giovedì 14 marzo 2013
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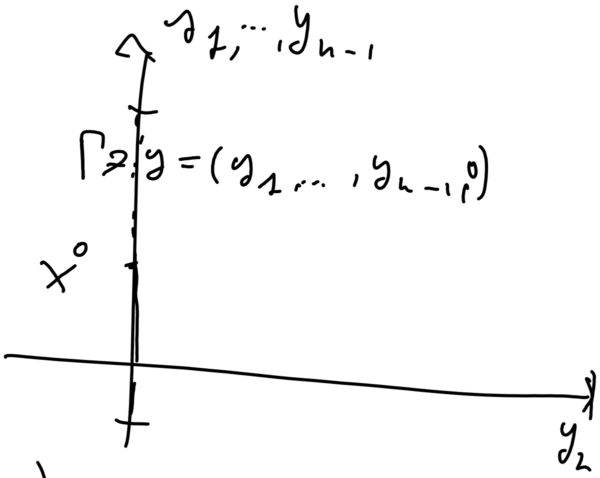
Lect. March 11, 2013 : ASSUME (p^0, z^0, x^0) ADMISS TRIPLE.

Look for initial conditions for the charact. eqs. (a - b - c)

$$\begin{cases} p(0) = q(y) \\ z(0) = g(y) \\ \bar{X}(0) = y \end{cases}$$



Look for $(q(y), g(y), y)$:

(1) $\quad q(x^0) = p^0$

(2) $\quad \begin{cases} q_i(y) = g_{x_i}(y) \qquad i = 1, \dots, n-1 \\ F(q(y), g(y), y) = 0 \end{cases}$

<u>Def.</u> (p^0, z^0, x^0) is NON-CHARACTERISTIC

if $\quad F_{p_n}(p^0, z^0, x^0) \neq 0$.

RMK. In general (p^0, z^0, x^0) is N-C if

$$F_p(p^0, z^0, x^0) \cdot \nu(x_0) \neq 0$$

↑ exterior normal.

Lemma. Ass. (p^0, z^0, x^0) ADMISS. & NON-CHARACT. $\Rightarrow \exists!\ q(\cdot)$ solving (locally) (1)–(2).

Pf. $G(y, q_n) := F(q_1(y), \dots, q_{n-1}(y), q_n, g(y), y)$

Look for $q_n(y)$: $\begin{cases} G(y, q_n(y)) = 0 \\ q_n(x^0) = p_n^0 \end{cases}$

Use Impl. Fn. Thm.: $G(x^0, p_n^0) \overset{?}{=} 0$

$$= F(p^0, \underset{\overset{\shortparallel}{z^0}}{g(x^0)}, x^0) = 0$$

$$\frac{\partial G}{\partial q_n}(x^0, p_n^0) = F_{p_n}(p^0, z^0, x^0) \neq 0 \quad \text{NON-CHARACTER.}$$

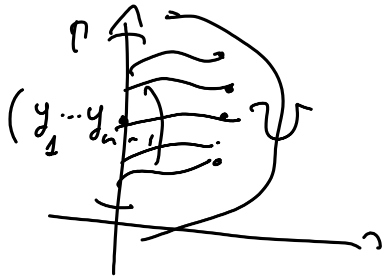
IFT $\Rightarrow \exists\ q_n \in C^1$: $\begin{cases} G(y, q_n(y)) = 0 \\ q_n(x^0) = p_n^0 \end{cases}$ ▨

I can solve (a-b-c) with INITIAL CONDITIONS

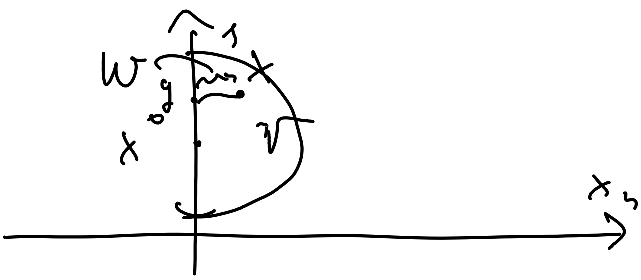
$$(IC)\begin{cases} p(0) = q(y) \\ z(0) = g(y) \\ \underline{X}(0) = y \end{cases}$$

We will assume $F \in C^3 \Rightarrow$ Solutions depend in a C^2 way from the data.

$$\begin{aligned} p(s) &= p(y_1, \dots, y_{n-1}, s) \\ z(s) &= z(\_\_\_\_\_\_) \\ \underline{X}(s) &= \underline{X}(\_\_\_\_\_\_) \end{aligned}$$



Lemma. Ass. $F \in C^3$, (p^0, z^0, x^0) ADM. & N-C. $\Rightarrow \exists\ I =]a, b[\ni 0$, V nbhd. of x^0, $W \subseteq \Gamma$ rel. nbhd of x_0 in Γ : $\forall x \in V$ $\exists$ UNIQUE $s \in I$, $y \in W$: $x = \underline{X}(y, s)$.



