ED 2 . Lecture March 18 - 2013

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Example C: Hamilton-Jacobi equations.

$$u_t + H(D_x u, x) = 0$$

$H : \mathbb{R}^n \times \overline{U} \to \mathbb{R}$ $\qquad q = (p, p_{n+1})$

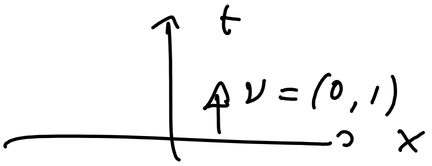
$\qquad y = (x, t)$

$$F(q, z, y) = p_{n+1} + H(p, x)$$

$$F_q = (H_p, 1), \quad F_z = 0, \quad F_y = (H_x, 0)$$

$$U = \mathbb{R}^n \times]0, +\infty[, \quad \Gamma = \mathbb{R}^n \times \{0\}$$

Non-characteristicity



$F_q \cdot \nu = 1 \implies$ all points of Γ are NON-CHARACT.

Charact. eqs.:

$$\begin{cases} \dot{\underline{p}} = -H_x(\underline{P}, \underline{X}) \\ \dot{p}_{n+1} = 0 \end{cases} \Bigg\} \; n+1 \text{ eqs}$$

$$\dot{z} = H_p(\underline{P}, \underline{X}) \cdot \underline{P} + P_{n+1} \qquad 1$$

$$\begin{cases} \dot{z} = H_p(\underline{p}, \underline{x}) \cdot \underline{p} + p_{n+1} & 1 \\ \dot{\underline{x}} = H_p(p, \underline{x}) \\ \dot{x}_{n+1} = 1 \end{cases} \quad \Big\} \; n+1 \text{ eqs.}$$

$$\Rightarrow \quad p_{n+1}(s) = p^0_{n+1}, \quad x_{n+1}(s) = s \quad (s = t!)$$

Admissible triples? $\quad p^0 = Dg(x^0)$

$$p^0_{n+1} + H(p^0, x^0) = 0 \quad \Rightarrow p^0_{n+1} = -H(p^0, x^0)$$

$$\Rightarrow \quad p_{n+1}(s) = -H(\underline{p}(s), \underline{x}(s)) \quad \forall s$$

$$\Rightarrow \quad H(\underline{p}(s), \underline{x}(s)) \equiv \text{cost} \quad \forall s.$$

& the CHAR. SYST REDUCES TO

$$\begin{cases} \dot{p} = -H_x(p, \underline{x}) & n \\ \dot{z} = H_p(p, \underline{x}) \cdot p - H(p, x) & 1 \\ \dot{\underline{x}} = H_p(p, \underline{x}) & n \end{cases}$$

& I can solve first the HAMILTONIAN

SYSTEM $\quad \dot{p} = -H_x(p, \underline{x})$

$$\dot{\underline{X}} = H_p(p, \underline{X})$$

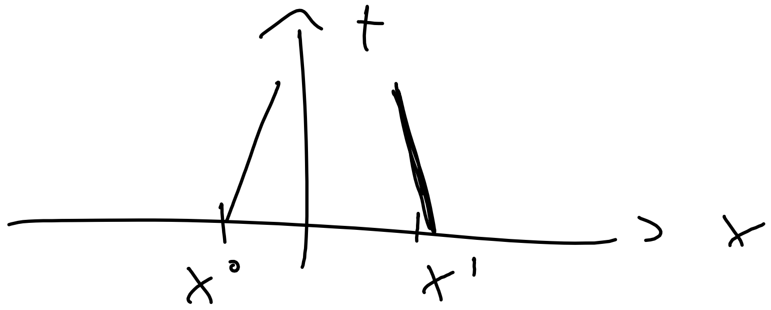
& then compute z.

<u>Subexample</u> $H = H(p)$, i.e. $H_x = 0$

$\Rightarrow$ $p(s) \equiv Dg(x^0)$ $\forall s$ &

$\underline{X}(s) = x^0 + s\, H_p(Dg(x^0))$

<u>$n=1$</u>

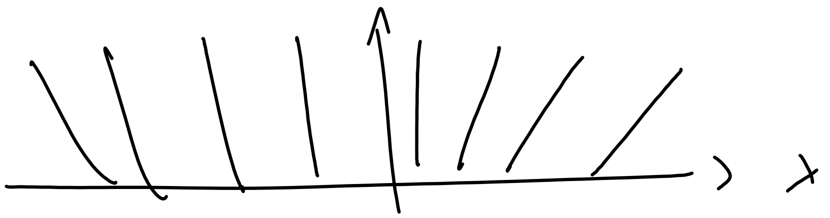


The projected characteristics intersect if

$$H_p(Dg(x^0)) > H_p(Dg(x^1)) \quad (x^0 < x^1)$$

Instead, they DON'T CROSS if "$\leq$"

$$H_p(Dg(x^0)) \leq H_p(Dg(x^1)) \quad \forall\, x^0 \leq x^1$$



Physically interesting case: $H_{pp} \geq 0$

$\Rightarrow H_p$ is INCREASING, $H_p(g'(x_0)) \leq H_p(g'(x'))$

$\forall x^\circ < x'$ if g' is also increasing.

