u-\varphi)$ has a MIN at x

Taylor $= D\varphi(x) \cdot (x-y) + o(|x-y|)$ as $\cancel{y} \to x$
$\Leftrightarrow t \to 0$

$\longrightarrow$

$\Leftarrow$ $t \to 0$

$$\Rightarrow \quad 1 \leq D\varphi(x) \cdot \frac{x-z}{|x-z|} + o(1) \quad \text{as } y \to x$$

$$\Rightarrow \quad 1 \leq \underbrace{\qquad\qquad}_{\;} \leq |D\varphi(x)| \quad . \quad \blacksquare_{S.2}$$

S.3 Of course $u(x) = 0$ if $x \in \partial\Omega$,

$u \in C(\bar{\Omega})$ (in fact Lip with Lip-constant $= 1$)

$\Rightarrow$ u solves visco-sense (DP).

S.4 Kružkov transform: $\Phi(r) = 1 - e^{-r}$

$\Phi \in C^\infty$, $\Phi' = e^{-r} > 0$, $\Psi := \Phi^{-1} :]0, +\infty[\to \mathbb{R}$

$\Psi(t) = -\log(1-t)$, $\Psi'(t) = \dfrac{1}{1-t}$. By Prop.

if v, w are $\overset{\text{cont. \&}}{\text{resp.}}$, sub & super sol. of (DP)

$\Rightarrow$ $V := \Phi(v)$, $W := \Phi(w)$ $\quad V = W = 0$ on $\partial\Omega$

V, W are, resp., sub. & supersol. of

$$\left| \frac{DV}{1-V} \right| - 1 = 0 \quad \Leftrightarrow \quad \frac{|DV|}{1-V} - 1 = 0$$

for $V < 1$

$\iff \quad |DV| + V - 1 = 0 \quad \text{in } \Omega.$

$1 - V < 0$ new term

The Comparison Principle proved yesterday $\Rightarrow$

$V \le W$ in Ω $\Rightarrow$ $v \le w$ in Ω.

then the Comparison Principle holds for

$|Du| - 1 = 0$ $\Rightarrow$ Dir. Probl. (DP) has

at most 1 visco-sol. $\blacksquare$

<u>RMK</u>. Same proof gives the Comp. Princ.

for $\quad |Du| - n(x) = 0 \quad$ if $n > 0$ in Ω.

If $n(x_0) = 0$, $x_0 \in \Omega$, $n(x) > 0 \ \forall x \neq x_0$

there are examples of NON-UNIQUENESS

(even classical soln, see [BCD]). $\blacksquare$

COMPARISON FOR CAUCHY PROBLEMS.

<u>Thm</u>. $u, v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ Lip. &

hold such a sub- & a supersol. of

$$u_t + H(D_x u) = 0 \quad \text{in } \mathbb{R}^n \times]0,T[\,,$$

$H \in C(\mathbb{R}^n)$, $u(x,0) \leq v(x,0) \quad \forall x \in \mathbb{R}^n$.

Then: $u \leq v \quad \forall x,t$.

Pf. postponed.

COROLLARY: H CONV. & SUPERL., $g : \mathbb{R}^n \to \mathbb{R}$ Lip & bdd, $u(x,t) := \begin{cases} \min_y \left\{ t H^*\left(\frac{x-y}{t}\right) + g(y) \right\}, & t > 0, \\ g(x) & \text{at } t = 0 \end{cases}$

the Hopf-Lax formula. Then u is the UNIQUE VISCO. SOL. of (CP) $\begin{cases} u_t + H(D_x u) = 0 & t > 0 \\ u = g & \text{at } t = 0 \end{cases}$

among functions Lip. & bdd. on $\mathbb{R}^n \times [0,T]$ $\forall T > 0$.

Proof of a variant of COMP. THM.

u, v Lip. & bdd. in $\mathbb{R}^n \times (0,\infty)$ & sub. & supersol. in $\mathbb{R}^n \times (0,\infty)$.

Hopf-L-formula is globally bdd. in time if, in addition, $H^*(q) \geq 0$: Proof Exercise.

Step. 0

$f(x) = \delta_0 (1 + |x|^2)$, $|Df| \leq 2$.

