Equazioni differenziali 2

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4.3.13 (March 4th)

ODE is equation where the unknown is a function u of 1-var

$$F(u^{(k)}(x), u^{(k-1)}(x), \dots, u'(x), u(x), x) = 0$$
$$\forall x \in I \subseteq \mathbb{R}$$

$u : I \supseteq J \to \mathbb{R}$.

PARTIAL DIFF. EQ. is an eq. whose unknown is a fn. u of SEVERAL VARS.

2^{nd} order PDEs:

$$F(D^2u, Du, u, x) = 0 \quad \text{in } U \subseteq \mathbb{R}^n$$

$U \subseteq \mathbb{R}^n$ is open UNKNOWN u

Ex heat eq., Laplace eq., Wave eq.
(see ED 1)

1st order PDEs

(1) $F(Du, u, x) = 0$ in $U \subseteq \mathbb{R}^n$

where $F: \mathbb{R}^n \times \mathbb{R}, \overline{U} \longrightarrow \mathbb{R}$

Goal: solving (1) with prescribed boundary conditions (condizioni al contorno)

B.C.: given $\Gamma \subseteq \partial U$

$g: \Gamma \to \mathbb{R}$

(BC) $\quad u(x) = g(x)$ on Γ

In particular: Dirichlet $\Gamma = \partial U$

Cauchy pb. for evolutive eqs.:

$X = (x_1, \dots, x_{n-1}, t)$, $\quad \mathcal{U} = \mathbb{R}^{n-1} \times]0, T[$

$$\Gamma = \mathbb{R}^{n-1} \times \{0\}$$

Linear eq. $\quad \displaystyle\sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} + c(x) u = f(x)$

Notations $\quad Du = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right) = \left(u_{x_1}, \dots, u_{x_n} \right)$

$$u_t = \frac{\partial u}{\partial t}$$

SEMILINEAR EQ. $\quad \sum a_i(x) u_{x_i} = f(x, u(x))$

QUASILINEAR EQ. $\quad \sum a_i(x, u) u_{x_i} = f(x, u(x))$

FULLY NONLINEAR: Eq. of the form (1) that cannot be put in quasilinear ... form.

Examples & MOTIVATIONS:

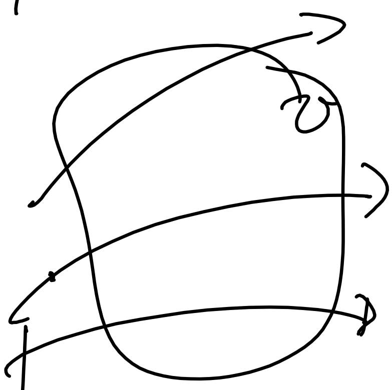
I Examples & MOTIVATIONS :

1. TRANSPORT EQ. :

Particles moving according to ODE $\dot{x}(t) = b(x(t))$

$b : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, $\underline{\text{Ass.}}$ $u \in C^1(V)$, $V \subseteq \mathbb{R}^n$,

is constant on trajectories of ODE :



$$u(x(t), t) = \text{cost.}$$

$$\frac{d}{dt} \Downarrow \quad = \nabla_x u \cdot \underset{\substack{\parallel \\ b(x(t))}}{\dot{x}(t)} + \frac{\partial u}{\partial t}\Big|_{(x(t),t)}$$

If the trajectories fill V $\Rightarrow$ u must satisfy

$$\text{(TS)} \quad \boxed{\frac{\partial u}{\partial t} + \nabla_x u \cdot b(x) = 0}$$

LINEAR, evolutive.

2. SCALAR CONSERVATION LAWS.

(CL) $$\frac{\partial u}{\partial t} + \operatorname{div}_x f(u) = 0$$

$$= \sum_{i=1}^{n-1} \frac{df_i(u)}{du} u_{x_i}$$

$f : \mathbb{R} \to \mathbb{R}^{n-1}$ smooth

QUASILINEAR. (DIV FORM).

$n-1 = 1$ $$u_t + f(u)_x = 0$$

Short derivation: $\forall\, \Omega \subseteq \mathbb{R}^{n-1}$ $\partial\Omega$ smooth

← flux density

$$\frac{d}{dt} \int_\Omega u(x,t)\, dx = -\int_{\partial\Omega} f(u) \cdot \nu_e \, d\sigma$$

$$\int_\Omega \frac{\partial u}{\partial t}(x,t)\, dt = -\int_\Omega \operatorname{div}_x f(u)\, dx$$

Ω arbitrary $\Rightarrow$ $$u_t + \operatorname{div}_x f(u) = 0$$

Most classical example : $n-1=1$

$f(u) = \frac{u^2}{2}$ $\qquad u_t + \left(\frac{u^2}{2}\right)_x = 0$

$u_t + u u_x = 0$ INVISCID BURGER EQUATION

in GAS DYNAMICS

3. HAMILTON-JACOBI EQS.

(HJ) $\quad u_t + H(D_x u, x) = 0$ in $U \times]0,T[$

$H : \mathbb{R}^n \times U \to \mathbb{R}$

FULLY NONLINEAR.