I proved

<u>Corollary</u> $n=1$, $H = H(P)$, $H_{PP} \geq 0$,

$g'' \geq 0 \Rightarrow$ Cauchy pb. for $u_t + H(u_x) = 0$

has a global C^2 solution (i.e. def. on $\mathbb{R} \times]0, +\infty[$).

<u>Exercise</u> Compute soln for $H(P) = \frac{P^2}{2}$,

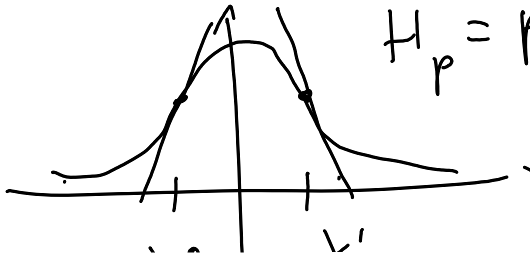
$g(x) = a\frac{x^2}{2} + d$, & compare with the soln.

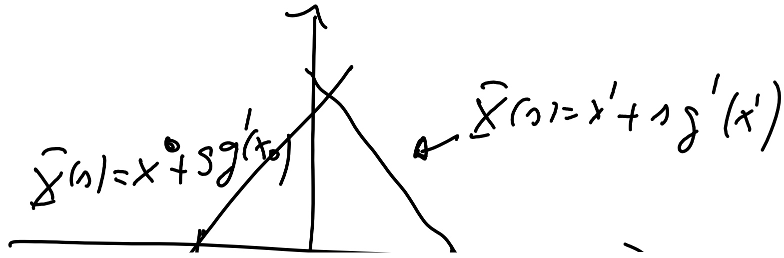
found by envelopes of complete integral.

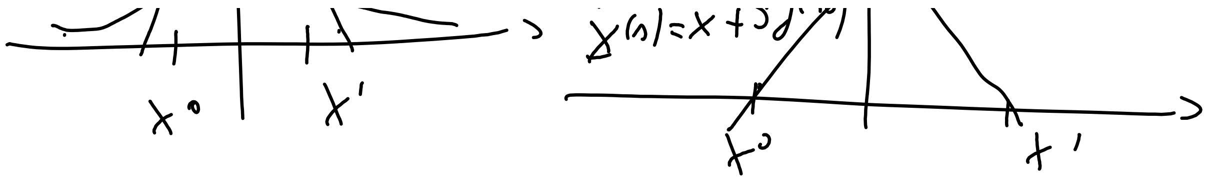
<u>Example of NON-existence of global soln</u>:

$n=1$, $H(P) = \frac{P^2}{2}$, $g'(x^\circ) > g'(x')$

$H_p = P$







$$x^0 + s g'(x^0) < x' + s g'(x') \iff s < \frac{x' - x^0}{g'(x^0) - g'(x')} =: \bar{t} > 0$$

Here z is NOT constant on CHAR.S, here u_x has a discontinuity at $t = \bar{t}$.

In fact $\quad u_t + \frac{u_x^2}{2} = 0$

$$\frac{\partial}{\partial x} \qquad (u_x)_t + \left(\frac{u_x^2}{2}\right)_x = 0$$

$$v = u_x \qquad v_t + \left(\frac{v^2}{2}\right)_x = 0 \qquad \text{Burger's Equation.}$$

$$v(x,0) = u_x(x,0) = g'(x)$$

$\Rightarrow$ time $\bar{t}$ when charact. cross = shock time for Burger's equation. ∎

Calculus of Variations & Hamilton's ODEs.

$L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, at least C^1,

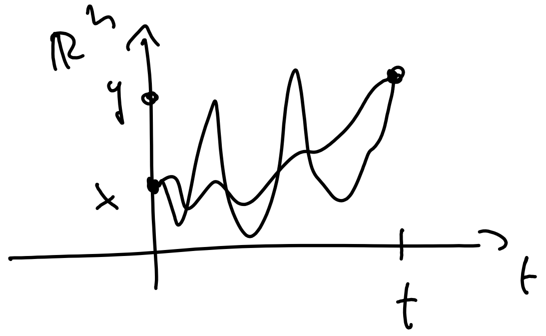
the Lagrangian , $L = L(q, x)$,

$L_x = D_x L = (L_{x_1}, \dots, L_{x_n}) \quad L_q = \left(L_{q_1}, \dots, L_{q_n} \right)$

<u>Def.</u> Action (or energy) functional

$$I[w(\cdot)] = \int_0^t L(\dot{w}(s), w(s))\, ds$$

$$w(\cdot) \in \mathcal{A} \ , \ \mathcal{A} := \left\{ w \in C^1([0, t], \mathbb{R}^n) : \begin{array}{l} w(0) = x \\ w(t) = y \end{array} \right\}$$



$$I : \mathcal{A} \to \mathbb{R}$$

<u>Basic problem</u> in CoV : find $\underline{x}(\cdot) \in \mathcal{A}$:

$$I[\underline{x}(\cdot)] = \min_{w(\cdot) \in \mathcal{A}} I[w(\cdot)]$$

minimization of the ACTION FUNCTIONAL.