I. o. w. $\underline{X}$ is invertible & the inverse $x \mapsto (y, s)$ is C^2.

Pf. Use the Inverse function theorem.

Recall. $\underline{X}(y,s)$ = solution of (a, b, c) with (IC) at "time" s with init. condition y

Must check $\det D\underline{X}(x^0, 0) \neq 0$

$$\underline{X}(y,0) = (y,0) \Rightarrow D_y \underline{X}(x^0, 0) = I_{(n-1)\times(n-1)}$$

$$\frac{\partial \underline{X}}{\partial s}(x^0, 0) = F_p(p^0, z^0, x^0)$$

$$D\underline{X}(x^0,0) = \left(\begin{array}{c|c} I & \begin{matrix} F_{p_1} \\ \vdots \\ F_{p_{n-1}} \end{matrix} \\ \hline 0 & F_{p_n}(p^0, z^0, x^0) \end{array}\right)$$

$n-1$ $\quad$ $n-1$

$\Rightarrow \det D\underline{X}(x^0, 0) = F_{p_n}(p^0, z^0, x^0) \neq 0$ by N-C condition.

$\Rightarrow$ by InFT inverse exist & it is at least C^2. $\boxtimes$

Def. $(\underline{Y}, S)$ the local inverse of $\underline{X}$, i.e.

$$x = \underline{X}(y,s) \iff y \in \underline{Y}(x), \; s = S(x),$$

&

$$u(x) := z(\underline{Y}(x), S(x))$$

($u \in C^2$ because $\underline{Y}, S$ are C^2 & $z \in C^2$).

<u>Thm.</u> ($\exists$ of local solution of $F(Du,u,x)=0$)

The function u just defined solves

$$\begin{cases} F(Du, u, x) = 0 & \text{in } V \\ u(x) = g(x) & x \in \Gamma \cap V . \end{cases}$$

<u>Pf.</u> 1. By last Lemma $u(x) = z(\underline{Y}(x), S(x))$ is well-defined in V &

$x \in \Gamma$ $\quad u(x) = z(x, 0) = g(x)$. ☒

2. <u>Claim</u> $f(y,s) := F(p(y,s), z(y,s), \underline{X}(y,s)) \equiv 0$

$\forall s \in I \ \forall y$

$f(y,0) = F(p(y,0), z(y,0), \underline{X}(y,0))$ by (IC),

$= F(q(y), g(y), y) = 0$ by construction of q

$$\frac{\partial f}{\partial s}(y,s) = \sum_{j=1}^{n} F_{p_j} \dot{p}^j + F_z \dot{z} + \sum_{j=1}^{n} F_{x_j} \dot{\underline{X}}_j$$

(a, b, c)

$$= \sum^{n} F_{p_j} \left(-F_z p^j - F_{x_j} \right) + F_z \sum F_{p_j} p^j + \sum^{n} F \ F$$

$$= \sum_{j=1} F_{p_j}\Big(\underbrace{-F_z p^j}_{\otimes} \underbrace{-F_{x_j}}_{+}\Big) + \underbrace{F_z \sum_j F_{p_j} p^j}_{\otimes} + \underbrace{\sum_{j=1}^n F_{x_j} F_{p_j}}_{+}$$

$$= 0 \quad \Rightarrow \quad f(y,s) \equiv 0 \qquad \blacksquare$$

3. <u>Def.</u> $P(x) = p(Y(x), S(x))$

Plugging in step 2 $y = Y(x)$, $s = S(x)$

$$\Rightarrow F(P(x), u(x), x) = 0 \quad \forall x \in V$$

To conclude It remains to show only

$$P(x) = Du(x).$$

<u>4.</u>

$$\frac{\partial z}{\partial s}(y,s) = \dot{x} \cdot p = \sum_{j=1}^n p^j(y,s) \frac{\partial X_j}{\partial s}(y,s) \qquad (\star)$$

Moreover

$$\frac{\partial z}{\partial y_i}(y,s) = \sum_{j=1}^n p^j(y,s) \frac{\partial X_j}{\partial y_i}(y,s) \qquad i = 1, \dots, n-1 \qquad (\star\star)$$

NOT TRIVIAL ! PF. POSTPONED

<u>5</u> Complete the pf. assuming $(\star\star)$.