They arise in

- analytical MECHANICS & Calculus of Variations

e.g. $\quad H(p,x) = \frac{1}{2}|p|^2 + V(x)$

NOTE $\quad p \mapsto H(p,x)$ is CONVEX $\forall x$

- OPTIMAL CONTROL. $\quad$ CONTROL &

H-J-BELLMAN

$\dot{x}(t) = f(x(t), a(t))$

$a(t) \in A$

$$H(p,x) = \max_{a \in A} \left[f(x,a) \cdot p + \ell(x,a) \right]$$

H convex in p

ℓ = running cost or payoff

- $\underline{\text{0-SUM}}$ DIFFERENTIAL GAMES

$$\dot{x}(t) = f(x(t), a(t), b(t))$$

1st player   2nd player

$$H(p,x) = \min_{b \in B} \max_{a \in A} \left[f(x,a,b) \cdot p + \ell(x,a,b) \right]$$

H-J-Isaacs   H <u>not</u> convex in p

$\underline{4}$. STATIONARY H-J equations

(SHJ) $\quad H(Du, x) = 0$

- GEOMETRIC OPTICS

<u>e.g.</u> EIKONAL EQ

(EI) $|Du| = 1$

or, more generally

$|Du| = n(x)$ = refraction index

- Fermat' principle (variational)
- from Maxwell eqs., see Born & Wolf "Principles of optics"
- from wave eq. [LCE] ch. 4 :

(WE) $u_{tt} = \Delta u \qquad \left(= \sum_{i=1} u_{x_i x_i}\right)$

Method of "stationary phase"

Look for sols. of the form

$$u^\varepsilon(x,t) = e^{\frac{i p^\varepsilon(x,t)}{\varepsilon}} a^\varepsilon(x,t)$$

GEOMETRIC OPTICS ANSATZ

phase

amplitude

"ε small": highly oscillatory solutions

Simplify $a^\varepsilon \equiv 1$

Simplify . .

$$u^\varepsilon_t = \frac{i\rho^\varepsilon_t}{\varepsilon} e^{i\frac{\rho^\varepsilon}{\varepsilon}} \qquad u^\varepsilon_{tt} = e^{i\frac{\rho^\varepsilon}{\varepsilon}}\left(\frac{i\rho^\varepsilon_{tt}}{\varepsilon} - \left(\frac{\rho^\varepsilon_t}{\varepsilon}\right)^2\right)$$

$$u^\varepsilon_{x_i} = \frac{i\rho^\varepsilon_{x_i}}{\varepsilon} e^{\cdots} \qquad u^\varepsilon_{x_i x_i} = e^{i\frac{\rho^\varepsilon}{\varepsilon}}\left(\frac{i\rho^\varepsilon_{x_i x_i}}{\varepsilon} - \left(\frac{\rho^\varepsilon_{x_i}}{\varepsilon}\right)^2\right)$$

$$0 = u^\varepsilon_{tt} - \Delta_x u^\varepsilon = e^{i\frac{\rho^\varepsilon}{\varepsilon}}\left(\frac{i}{\varepsilon}\left(\rho^\varepsilon_{tt} - \Delta\rho^\varepsilon\right) - \frac{1}{\varepsilon^2}\left(\left(\rho^\varepsilon_t\right)^2 - |D_x\rho^\varepsilon|^2\right)\right)$$

$$\mathrm{Re}[\quad] = 0 \iff |\rho^\varepsilon_t|^2 = |D_x \rho^\varepsilon|^2$$

Look for special solutions

$$\rho(x,t) = ct + u(x), \quad c > 0$$

$$c = |Du| \implies \frac{u(x)}{c} \text{ solves (EI)}$$

See [LCE] for much more!

LECT. March 5, 2013

~~Df~~ Complete integral of the PDE (1) $F(Du, u, x) = 0$

is a "family of solutions"

$u : V \times A \to \mathbb{R}$, C^2, $u(x; a)$ ← parameter

$V \subseteq \mathbb{R}^n$, $A \subseteq \mathbb{R}^n$ open

$x \mapsto u(x; a)$ solves (1) $\forall a \in A$

&

$$rk\left(D_a u, D^2_{xa} u\right) = rk \begin{pmatrix} u_{a_1} & u_{x_1 a_1} & \cdots & u_{x_n a_1} \\ \vdots & \vdots & & \\ u_{a_n} & u_{x_1 a_n} & \cdots & u_{x_n a_n} \end{pmatrix}$$

$= n$ (maximal)

MEANING of the completeness condition:

$u(x; a)$ really depends on n parameters & NOT on $m < n$. Can prove [LCE] that: if $v(x;b)$ of sols. with $b \in B \subseteq \mathbb{R}^{n-1}$ open & $\psi: A \to B$ C^1

s.t. $u(x;a) = v(x;\psi(a))$ $\forall x, a$

$\Rightarrow \quad \operatorname{rk}(D_a u, D_{xa} u) < n$.