<u>Thm.</u> (Necessary condition of MINIMALITY)

If $\bar{x}(\cdot)$ is a minimizer of I, then it solves

$$-\frac{d}{ds} L_q(\dot{\bar{x}}(s), \bar{x}(s)) + L_x(\dot{\bar{x}}(s), \bar{x}(s)) = 0 \qquad 0<s<t.$$

(E-L)

the Euler-Lagrange eqs associated to I.

<u>Proof</u> I Construct an arbitrary "variation" $w(\cdot)$ of $\bar{x}(\cdot)$. $v:[0,t]\to\mathbb{R}^n$, C^1, $v(0)=0=v(t)$

$\Rightarrow$ $w(s) = \bar{x}(s)+\tau v(s) \in \mathcal{A}$, $\tau\in\mathbb{R}$.

$\Rightarrow$ $I[\bar{x}(\cdot)] \leq I[w(\cdot)] \quad \forall \tau\in\mathbb{R}$

$i(\tau) := I[\bar{x}(\cdot)+\tau v(\cdot)]$ has a MIN at $\tau=0$

$\Rightarrow$ $i'(0)=0$ if $i'(0)$ exists.

$$i(\tau) = \int_0^t L(\dot{\bar{x}}(s)+\tau\dot{v}(s), \bar{x}(s)+\tau v(s))\,ds$$

$$\Rightarrow i'(\tau) = \int_0^t \sum_{i=1}^n \left[L_{q_i}(\dot{\bar{x}}+\tau\dot{v}, \bar{x}+\tau v)\dot{v}_i + L_{x_i}(\dots)v_i \right] ds$$

$$0 = i'(0) = \int^t \sum^n [L \ (\dot{\bar{x}}, \bar{x})\dot{v}_i + L \ (\dot{\bar{x}}, \bar{x}) v_i] ds$$

$$0 = i'(0) = \int_0^t \sum \; L_{q_i}(\dot{\bar{x}}, \bar{x})\dot{v}_i + L_{x_i}(\dot{\bar{x}}, \bar{x}) v_i \Big] ds$$

by parts
$$= \int_0^t \sum_{i=1}^n \left[-\frac{d}{ds} L_{q_i}(\dot{\bar{x}}, \bar{x}) + L_{x_i}(\dot{\bar{x}}, \bar{x}) \right] v_i \; ds$$

Since v_i are "arbitrary" $\Rightarrow$

$$\forall i = 1, \dots, n \qquad -\frac{d}{ds} L_{q_i}(\dot{x}, x) + L_{x_i}(\dot{x}, x) = 0$$ ∎

<u>Ex.</u> $L(q, x) = \frac{m}{2}|q|^2 - \phi(x)$ = kinetic en. − pot. energy

$L_q = mq \quad L_x = \nabla\phi$, (E-L) is

$$-\frac{d}{ds} m\dot{\bar{x}} - \nabla\phi(\bar{x}) = 0 \quad \Rightarrow \quad m\ddot{\bar{x}} = f(\bar{x})$$
$$f = -\nabla\phi$$

Newton's law. ∎

<u>CONNECTION</u> with <u>Hamilton's ODEs</u>.

Ass. $\bar{x}(\cdot)$ soln. of (E-L).

Generalized momentum

$$p(s) := L_q(\dot{\bar{x}}(s), \bar{x}(s)) \qquad 0 < s \le t$$

Assumption on L: $\forall\, x, p \in \mathbb{R}^n$ $\exists$ UNIQUE sol. q of (I) $p = L_q(q, x)$,

call it $q = Q(p, x)$, Q at least C^1.

Ex. $n=1$ $L_{qq} > 0$ $\Rightarrow$ L_q is strictly increasing.
$\Rightarrow$ Ass. holds.

N.B. (I) $\Rightarrow$ $\dot{\underline{x}}(s) = Q(\underline{p}(s), \underline{x}(s))$

Def. The Hamiltonian H associated to L is

$$H(p, x) := p \cdot Q(p, x) - L(Q(p, x), x)$$
$$x, p \in \mathbb{R}^n$$

Ex. (Newton). $L = \frac{m}{2}|q|^2 - \phi(x)$, $L_q = mq$
$m \neq 0$

$$Q(p, m) = \frac{p}{m} \Rightarrow H = \frac{|p|^2}{m} - \frac{\cancel{m}}{2}\frac{|p|^2}{m^{\cancel{2}}} + \phi(x)$$
$$= \frac{|p|^2}{2m} + \phi(x).$$

Thm. $\underline{x}(\cdot)$ sol. of (E-L), $\underline{p} = L_q(\dot{\underline{x}}, \underline{x})$

$$\Rightarrow \quad (HS) \begin{cases} \dot{\underline{X}}(s) = H_p(\underline{P}(s), \overline{X}(s)) \\ \dot{\underline{P}}(s) = -H_x(\underline{P}(s), \overline{X}(s)) \end{cases} \quad \forall\, 0<s<t$$

Moreover $s \mapsto H(\underline{P}(s), \overline{X}(s)) \equiv \text{const}$. $(+)$

<u>Lect. March 19, 2013</u>

Pf. $(+)$

$$\frac{d}{ds} H(P(s), \overline{X}(s)) = \sum_{i=1}^{n} \left[\frac{\partial H}{\partial p_i} \dot{p}^i + \frac{\partial H}{\partial x_i} \dot{\overline{X}}_i \right]$$

$$= \sum_{i=1}^{n} \left[-H_{p_i} H_{x_i} + H_{x_i} H_{p_i} \right] = 0 \qquad \square \ \tfrac{1}{3}$$