Goal: $Du = P$

$$\frac{\partial u}{\partial x_j} = \frac{\partial z}{\partial s}\frac{\partial S}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial z}{\partial y_i}\frac{\partial Y_i}{\partial x_j}$$

$(\star)$ $(\star\star)$

$$= \frac{\partial S}{\partial x_j} \sum_{k=1}^{n} p^k \frac{\partial X_k}{\partial s} + \sum_{i=1}^{n-1} \frac{\partial Y_i}{\partial x_j}\left(\sum_{k=1}^{n} p^k \frac{\partial X_k}{\partial y_i}\right)$$

$$= \sum_{k=1}^{n} p^k \left(\frac{\partial X_k}{\partial s}\frac{\partial S}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial X_k}{\partial y_i}\frac{\partial Y_i}{\partial x_j}\right) = (S)$$

Use $X(Y(x), S(x)) = x$

$$\frac{\partial X_k}{\partial x_j} = \sum_{i=1}^{n-1} \frac{\partial X_k}{\partial y_i}\frac{\partial Y_i}{\partial x_j} + \frac{\partial X_k}{\partial s}\frac{\partial S}{\partial x_j} = \delta_{kj} = \begin{cases} 1 & j=k \\ 0 & j\neq k \end{cases}$$

$\Rightarrow (S) = p_j \qquad \Rightarrow \frac{\partial u}{\partial x_j} = p_j \quad \forall j$

$\Rightarrow Du = p$. $\blacksquare$

Remaining step (next time):

prove $(\star\star)$. $\blacksquare$

Lect. March 13 - 2013

Complete the proof of main thm. by

Showing (☆☆).

$$r^i(s) \doteq \frac{\partial z}{\partial y_i}(y,s) - \sum_{j=1}^{n} p^j \frac{\partial \bar{X}_j}{\partial y_i}(y,s) \qquad i = 1,\dots,n-1$$

Goal: $r^i \equiv 0$. Note $\frac{\partial \bar{X}_j}{\partial y_i}(y,0) = \delta_{ij}$

$$r^i(0) = g_{x_i}(y) - p^i(0)$$

$$= g_{x_i}(y) - q^i(y) \underset{\uparrow}{=} 0$$

compatibil. cond.

$$\dot{r}^i(s) = \frac{\partial^2 z}{\partial y_i \partial s} - \sum_{j=1}^{n}\left[\frac{\partial p^j}{\partial s}\frac{\partial \bar{X}_j}{\partial y_i} + p^j \frac{\partial^2 \bar{X}_j}{\partial y_i \partial s}\right]$$

To simplify it: $\frac{\partial}{\partial y_i}$(☆)

the same ($\bar{X} \in C^2$)

$$\frac{\partial^2 z}{\partial s \partial y_i} = \sum_{j=1}^{n}\left[\frac{\partial p^j}{\partial y_i}\frac{\partial \bar{X}_j}{\partial s} + p^j \frac{\partial^2 \bar{X}_j}{\partial s \partial y_i}\right]$$

$$\Rightarrow \dot{r}^i(s) = \sum_{j=1}^{n}\left[\frac{\partial p^j}{\partial y_i}\frac{\partial \bar{X}_j}{\partial s} - \frac{\partial p^j}{\partial s}\frac{\partial \bar{X}_j}{\partial y_i}\right]$$

$$= \sum_j \left[\underbrace{\frac{\partial p^j}{\partial y_i} F_{p_j}}_{①} - \underbrace{\frac{\partial \bar{X}_j}{\partial y_i}\left(-F_z p^j - F_{x_j}\right)}_{②}\right]$$

To simplify: $\frac{\partial}{\partial y_i} F\left(p(y,s), z(y,s), \bar{X}(y,s)\right) = 0$

To simplify! $\frac{\partial}{\partial y_i} F(p(y,s), z(y,s), \bar{x}(y,s)) = 0$

$$\underbrace{\sum_j F_{p_j} \frac{\partial p^j}{\partial y_i}}_{1} + F_z \frac{\partial z}{\partial y_i} + \underbrace{\sum_j F_{x_j} \frac{\partial \bar{x}_j}{\partial y_i}}_{2} = 0$$

$$\Rightarrow \sum_j (1 + 2) = -F_z \frac{\partial z}{\partial y_i}$$

$$\Rightarrow \dot{r}^i(s) = F_z \left[\sum_j \frac{\partial \bar{x}_j}{\partial y_i} p^j - \frac{\partial z}{\partial y_i} \right] = -F_z\, r^i(s)$$

is LINEAR HOMOGENEOUS ODE, $r^i(0) = 0$

$\Rightarrow r^i(s) \equiv 0$. ∎

The proof of thm. is complete!! ∎

Applications & examples.