Ex. 1 Eikonal equation $|Du| = 1$.

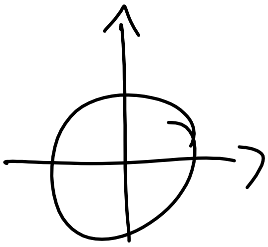
Try: affine solutions

$$u(x;a,b) = a \cdot x + b$$

$Du = a \qquad |Du| = 1 \iff |a| = 1$

$a \in \partial B_1(0)$ must parametrize $\partial B_1(0)$

with $n-1$ parameters: let's do it for $n=2$



$a_1 \in]0, 3\pi[$, $a_2 \in \mathbb{R}$

$$u(x; a_1, a_2) = x_1 \cos a_1 + x_2 \sin a_1 + a_2$$

$D_x u = (\cos a_1, \sin a_1)$

$(\ldots ^2 \ldots) \quad \begin{pmatrix} -x_1 \sin a_1 + x_2 \cos a_1 & -\sin a_1 & \cos a_1 \end{pmatrix}$

$$\left(D_a u,\ D^2_{xa} u \right) = \begin{pmatrix} -x_1 \sin a_1 + x_2 \cos a_1 & -\sin a_1 & \cos a_1 \\ 1 & 0 & 0 \end{pmatrix}$$

$\forall a_1$ $\qquad\qquad\qquad\qquad$ det $= \sin a_1$ $\qquad$ det $= -\cos a_1$

$\forall$ at least one of them is $\neq 0$.

<u>Ex. 2</u>: Clairaut eq.:

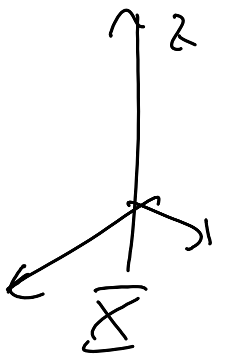
(CE) $\quad x \cdot Du + F(Du) = u \qquad F : \mathbb{R}^n \to \mathbb{R}$

from diff. geom.: <u>Pb</u> find $u : V \to \mathbb{R}$

s.t. $\forall x \in V$ tangent plane to graph u

at $(x, u(x))$ has intersection with

z-axis $= F(Du)$



Tg plane in $(\underline{X}, z)$ coord. is

$$z = u(x) + Du(x) \cdot (\underline{X} - x)$$

Intersect. with z-axis is $\underbrace{u(x) - x \cdot Du(x)}$

Impose $\qquad\qquad \overset{\shortparallel}{F(Du)} \implies$ (CE)

Try: affine solutions

$$u(x;a,b) = a\cdot x + b$$

Plug into (CE)

$$\cancel{x\cdot a} + F(a) = \cancel{a\cdot x} + b \qquad b = F(a)$$

Candidate C.I. $\quad u(x;a) = a\cdot x + F(a)$

$D_x u = a \qquad D^2_{xa} = Id_{n\times n} \Rightarrow$ the integral is complete.

Exa. 3 H-J equation with $H = H(p)$, $H \in C^1$

$$(HJ) \qquad u_t + H(D_x u) = 0 \qquad U \subseteq \mathbb{R}^{n+1}$$

$H : \mathbb{R}^n \to \mathbb{R}$. Try with SEPARATION OF VARIABLES: $u(x,t) = \varphi(t) + w(x)$

Plug into HJ $\Rightarrow \quad \varphi' + H(Dw) = 0$

$$\varphi'(t) = -H(Dw(x)) \quad \Rightarrow \begin{cases} \varphi' = \text{const} \\ Dw = \text{const}. \end{cases}$$

$$w(x) = a \cdot x + d \qquad \varphi(t) = ct + b$$

& $\quad c = -H(a)$

$$u(x,t;a,b) = a \cdot x - H(a)t + b$$

$$a \in \mathbb{R}^n, \; b \in \mathbb{R}$$

Is it COMPLETE? $D_x u = a \qquad D_t u = -H(a)$

$$\left(D_{(a,b)} u, \; D^2_{(x,t)(a,b)} u \right) =$$

$$= \begin{array}{c} \\ n \end{array} \left(\begin{array}{c|c|c} x - t\nabla H(a) & I_{n \times n} & -\nabla H(a) \\ \hline 1 & 0 & 0 \end{array} \right).$$

$$\underbrace{\qquad\qquad\qquad}_{\det = -1 \neq 0}$$

∎

Construction of NEW solutions by the method of ENVELOPES (INVILUPPI)

<u>Def.</u>: Given $\{u(\cdot\,;a) : a \in A\}$ V, A open $V \subseteq \mathbb{R}^n$, $A \subseteq \mathbb{R}^m$, $u \in C^1(V \times A)$, supp.

the m equation

$$D_a u(x;a) = 0$$ (STATIONARIZATION W.R.T. PARAM. a)

define $a = \phi(x)$, $\phi \in C^1$, i.e.

$$D_a u(x;\phi(x)) = 0 \quad \forall x \in V.$$

Then $v(x) := u(x, \phi(x))$ is the ENVELOPE of $\{u(x;a)\}$.