$\dot{\underline{X}} = H_p$:

$$\frac{\partial H}{\partial p_i} = q^i(p,x) + \sum_{k=1}^{n} \left[p_k \frac{\partial q^k}{\partial p_i}(p,x) - \frac{\partial L}{\partial q^k}(Q,x) \frac{\partial q^k}{\partial p_i}(p,x) \right] = q^i(p,x)$$

$Q = (q^1, \dots, q^n)$

$$\Rightarrow \frac{\partial H}{\partial p_i}(\underline{P}(s), \overline{X}(s)) = q^i(\underline{P}(s), \overline{X}(s)) = \dot{\overline{X}}_i(s)$$

$\square \ \tfrac{1}{3}$

$\dot{\underline{P}} = -H_x$:

$$\frac{\partial H}{\partial x_i} = \sum_{k=1}^{n} \left[p^k \frac{\partial q^k}{\partial x_i}(p,x) - \frac{\partial L}{\partial q_k}(Q,x) \frac{\partial q^k}{\partial x_i}(p,x) \right]$$

$$-\frac{\partial L}{\partial x_i}(Q,x) = -\frac{\partial L}{\partial x_i}(q,x)$$

$$\frac{\partial H}{\partial x_i}\left(\underline{P}(s), \underline{\bar{X}}(s)\right) = -\frac{\partial L}{\partial x_i}\left(Q\left(\underline{P}(s), \underline{\bar{X}}(s)\right), \underline{\bar{X}}(s)\right)$$

$$= -\frac{\partial L}{\partial x_i}\left(\underline{\dot{\bar{X}}}(s), \underline{\bar{X}}(s)\right) = -\frac{d}{ds} L_{q_i}\left(\underline{\dot{\bar{X}}}(s), \underline{\bar{X}}(s)\right)$$

$$= -\dot{p}_i(s) \qquad . \quad \blacksquare$$

<u>Rmk</u>. Thm. is a connection between the pb. in CoV. & the (H-J) eq. $u_t + H(D_x u, x) = 0$ because (HS) are the characteristics of (H-J).

What about the eq. for $z(\cdot)$?

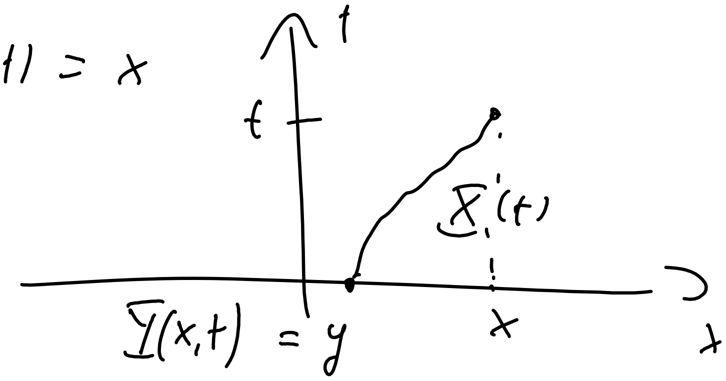
$$\dot{z} = \underline{\dot{\bar{X}}} \cdot \underline{p} - H(\underline{P}, \underline{\bar{X}}) = \cancel{p \cdot Q(\underline{P}, \underline{\bar{X}})} - \cancel{p \cdot Q(P, x)}$$
$$+ L(Q(\underline{P}, \underline{\bar{X}}), \underline{\bar{X}})$$

$$\Rightarrow \quad z(y,t) = g(y) + \int_0^t L\left(\underline{\dot{\bar{X}}}(s), \underline{\bar{X}}(s)\right) ds$$

with $\underline{\bar{X}}(\cdot)$ minimizer of $\int_0^t L(\underline{\dot{\bar{X}}}, \underline{\bar{X}})\, ds$

with $\bar{X}(0) = y$, $\underline{\bar{X}}(t) = x$ $\quad \uparrow^{\,t}$

with $\bar{X}(0) = y$, $\bar{X}(t) = x$



$$u(x,t) = z(\bar{Y}(x,t), t)$$

This suggests

$$u(x,t) = \min \left\{ \int_0^t L(\dot{w}, w)\, ds + g(y) \; : \; \begin{array}{l} w(0) = y \in \mathbb{R}^n \\ w(t) = x \\ w \in C^1 \end{array} \right\}$$

as a "solution" of

$$(CP) \quad \begin{cases} u_t + H(D_x u, x) = 0 & , \; t > 0 \\ u = g & \text{at } t = 0 \end{cases}$$

for H associated to L as above.

Will see that

- $p \mapsto H(p,x)$ is convex can find L such that H is assoc. to L.
- the "candidate formula" gives a <u>generalized</u> soln. of (CP).