A. F LINEAR $\quad b(x) \cdot Du + c(x) u - f(x) = 0$

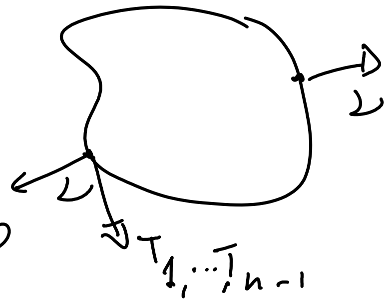
Already seen that it's enough to solve

$$\Big) \; \dot{\bar{x}} = b(\bar{x})$$

$$\begin{cases} \dot{X} = b(X) \\ \dot{z} = \ell(X) - c(X)\, z \end{cases}$$

Who are the non-charact. points of $\partial\mathcal{V}$?

$$F_p \cdot \nu \neq 0$$



Answ.: x^0: $\circledast$ $b(x^0) \cdot \nu(x^0) \neq 0$

ADMISSIBLE TRIPLES? (p^0, z^0, x^0)

$z^0 = g(x^0)$. Sie $\nu, T_1, \dots, T_{n-1}$ base ortonorm. of $\mathbb{R}^n$

$$p^0 \cdot T_i = \frac{\partial g}{\partial T_i}(x^0) \qquad i = 1, \dots, n-1$$

$$F(p^0, z^0, x^0) = 0 \quad : \quad b(x^0) \cdot p^0 + c(x^0)\, z^0 - \ell(x^0) = 0$$

$$b = b_\nu \nu + \sum b_{T_i} T_i \quad , \quad b_\nu = b \cdot \nu \neq 0$$

$$\Rightarrow \quad b_\nu p_\nu^0 + \sum_{i=1}^{n-1} b_{T_i} p_{T_i}^0 = \ell - c z^0$$

$\Rightarrow$ There is a unique p_ν^0 which makes (p^0, z^0, x^0)

ADMISS.:
$$p_\nu^0 = \frac{\ell - c z^0 - \sum b_{T_i} p_{T_i}^0}{b_\nu}$$

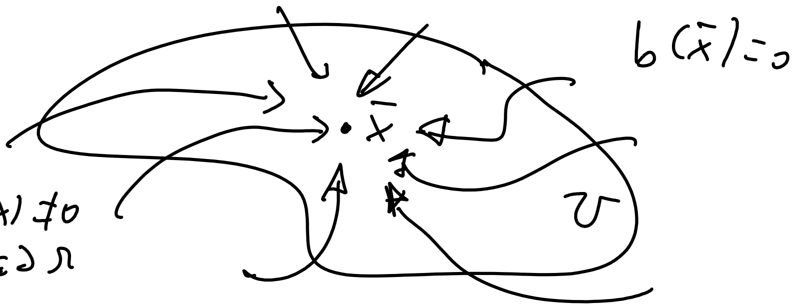
CONCLUSION! $\forall x^0$ s.t. $\circledast$ (NON-CHAR.)

$\forall g \in C^1$ we can build an ADMISS. TRIPLE & therefore the Thm. applies.

<u>N.B.</u> Projected characteristics $\underline{X}(\cdot)$ don't cross!

<u>Q</u> Can we build GLOBAL SOLUTIONS?

<u>Es. 1</u> $n=2$



$\dot{x} = b(x)$, $b(x)\cdot\nu(x) \neq 0$ $\forall x \in \partial\Omega$

$\exists$? sol. of $\begin{cases} b(x)\cdot Du = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$

Note that $u(\underline{X}(1)) \equiv g(\underline{X}(0))$, if $g \not\equiv$ cost.

u cannot be extended continuously in $\bar{x}$.

ANSW: No, there is no global solution.