<u>Thm.</u> (The envelope of sols. of (1) is a sol'n)

If $\forall a \in A$ $u(\cdot\,;a)$ solves (1) & the env. v $\exists$ and $v \in C^1(V)$ $\Rightarrow$ $F(Dv, v, x) = 0$ in V

(v is called SINGULAR INTEGRAL).

<u>Pf.</u> $$v_{x_i} = u_{x_i}(x, \phi(x)) + \sum_{j=1}^m \underbrace{u_{a_j}(x, \phi(x))}_{=0} \phi^j_{x_i}$$

<u>Not</u>: $\phi = (\phi^1, \dots, \phi^m)$ $$= u_{x_i}(x, \phi(x))$$

$$F(Dv(x), v(x), x) = F(Du(x,\phi(x)), u(x,\phi(x)), x)$$
$$= 0$$ because $u(\cdot, a)$ solves (1) $\forall a \in A$ ☒

$= 0$ because $u(\cdot, a)$ solves (1) $\forall a \in A$ ∎

GEOMETRIC IDEA! $\forall x$ fixed graph v is tangent to graph of $u(\cdot, a)$ for $a = \phi(x)$

because $Dv(x) = Du(x, \phi(x))$.

<u>Exe</u> [CCE] p. 95 $u^2(1 + |Du|^2) = 1$

<u>Exa.</u> Clairaut eq. for $F(p) = |p|^2$:

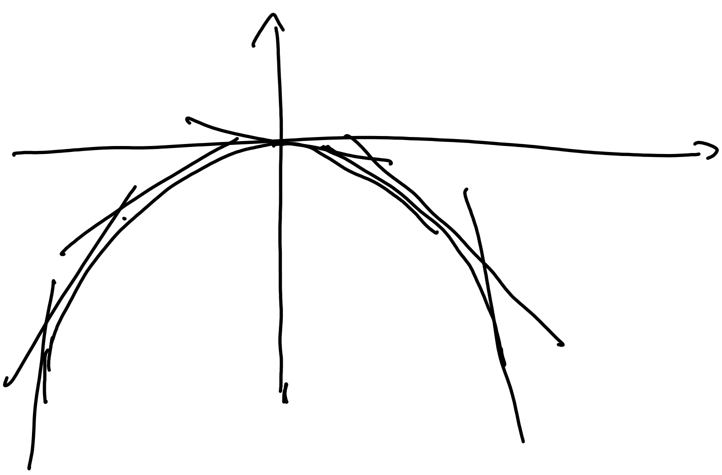
$$x \cdot Du + |Du|^2 = u$$

Comp. Int.: $u(x; a) = a \cdot x + |a|^2$

$D_a u = x + 2a = 0 \qquad a = -\frac{x}{2} =: \phi(x)$

$$v(x) = u\left(x; -\frac{x}{2}\right) = -\frac{|x|^2}{2} + \frac{|x|^2}{4} = -\frac{|x|^2}{4}$$

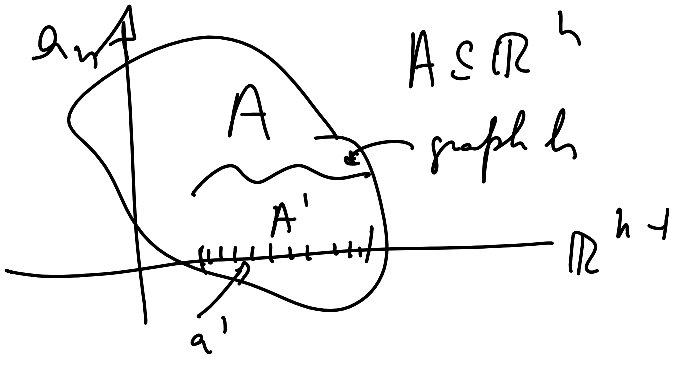
<u>$n = 1$</u>



Note Even for $n=1$: $xu' + (u')^2 = u$
result is remarkable because this ODE is NOT in normal form & the Cauchy Pb. has at least 2 solutions at each point

Exe $n=1$ $\quad xu' + (u')^4 = u$.

Variant of the envelope contruction.



$A \subseteq \mathbb{R}^n \quad A' \subseteq \mathbb{R}^{n-1} \quad a = (a', a_n)$

$h : A' \to \mathbb{R} \quad a' = (a_1, \dots a_{n-1})$

graph $h \subseteq A$

$u(x; a)$ complete integral of (1).

Def. GENERAL INTEGRAL of (1) depending on h is the envelope v' of

$$u'(x; a') := u(x; a', h(a')) , \; a' \in A'$$

provided v' $\exists$ & is C^1 .