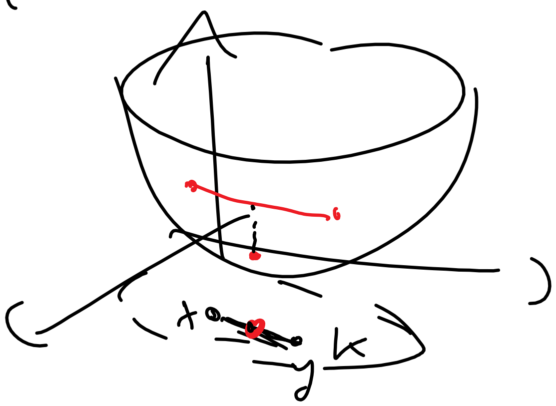
Review of CONVEX ANALYSIS

$K \subseteq \mathbb{R}^n$ convex if $\forall x, y \in K$

$$tx + (1-t)y \in K \quad \forall t \in [0,1]$$

K convex, $f : K \to \mathbb{R}$ is convex

$$f(tx + (1-t)y) \le t f(x) + (1-t) f(y) \quad \forall x, y \in K$$
$$\forall t \in [0,1]$$

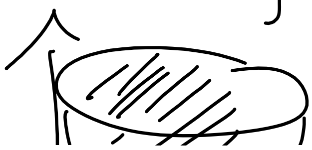


Basic properties

1. f convex $\Rightarrow$ $\forall c \in \mathbb{R}$

   $K_c = \{ x \in K : f(x) \le c \}$ is convex

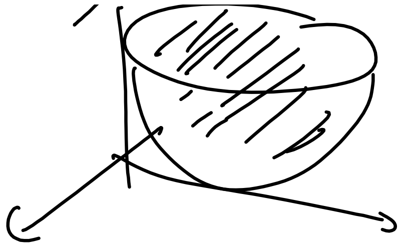
2. $\mathrm{epi}\, f = \{ (x,t) \in K \times \mathbb{R} : t \ge f(x) \}$



$\overset{\circ}{\mathrm{epi}}\, f = \{ (x,t) \in K \times \mathbb{R} :$

$\overset{\circ}{\mathrm{epi}}\, f = \{ (x,t) \in K \times \mathbb{R} :$

$t > f(x) \}$



are both convex sets in $\mathbb{R}^{n+1}$

3. Separation theorem for convex sets

(Hahn-Banach geometric in dim $< +\infty$)

$A, B \subseteq \mathbb{R}^m$ convex, $A, B \neq \emptyset$, $A \cap B = \emptyset$,

A open $\Rightarrow \exists\, v \in \mathbb{R}^m, \alpha \in \mathbb{R}$:

$$v \cdot x \leq \alpha \leq v \cdot y \quad \forall x \in A \ \forall y \in B$$

($\{ z : v \cdot z = \alpha \}$ is a hyperplane in $\mathbb{R}^m$ that separates A & B).

<u>Ref.</u> : Brezis : Analisi Funzionale.

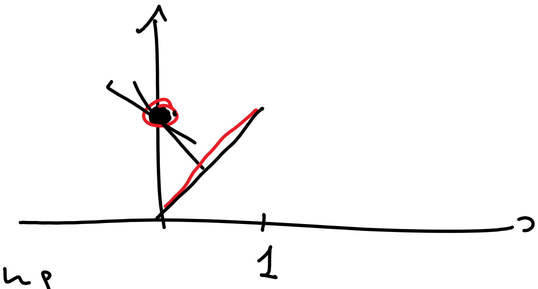
<u>Teor.</u> $K \subseteq \mathbb{R}^n$ convex, $f : K \to \mathbb{R}$ convex

$\Rightarrow \forall x_0 \in \overset{\circ}{K} \ \exists\, r \in \mathbb{R}^n$:

$$f(x) \geq f(x_0) + r \cdot (x - x_0) \quad \forall x \in K$$

Rmk $x_0 \in \overset{\circ}{K}$ is essential, Ex.:

$$f(x) = \begin{cases} 1 & x=0 \\ x & 0<x\le 1 \end{cases} \qquad K=[0,1]$$

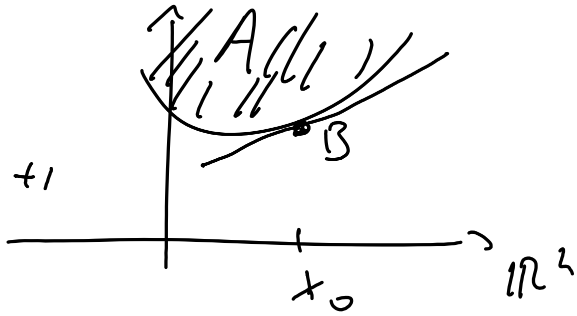


is convex.

but there is supporting line at $x_0 = 0$

<u>Proof</u> $A = \overset{\circ}{\text{epi}} f = \{(x,t) \in K \times \mathbb{R} : t > f(x)\}$.

$B = \{(x_0, f(x_0))\}$



H-B thm. $\Rightarrow \exists\, v=(p,\gamma) \in \mathbb{R}^{n+1}$

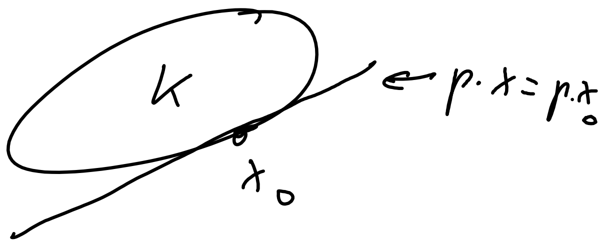
(IN) $p\cdot x + \gamma t \le p\cdot x_0 + \gamma f(x_0) \quad \forall x \in K$
$\forall t > f(x)$.