<u>Ex. 2</u> $n=2$

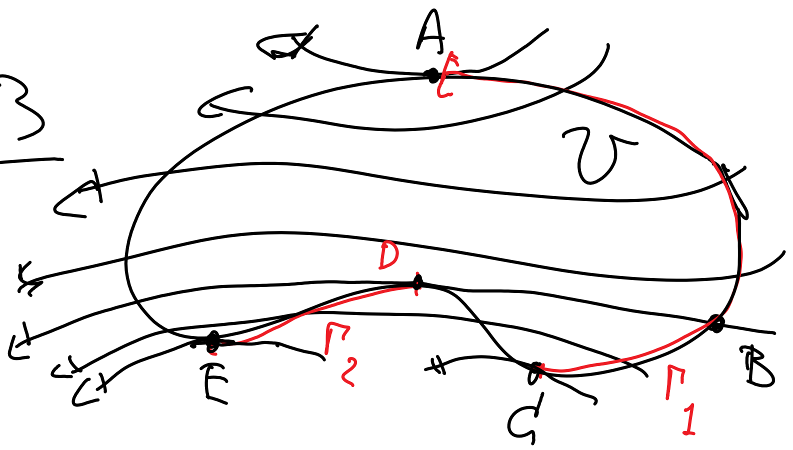
in A & B $b\cdot\nu = 0$

Then I choose $\Gamma = \{x \in \partial U : b(x)\cdot\nu(x) < 0\}$

The method gives global solution of

$$\begin{cases} b(x)\cdot Du = 0 & \text{in } U \\ u = g & \Gamma \end{cases}.$$

Es. 3



u discont. if $g(B) \neq g(D)$

Loc. existence fails near D. ∎

B. QUASILINEAR EQUATIONS

$$b(x,u)\cdot Du + c(x,u) = 0$$

Non-charact. points are such that

$$b(x^0, g(x^0))\cdot \nu(x^0) \neq 0$$

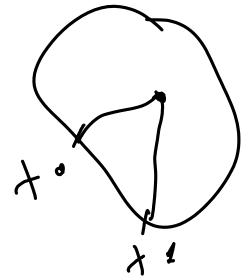
Easy to check that, if x^0 is N.C., $\exists$ unique ADMISS. TRIPLE (p^0, z^0, x^0).

$$\begin{cases} \dot{X} = b(X, z) \\ \dots \end{cases}$$

Here projected CHARACTERISTICS CAN CROSS.

$$\begin{cases} \dot{X} = b(X, z) \\ \dot{z} = -c(X, z) \end{cases}$$

where Γ ... CAN CROSS.



This is another cause of nonexistence of smooth solutions.

Example: SCALAR CONSERV. LAW (in n space vars.)

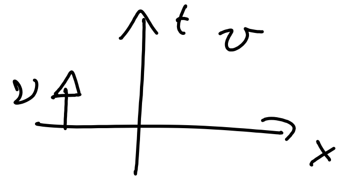
$$F(\underset{\shortparallel}{D_x u}\!\!\!\!\!\!\!\!Du, u_t, u, x, t) = u_t + \operatorname{div}_x f(u)$$

$$= u_t + f'(u) \cdot Du$$

for some $f : \mathbb{R} \to \mathbb{R}^n$ smooth.

Cauchy pb.: $V = \mathbb{R}^n \times]0, \infty[$, $\Gamma = \mathbb{R}^n \times \{0\}$.

$$\nu = (0, 1)$$



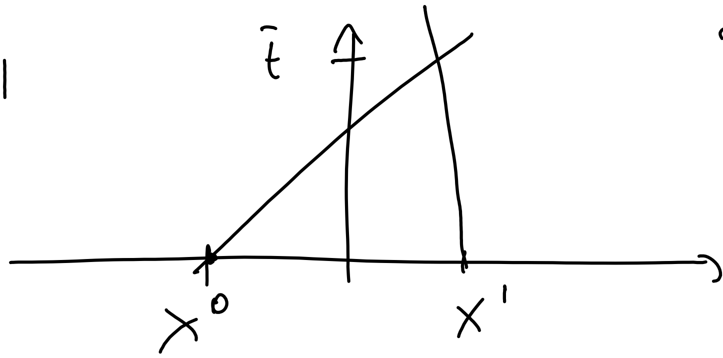
$$b(u) = (f'(u), 1) \qquad c \equiv 0$$

$$\begin{cases} \dot{\bar{X}}_i(s) = f'(z(s)) & i = 1, \dots, n \\ \dot{\bar{X}}_{n+1} = 1 & \Rightarrow \bar{X}_{n+1} = s \\ \dot{z} = 0 & \Rightarrow z(s) = z^0 = g(x^0) \end{cases}$$

$$\bar{X}(s) = x^0 + f'(g(x^0))\, s$$

Projected charact. are STRAIGHT LINES!