Note: From complete integral we want to construct new solutions depending on $h \in C^1$

Exa 1 EIKONAL EQ. $|Du| = 1$ in $\mathbb{R}^2$

compl. int. $u(x; a_1, a_2) = x_1 \cos a_1 + x_2 \sin a_1 + a_2$

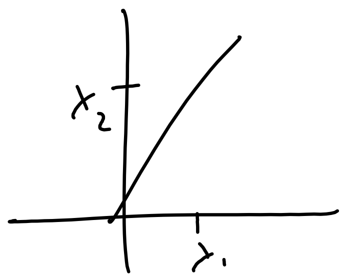
Try $a_2 = h(a_1) = 0$ $\qquad a' = a_1$

$$u'(x; a_1) = x_1 \cos a_1 + x_2 \sin a_1$$

$$\frac{\partial u'}{\partial a_1} = -x_1 \sin a_1 + x_2 \cos a_1 = 0$$

$a_1 \neq \pm \frac{\pi}{2}$

$$\operatorname{tg} a_1 = \frac{\sin a_1}{\cos a_1} = \frac{x_2}{x_1}$$

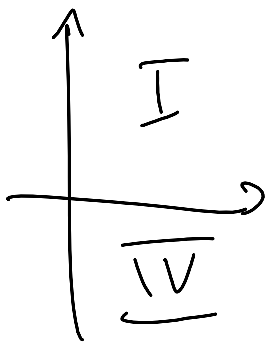


$$a_1 = \arctan \frac{x_2}{x_1}$$

If. $x_1 > 0$

$$\cos a_1 = \frac{x_1}{|x|}$$

$$\sin a_1 = \frac{x_2}{|x|}$$

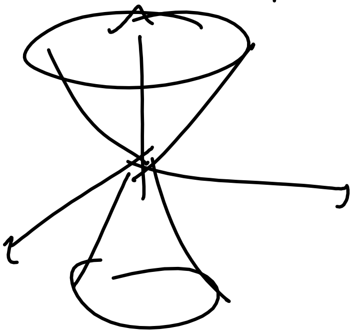


$$v(x) = x_1 \cdot \frac{x_1}{|x|} + x_2 \cdot \frac{x_2}{|x|} = |x|$$

Note $Dv = \frac{x}{|x|}$ $|Dv| = 1$ $\forall x \neq 0$

$\pm |x|$ are solutions of $\begin{cases} |Du| = 1 & \text{in } \mathbb{R}^2 \setminus \{0\} \\ u = 0 & \text{in } 0 \end{cases}$

Same holds in $\mathbb{R}^n \setminus \{0\}$ $\forall n$



Remarks $V \subseteq \mathbb{R}^n$ open $u(x) = \text{dist}(x, \partial V)$

Exer. $\forall x$: $Du(x) \exists$ $|Du(x)| = 1$

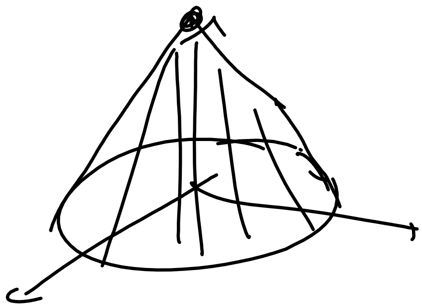
Then have 2 candidate solutions for

Dir. problem $\begin{cases} |Du| = 1 & \text{in } V \\ u = 0 & \partial V \end{cases}$

Try to think examples when $\text{dist}(x, \partial V)$ is or is not differentiable everywhere

Ex.: $V = B_1(0)$

$u(x) = \text{dist}(x, \partial u) = 1 - |x|$



Exa: (H-J) $\quad u_t + |D_x u|^2 = 0$

Compl. int. $\quad u(x,t;a,b) = a\cdot x - t|a|^2 + b$

$b = h(a) = c|a|^2$

$$u'(x,t;a) = a\cdot x + (c-t)|a|^2$$

$D_a u' = x + 2(c-t)a = 0 \qquad a = \dfrac{x}{2(t-c)} = \phi(x,t)$

$c \neq t$

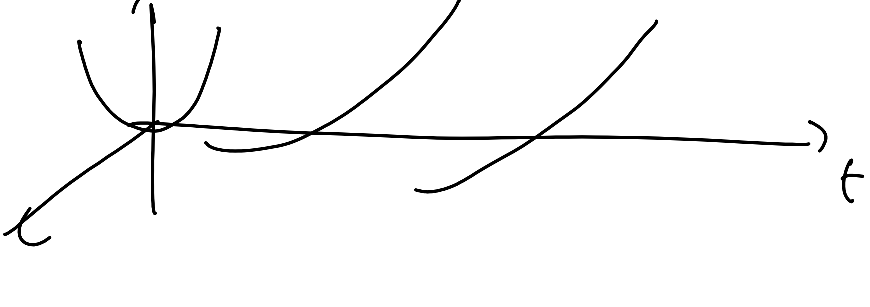
$$V(x,t) = u'(x,t;\phi(x,t)) = \frac{|x|^2}{2(t-c)} + (c-t)\frac{|x|^2}{2(t-c)^2}$$