Observe that $\gamma \le 0$, if not $\gamma t \to +\infty$ as $t \to +\infty$

$\otimes$ with (IN)

Also, $\gamma \neq 0$. If it were $\gamma = 0$ (IN)

$p\cdot x \le p\cdot x_0 \quad \forall x \in K$



$\otimes$ with $x_0 \in \overset{\circ}{K}$

<u>Then</u> $\gamma < 0$ Divide (IN) by $|\gamma|$

$$\Rightarrow \quad \frac{p}{|\gamma|}\cdot x - t \leq \frac{p}{|\gamma|}\cdot x_0 - f(x_0) \qquad \forall t > f(x) \quad \forall x \in K$$

$$r = \frac{p}{|\gamma|}, \; t \geq f(x) \Rightarrow \; r\cdot x - f(x) \leq r\cdot x_0 - f(x_0)$$

$$\Rightarrow \quad f(x) \geq f(x_0) + r\cdot(x - x_0) \qquad \forall x \in K \qquad \blacksquare .$$

<u>An application</u>.

<u>Thm</u> (Jensen inequality). $f: \mathbb{R}^n \to \mathbb{R}$ convex, $u: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ INTEGRABLE.

$$\Rightarrow \quad f\left(⨍_U u\, dx \right) \leq ⨍_U f(u(x))\, dx$$

where $⨍_U \doteq \frac{1}{|U|}\int_U$.

<u>Pf.</u> $p := ⨍_U u\, dx \in \mathbb{R}^n$. Previous thm.

$$\Rightarrow \exists\, r \in \mathbb{R}^n : \quad f(q) \geq f(p) + r\cdot(q-p) \quad \forall q \in \mathbb{R}^n$$

$$\Rightarrow \quad f(u(x)) \geq f(p) + r\cdot(u(x) - p) \qquad \forall x \in U$$

$$\Rightarrow \quad \fint_V f(u(x)) \geq f\left(\fint_V u\,dx\right) + r\cdot\underbrace{\left(\fint_V u\,dx - \fint_V u\,dx\right)}_{=0} \quad \blacksquare$$

Legendre (-Fenchel) transform.

We are interested in Lagrangians s.t.

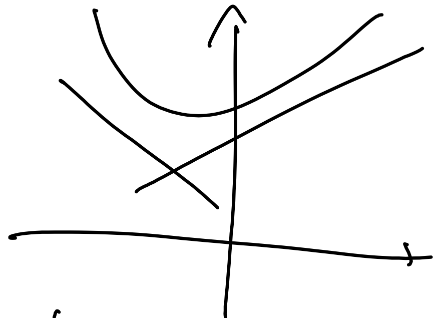
$q \mapsto L(q,x)$ is convex $\forall x$.

Since $x \in \mathbb{R}^n$ is fixed, I drop it.

Now $L : \mathbb{R}^n \to \mathbb{R}$

(C) $\quad q \mapsto L(q)$ convex

(S) $\quad \displaystyle\lim_{q\to\infty} \frac{L(q)}{|q|} = +\infty$



Def. Legendre transform L^* of L is

$$L^*(p) := \sup_{q\in\mathbb{R}^n} \left\{ q\cdot p - L(q) \right\}, \quad p \in \mathbb{R}^n.$$

Motivation (S) $\Rightarrow \dfrac{q\cdot p - L(q)}{|q|} \to -\infty$

$|q|$

$\Rightarrow p \cdot q - L(q) \to -\infty$ as $|q| \to +\infty$

Since L convex $\Rightarrow L$ cont., sup is a max.

If $L \in C^1$ at the max point $p - DL(q) = 0$

If this eq. $p = DL(q)$ has a unique soln.

$Q(p) \Rightarrow L^*(p) = Q(p) \cdot p - L(Q(p)) = H(p)$

<u>Conclusion</u> If $L \in C^1$ & $p = DL(q)$ has a unique soln. $\Rightarrow L^* = H$, but the current def. is MORE GENERAL.

<u>Thm.</u> (CONVEX DUALITY). Ass. (C)(S)

$\Rightarrow H := L^*$ satisfies $p \mapsto H(p)$ convex

$$\lim_{p \to \infty} \frac{H(p)}{|p|} = +\infty$$

& $H^*(q) = L(q) \quad \forall q \in \mathbb{R}^n$.

<u>Rmk</u>. Leg. transf. is INVOLUTIVE! $L^{**} = L$ ∎

Proof 1. $H = L^*$ is convex: $\tau \in [0,1]$, $p, \hat{p} \in \mathbb{R}^n$

$$H(\tau p + (1-\tau)\hat{p}) = \sup_q \left\{ q \cdot (\tau p + (1-\tau)\hat{p}) - \underset{(\tau + 1 - \tau)}{L(q)} \right\}$$

$$\leq \sup_q \{ q \cdot \tau p - \tau L(q) \} + \sup_q \{ q \cdot (1-\tau)\hat{p} - (1-\tau) L(q) \}$$

$$= \tau \sup_q \{ q \cdot p - L(q) \} + (1-\tau) \sup_q \{ q \cdot \hat{p} - L(q) \}$$

$$= \tau H(p) + (1-\tau) H(\hat{p}) .$$ $\blacksquare$ 1st statement

[LECT. March 20, 2013].