$n=1$

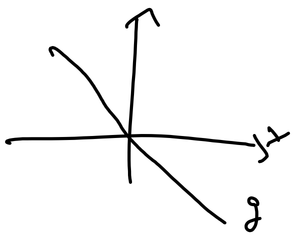


They can cross if $f'(g(x^0)) \neq f'(g(x^1))$

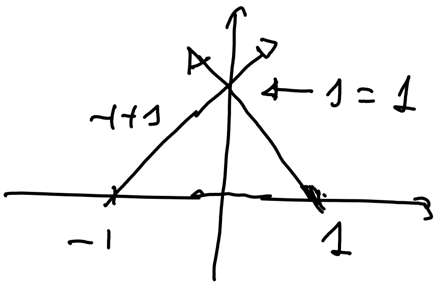
In such a case there is a discontinuity at $t = \bar{t}$ (this is a SHOCK!!).

<u>Sub Example</u> $n=1$ $f(u) = \frac{u^2}{2}$ $f'(u) = u$

Burger eq. $\begin{cases} u_t + u u_x = 0 \\ u(x,0) = g(x) = -x \end{cases}$



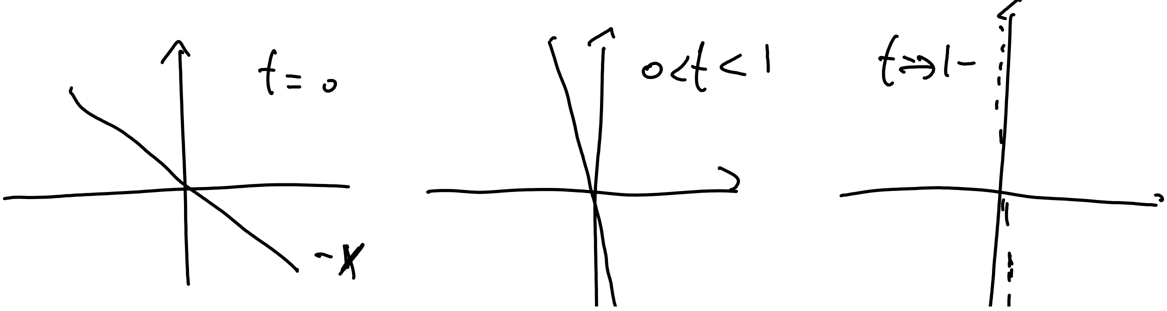
$x^0 = -1$, $x^1 = 1$.

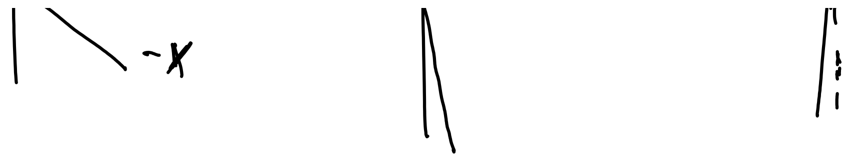


$-1 + s < 1 - s$

$\Leftrightarrow 2s < 2 \Leftrightarrow s < 1$

SHOCK occurs at time $t = 1$

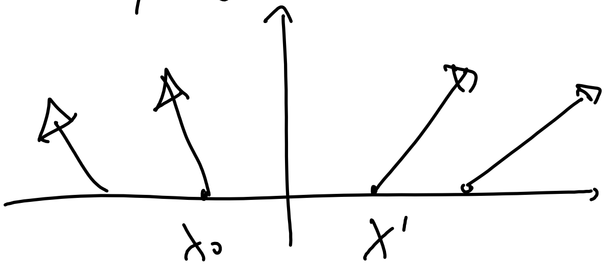




For $x^0 < x^1$, g strictly decreasing.

$$x^0 + s\,g(x^0) < x^1 + s\,g(x^1) \iff s < \frac{x^1 - x^0}{g(x^0) - g(x^1)} =: \bar{t} > 0$$

If, instead, g is strictly INCREASING



$$x^0 < x^1 \Rightarrow x^0 + s\,g(x^0) < x^1 + s\,g(x^1) \quad \forall s > 0$$

Proj. char. don't cross & I get a GLOBAL SOLUTION.

<u>Exercise</u>: Fact that $g' < 0 \Rightarrow$ shocks

& $g' > 0 \Rightarrow \exists$ global solution

remains true for $u_t + f(u)_x = 0$

if $f'' > 0$. ∎