$$= \frac{|x|^2}{4(t-c)}$$

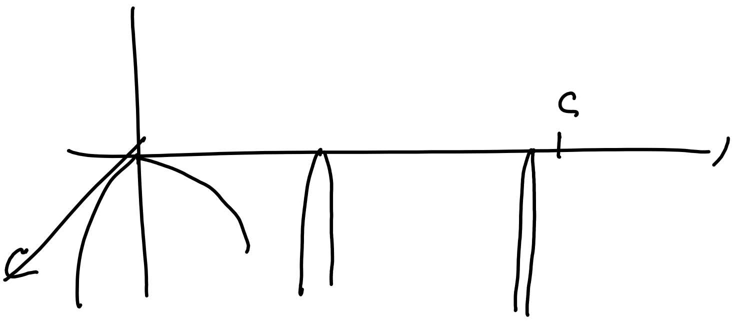
NEW sol. of $u_t + |Du|^2 = 0$

$t \neq c$

$c < 0$ $\quad V$ solves (H-J) on $]c, +\infty[$



$c > 0$ $\quad V$ solves (H-J) on $]-\infty, c[$



$$\lim_{t\to c^-} v(x,t) = \begin{cases} -\infty & x\neq 0 \\ 0 & x=0 \end{cases}$$

Lecture 3: March 6 - 2013

Method of characteristics.

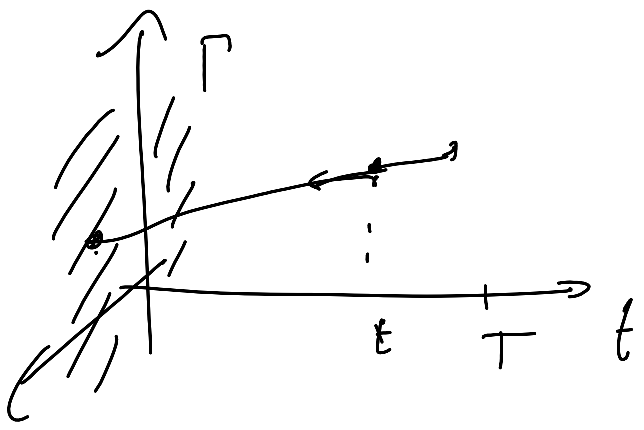
Motivation to the method.

The linear transport equation.

$$\begin{cases} u_t + b(x)\cdot D_x u = 0 & t>0 \\ u(x,0) = g(x) & \forall x\in\mathbb{R}^n \end{cases} \qquad (CP)$$

$U = \mathbb{R}^n \times]0,T[$

$\Gamma = \mathbb{R}^n \times \{0\}$



We know that a solution $u\in C^1$ satisfies

$u(x(t),t) = \text{cost}$ $\quad \forall$ traj. of

(ODE) $\dot{x} = b(x)$

Suppose $b \in C^1$ & such that the Cauchy pb. for (ODE) has unique solution defined globally in time.

$$u(x(0),0) = u(x(t),t) \quad \forall t > 0$$

$$u(x(0),0) = g(x(0))$$

Call $\Phi_t(x_0) = x(t;x_0)$ the flow ass. to the (ODE)

$$u(\Phi_t^{-1}(x),0) = u(x,t)$$

$\forall t$ fixed $\Phi_t(\cdot)$ is a diffeomorphism by the C^1 dependence of solutions of ODEs upon initial data.

Then I define $u(x,t) := g(\Phi_t^{-1}(x))$

Supp $g \in C^1 \Rightarrow u \in C^1$ solves (CP).

<u>Es</u> $\quad b(x) \equiv b \qquad \Phi_t(x_0) = x_0 + tb$

$\Phi_t^{-1}(x) = x - tb \quad \Rightarrow u(x,t) = g(x - tb)$.

The general method for

$$(1) \quad F(Du, u, x) = 0 \quad \text{in } U \subseteq \mathbb{R}^n$$

$$u = g \quad \text{on } \Gamma \subseteq \partial U$$

Assume $u \in C^2$ is solution

I look for curves

$\bar{X}(s)$ s.t. $\bar{X}(0) = x^0$

that allow me to compute

$$u(\bar{X}(s)) =: z(s)$$

$$Du(\bar{X}(s)) =: p(s)$$

Look for a system of ODEs satisfied by

$(p(s), z(s), \bar{X}(s))$. Compute

$$\dot p^i(s) = \sum_{j=1}^{n} u_{x_i x_j}(\underline{\tilde X}(s))\, \dot{\underline{\tilde X}}^j(s)$$

Assume $F \in C^1$ & differentiate (1) w.r.t. x_i

$$\sum_{j=1}^{n} F_{p_j} u_{x_j x_i} + F_z u_{x_i} + F_{x_i} = 0 \qquad (2)$$

Assume

$$(\ast) \qquad \dot{\underline{\tilde X}}^j(s) = F_{p_j}\left(p(s), z(s), \underline{\tilde X}(s)\right)$$

cioè $\dot{\underline{X}} = F_p(p, z, \underline{X})$. Evaluate (2) on $\underline{\tilde X}(s)$