2. H satisfies (S). Fix $\lambda > 0$,

$p \neq 0$ $\quad H(p) \geq \lambda |p| - L\left(\lambda \frac{p}{|p|}\right)$

$\left(q = \lambda \frac{p}{|p|}\right)$ $\quad \geq \lambda |p| - \max_{\partial B(0,\lambda)} L$

$$\Rightarrow \frac{H(p)}{|p|} \geq \lambda - \frac{C_\lambda}{|p|} \Rightarrow \liminf_{p \to \infty} \frac{H(p)}{|p|} \geq \lambda$$

$$\Rightarrow \lim_{|p| \to +\infty} \frac{H(p)}{|p|} = +\infty$$ by the arbitrariness of $\lambda > 0$.

3. $H(p) + L(q) = \sup_{\xi} \{ \xi \cdot p - L(\xi) \} + L(q)$

$$(\xi = q) \quad \geq q \cdot p \qquad \forall p, q \in \mathbb{R}^n$$

$$\Rightarrow L(q) \geq q \cdot p - H(p) \qquad \forall p$$

$$\Rightarrow L(q) \geq \sup_{p} \{ q \cdot p - H(p) \} = H^*(q)$$

It remains to prove: $L(q) \leq H^*(q)$.

$$H^*(q) = \sup_{p} \{ p \cdot q - \sup_{\xi \in \mathbb{R}^n} \{ \xi \cdot p - L(\xi) \} \}$$

$$(*) = \sup_{p} \inf_{\xi} \{ p \cdot (q - \xi) + L(\xi) \}$$

Use the supporting hyperplane to the graph of L at q: $\exists s \in \mathbb{R}^n$:

$$(**) \quad L(\xi) \geq L(q) + s \cdot (\xi - q) \qquad \forall \xi \in \mathbb{R}^n$$

Use $p = s$ in $(*)$ & get

$$H^*(q) \geq \inf_{\xi} \{ s \cdot (q - \xi) + L(\xi) \} \overset{(**)}{\geq} L(q).$$

$\blacksquare$

$|q|^2$ $|p|^2$

<u>Exercise</u> $L(q) = \frac{''}{2} \Rightarrow H(p) = ''$

<u>HW</u> L^* for $L(q) = \frac{|q|^\alpha}{\alpha}$, $\alpha > 1$.

Hopf-Lax formula

Here we study the special H-J equation

$$(CP) \begin{cases} u_t + H(D_x u) = 0 & , t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R}^n \end{cases}$$

for $\begin{cases} H \text{ convex} \\ \lim_{|p| \to +\infty} \frac{H(p)}{|p|} = +\infty \end{cases}$

$g : \mathbb{R}^n \to \mathbb{R}$ Lipschitz.

Recall: from the connection between Calc. of Vars. & H-J equations we "guessed" as solution of (CP), $L = H^*$,

$$u(x,t) := \inf\left\{ \int_0^t L(\dot{w}(s))\,ds + g(y) : w \in C^1([0,T], \mathbb{R}^n) : w(0) = y,\ w(t) = x \right\}$$

$$w(0) = y \ , \ w(t) = x \ \}$$

Note that $u(x,0) = g(x)$. (Here u is defined as the "VALUE FUNCTION" of a problem in Calc. of Vars.)

<u>Thm</u> (Hopf-Lax formula) $\forall x \in \mathbb{R}^n$ $\forall t > 0$

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\} \qquad (HL)$$

<u>Pf.</u> 1. "$\leq$". $w(s) = y + \frac{s}{t}(x-y)$, $0 \leq s \leq t$

$$\dot{w}(s) = \frac{x-y}{t} \quad \forall s.$$

$$u(x,t) \leq \int_0^t L\left(\frac{x-y}{t}\right) ds + g(y) = tL\left(\frac{x-y}{t}\right) + g(y) \quad \forall y \in \mathbb{R}^n$$

$$\Rightarrow u(x,t) \leq \inf_y \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}. \quad \blacksquare$$

<u>2.</u> "$\geq$" Fix $w \in C^1$, Jensen $\leq$ $\Rightarrow$

$$L\left(\frac{1}{t}\int_0^t \dot{w}(s)\,ds\right) \leq \frac{1}{t}\int_0^t L(\dot{w}(s))\,ds$$

$w(0) = y, \quad w(t) = x \quad \Rightarrow \quad \int_0^t \dot{w}(s)\, ds = x - y$

$$\Rightarrow \quad t L\left(\frac{x-y}{t}\right) \underset{+g(y)}{\wedge} \le \int_0^t L(\dot{w}(s))\, ds \underset{+g(y)}{\wedge} \qquad \forall y$$

$$\Rightarrow \quad \inf_y \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\} \le u(x,t). \qquad \blacksquare$$

<u>3.</u> For $t > 0$ inf = min in (HL).