$\beta, \eta, \varepsilon > 0$

$$\Phi_\varepsilon(x,y,t,s) \doteq u(x,t) - v(y,s) - \frac{|x-y|^2}{2\varepsilon} - \frac{|t-s|^2}{2\varepsilon}$$
$$- \beta\left(f(x)+f(y)\right) - \eta(t+s)$$

$\Phi_\varepsilon \to -\infty$ as $|x|$ or $|y|$ or t or $s \to +\infty$.

S.1 By contradiction ass. $\exists\, (x_0,t_0),\ \delta > 0$,

$$(u-v)(x_0,t_0) = \delta > 0. \quad \Rightarrow$$

$$\Phi(x_0,x_0,t_0,t_0) = \delta - 2\beta f(x_0) - 2\eta t_0 \geq \frac{\delta}{2}$$
for $\beta, \eta \leq \bar{\beta}$

S.2 $\exists\, (\bar{x}, \bar{y}, \bar{t}, \bar{s})$ depending on ε, β, η

MAX of Φ , $\Phi(\bar{x},\bar{y},\bar{t},\bar{s}) \geq \frac{\delta}{2} > 0$.

S.3 $$\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} + \frac{|\bar{t}-\bar{s}|^2}{2\varepsilon} + \beta\left(f(\bar{x}) + f(\bar{y})\right) + \eta\,(\bar{t}+\bar{s}) \leq \sup(u-v) = C$$

$$\Rightarrow \quad |\bar{x}-\bar{y}| \leq \sqrt{2\varepsilon C}\ , \quad |\bar{t}-\bar{s}| \leq \sqrt{2\varepsilon C} \xrightarrow[\varepsilon \to 0]{} 0$$

S.4 $\Phi(\bar x,\bar x,\bar t,\bar t)+\Phi(\bar y,\bar y,\bar s,\bar s)\le 2\Phi(\bar x,\bar y,\bar t,\bar s)$

$$u(\cancel{\bar x,\bar t})-v(\bar x,\bar t)-2\cancel{\beta f(\bar x)}-2\cancel{\eta\bar t}+u(\bar y,\bar s)-v(\cancel{\bar y,\bar s})$$
$$-2\cancel{\beta f(\bar y)}-2\cancel{\eta\bar s}\le \cancel{2}u(\bar x,\bar t)-\cancel{2}v(\bar y,\bar s)-\frac{|\bar x-\bar y|^2}{\varepsilon}-\frac{|\bar t-\bar s|^2}{\varepsilon}$$
$$\cancel{-2\beta(f(\bar x)-f(\bar y))}\cancel{-2\eta(\bar t+\bar s)}$$

$$\underbrace{\frac{|\bar x-\bar y|^2}{\varepsilon}-\frac{|\bar t-\bar s|^2}{\varepsilon}}\le u(\bar x,\bar t)-u(\bar y,\bar s)+v(\bar x,\bar t)-v(\bar y,\bar s)$$
$$\le 2L\underbrace{(|\bar x-\bar y|+|\bar t-\bar s|)}\xrightarrow[\varepsilon\to 0]{}0$$

$$\frac{(|\bar x-\bar y|+|\bar t-\bar s|)^2}{2\varepsilon}\le\ \nabla\ \le$$

$$(a+b)^2\le 2(a^2+b^2)$$

$$\Rightarrow\ \frac{|\bar x-\bar y|+|\bar t-\bar s|}{2\varepsilon}\le 2L\qquad\Rightarrow\ \frac{|\bar x-\bar y|}{\varepsilon}\le 4L$$

Step. 4 Spse. $\bar t=0$

$$\underbrace{u(\bar x,0)-v(\bar y,\bar s)}\ge\Phi(\bar x,\bar y,0,\bar s)\ge\frac{\delta}{2}>0$$

Init. Cond. $\le v(\bar x,0)-v(\bar y,\bar s)\xrightarrow[\varepsilon\to 0]{}0$ $\otimes$

The case $\bar s=0$ is analogous.

Step. 5 $\quad \bar{t}, \bar{s} > 0$: I use the equation