& use $(\ast)$ to get

$$\dot p^i = -F_z(p, z, \underline{X})\, p^i - F_{x_i}(p, z, \underline{X})$$

$$z(s) = u(\underline{\tilde X}(s)) \Rightarrow \dot z(s) = Du(\underline{\tilde X}(s)) \cdot \dot{\underline{\tilde X}}(s)$$

$$= p(s) \cdot F_p(p, z, \underline{X})$$

SISTEM OF CHARACTERISTIC ODE

$$\begin{cases} \dot p = -F_z(p, z, \underline{X})\,p - F_x(p, z, \underline{X}) & \text{(a)} \quad n \text{ eq.} \\ \dot z = F_p(p, z, \underline{X}) \cdot p & \text{(b)} \quad 1 \text{ eq.} \end{cases}$$

(b) $\begin{cases} \dot{z} = F_p(p, z, \bar{X}) \cdot p & 1 \text{ eq.} \\ \dot{\bar{X}} = F_p(p, z, \bar{X}) & n \text{ eq.} \end{cases}$ (c)

Traject. of this system are called characteristic curves, $\bar{X}(\cdot)$ are called projected characteristics.

We have proved:

Thm. $F \in C^1$, $u \in C^2$ sol. of (1).
If $\bar{X}(\cdot)$ solves (c) with $p(s) = Du(\bar{X}(s))$
& $z(s) = u(\bar{X}(s)) \Rightarrow p(\cdot)$ solves (a)
& $z(\cdot)$ solves (b), $\forall s : \bar{X}(s) \in V$.

RMK. Can solve (a - b - c) without knowing u !!

Idea of the method: solve (a - b - c) (with suitable INITIAL CONDITIONS!) & reconstruct $u(x) = z(t)$ if $\bar{X}(t) = x$.

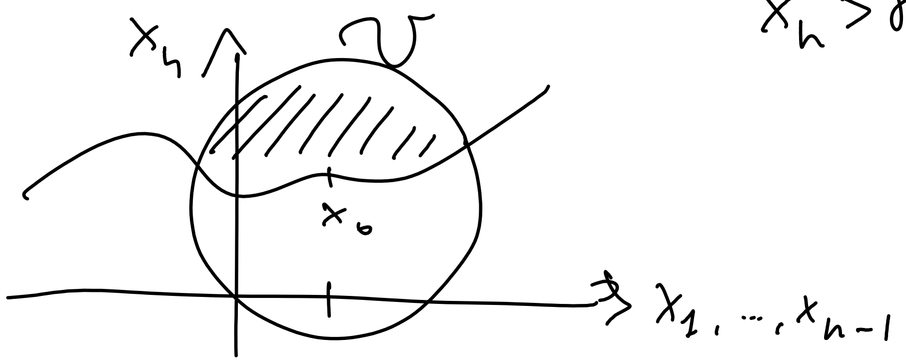
Boundary conditions

Def. $\partial U \in C^k$ if $\forall x^0 \in \partial U$ $\exists \gamma : \mathbb{R}^{n-1} \to \mathbb{R}$

s.t., after some rotation of axes if necessary,

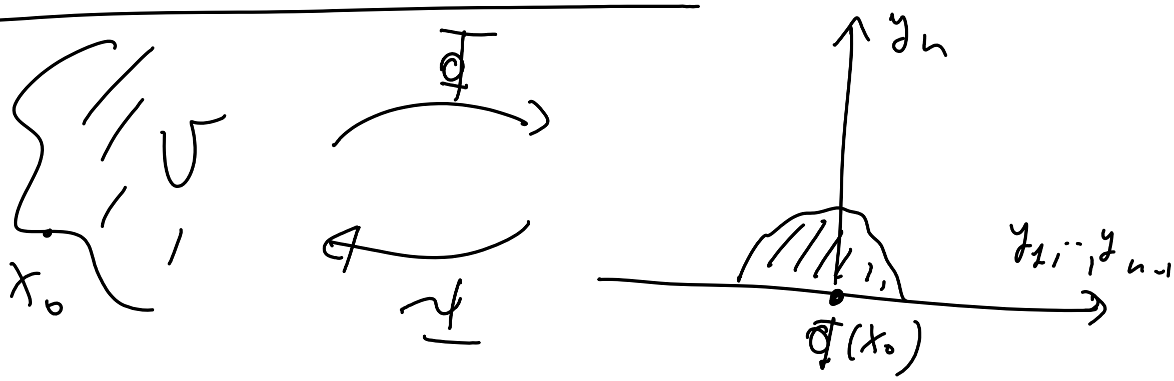
$\gamma \in C^k$

$$U \cap B(x^0, r) = \{x \in B(x^0, r) : x_n > \gamma(x_1, \dots, x_{n-1})\}$$



($\Rightarrow$ ∂U is locally the graph of a C^k function).