$\dfrac{L(q)}{|q|} \xrightarrow{q \to \infty} +\infty$, $\quad \left|\dfrac{x-y}{t}\right| = |y| + o(|y|)$ as $y \to \infty$

$\dfrac{L\left(\frac{x-y}{t}\right)}{|y|} \xrightarrow{y \to \infty} +\infty$. On the other hand, $g \in Lip \Rightarrow$

$$|g(y)| \le |g(0)| + Lip_g |y|$$

$$\Rightarrow \quad L\left(\frac{x-y}{t}\right) + g(y) \xrightarrow[y \to \infty]{} +\infty \quad \Rightarrow \exists \min \text{ in (HL)}.$$

$\blacksquare$

<u>Lemma</u> (A simplified version of Dynamic Programming) $\forall x \in \mathbb{R}^n \quad \forall\ 0 \le s < t$

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s) L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}$$

<u>RMK</u> For $s=0$ this reduces to (HL).

<u>Pf.</u> "$\leq$" Fix $y \in \mathbb{R}^n$, $0 \leq s < t$ $\Rightarrow \exists z$:

$$u(y,s) = sL\left(\frac{y-z}{s}\right) + g(z).$$

$$u(x,t) \leq tL\left(\frac{x-z}{t}\right) + g(z) \quad . \quad \oplus$$

Want to use convexity of L :

$$\frac{x-z}{t} = \frac{x-y}{t} + \frac{y-z}{t} = \left(1-\frac{s}{t}\right)\frac{x-y}{t-s} + \frac{s}{t}\,\frac{y-z}{s}$$

Use in $\oplus$

$$u(x,t) \leq (t-s)L\left(\frac{x-y}{t-s}\right) + \underbrace{sL\left(\frac{y-z}{s}\right) + g(z)}_{u(y,s)}$$

$$\Rightarrow u(x,t) \leq \inf_y \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}$$ ∎ "$\leq$"

"$\geq$" OPTIONAL (see [Evans])

<u>Conclusion:</u> inf = min because $y \mapsto u(y,s)$ is Lipschitz, see next Lemma, & the proof is the same as for (HL). ∎

Lemma u is Lipschitz in $\mathbb{R}^n \times [0, +\infty[$

e $\lim_{t \to 0^+} u(x,t) = g(x)$.

Dim. Lip in x: Fix $t>0$, $x, \hat{x} \in \mathbb{R}^n$, $\exists y$:

$$u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$$

$$u(\hat{x},t) - u(x,t) \leq tL\left(\frac{\cancel{\hat{x}} - (\cancel{\hat{x}} + y - x)}{t}\right) + g(\hat{x}+y-x)$$

$$- tL\left(\frac{x-y}{t}\right) + g(y)$$

$$= g(\hat{x} - x + y) - g(y) \leq Lip_g |\hat{x} - x|$$

Interchanging x & $\hat{x}$ $\Rightarrow$

$$|u(\hat{x},t) - u(x,t)| \leq Lip_g |\hat{x} - x|. \quad \blacksquare$$

2. Lip in t for $t=0$. For $t>0$ choose $y=x$

$\Rightarrow u(x,t) \leq tL(0) + g(x)$.

$$u(x,t) = \min_y \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} \pm g(x)$$

$$\left[g(y) - g(x) \geq -Lip_g |y-x| \right]$$

$$\geq g(x) + \min_y \left\{ tL\left(\frac{x-y}{t}\right) - Lip_g |y-x| \right\}$$

$$\left[z = \frac{x-y}{t} \right]$$

$$= g(x) + t \min_z \{ L(z) - \mathrm{Lip}_g |z| \}$$

$$\Big[C|z| = \max_{|w| \le C} z \cdot w \Big]$$

$$= g(x) - t \max_z \{ \mathrm{Lip}_g |z| - L(z) \}$$

$$= g(x) - t \max_{|w| \le \mathrm{Lip}_g} \underbrace{\max_z \{ z \cdot w - L(z) \}}_{H(w)}$$

$$\ge g(x) - t \max_{B(0, \mathrm{Lip}_g)} H$$

$$\Rightarrow \quad -t \max_{B(0,\mathrm{Lip}_g)} H \le u(x,t) - g(x) \le t L(0)$$

$$\Rightarrow \quad |u(x,t) - g(x)| \le C|t| \quad .$$

3. Lip in $t > 0$: $0 < \hat{t} < t$. Use

$$u(x,t) = \min_y \left\{ (t - \hat{t}) L\left(\frac{x-y}{t-\hat{t}}\right) + u(y, \hat{t}) \right\}$$

& $\mathrm{Lip}\, u(\cdot, \hat{t}) = \mathrm{Lip}_g$. Same calculation

as in 2 $\Rightarrow$

$$|u(x,t) - u(x,\hat{t})| \le C|t - \hat{t}|$$

$$C_1 = \max \left\{ L(0), \max_{B(0, \mathrm{Lip}\, g)} H \right\}. \quad \blacksquare$$

<u>Teor.</u> $\forall\, x \in \mathbb{R}^n, t > 0$: u diff.le in (x,t)

$$\Rightarrow \quad u_t + H(D_x u)\Big|_{(x,t)} = 0 .$$

Combining with Rademacher's Thm.

(Lip fns. are diff.le a.e.)

<u>Cor.</u> The (HL) formula $u(x,t)$ solves

$u_t + H(D_x u) = 0$ a.e. in $\mathbb{R}^n \times]0, +\infty[$.