Straightening of the boundary



$$\begin{cases} y_i(x) = \Phi^i(x) = x_i & i = 1, \dots, n-1 \\ y_n(x) = \Phi^n(x) = x_n - \gamma(x_1, \dots, x_{n-1}). \end{cases}$$

$$x \in \partial U \cap B \iff \Phi^n(x) = 0 \iff y_n = 0$$

$x \in \mathcal{V} \cap B \iff \Phi(x) > 0$

$$\Psi(y) = \begin{cases} y_i & i = 1, \dots, n-1 \\ y_n + \gamma(y_1, \dots, y_{n-1}) = x_n \end{cases}$$

N.B. $\Phi, \Psi \in C^k$

Change variables in the equation (1):

$V(y) = u(\Psi(y)) \iff u(x) = V(\Phi(x))$

What equation is solved by V?

$$u_{x_i} = \sum_{k=1}^{n} V_{y_k} \Phi^k_{x_i} \Rightarrow Du = (D\Phi)^T DV$$

$$D\Phi = \mathrm{Jac}\,\Phi = \begin{pmatrix} \Phi^1_{x_1} & \cdots & \Phi^1_{x_n} \\ \vdots & & \\ \Phi^n_{x_1} & \cdots & \Phi^n_{x_n} \end{pmatrix}$$

$$\Rightarrow F\left((D\Phi(y))^T DV(y), V(y), \Psi(y) \right) = 0$$

$$=: \tilde{F}(DV, V, y)$$

Locally $\begin{cases} F(Du, u, x) = 0 & \mathcal{U} \\ u = g & \Gamma \end{cases}$

is equivalent to $\begin{cases} \tilde{F}(Dv, v, y) = 0 & \text{in } \tilde{\mathcal{U}} = \Phi(\mathcal{U}) \\ v(y) = g(\Psi(y)) & \text{in } \tilde{\Gamma} = \Phi(\Gamma). \end{cases}$

$$g(\Psi(y)) \stackrel{!!}{=} \tilde{g}(y)$$

Conclusion: from now on we will assume that Γ is flat (& keep calling the data F, g, Γ ....)

Admissibility condition on boundary data.

Look for initial data for (a - b - c) :

$$\begin{cases} p(0) = p^0 \\ z(0) = z^0 \\ X(0) = x^0 \end{cases} \qquad x_0 \in \partial \mathcal{U}$$

Obvious choice : $z^0 = g(x^0)$.

Assume $g \in C^1$. Since $u(x_1, .., x_{n-1}, 0) = g(x_1, .., x_{n-1})$

we choose $p^0_i = g_{x_i}(x^0)$ $\quad i = 1, .., n-1$

Must choose p^0_n : $\boxed{F(p^0, z^0, x^0) = 0}$

Must choose p_n^0 : $\boxed{F(p^0, z^0, x^0) = 0}$

<u>Assume</u> $\exists\ (p^0, z^0, x^0)$ ADMISSIBLE TRIPLE

i.e. s.t. !

$$\begin{cases} z^0 = g(x^0) \\ p_i^0 = g_{x_i}(x^0) \\ F(p^0, z^0, x^0) = 0 . \end{cases}$$

continue next time ...

— o —

Examples of characteristic systems :

<u>Ex 1</u> Linear PDE :

$$F(Du, u, x) = b(x) \cdot Du + c(x) u - \ell(x) = 0 \qquad (LIN)$$

$F_p = b(x) \quad \Rightarrow \quad (c) \quad \dot{\underline{X}}(s) = b(\underline{X}(s))$

don't depend on $z(\cdot)$ & $p(\cdot)$

$$(b) \quad \dot{z}(s) = b(\underline{X}(s)) \cdot \underset{\substack{\| \\ Du(\underline{X}(s))}}{p(s)} = \ell(\underline{X}(s)) - c(\underline{X}(s)) z(s)$$

$$\Rightarrow \quad z(s) = \int_0^s \ell(\underline{X}(\tau)) e^{-\int_\tau^s c(\underline{X}(r))\, dr}\, d\tau$$

N.B. Don't need to solve the equations for

$p(\cdot)$!

Caso partic: $\ell \equiv 0 \equiv c \Rightarrow z(s) = \text{const} =$

$= u(\underline{X}(0)) = g(x^0)$.

//

Exa 2 QUASILINEAR PDE

$$b(x,u) \cdot Du + c(x,u)\, u = \ell(x,u) \qquad (QL)$$

$F(p,z,x) = b \cdot p + cz - \ell$

$F_p = b \Rightarrow$ (c) $\dot{\underline{X}} = b(\underline{X}, z)$

(b) $\dot{z} = b(\underline{X}, z) \cdot \underset{\underset{Du}{\parallel}}{p} = \ell(\underline{X}, z) - c(\underline{X}, z)\, z$

As before, don't need to solve eqs. (a) for $p(\cdot)$.

However (b) & (c) now are coupled.