Introduction to homogenization

Pb. Find macroscopic equations describing phenomena with fast oscillations at the microscopic scale:

PDEs with oscillating coeff.: $\begin{cases} F\left(\frac{x}{\varepsilon}, x, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon\right) = 0 & \text{in } \Omega \subseteq \mathbb{R}^n \\ + BC \end{cases}$

Q! $u^\varepsilon \to u$ as $\varepsilon \to 0$ & u solves $\begin{cases} \bar{F}(x, u, Du, D^2u) = 0 \\ + BC \end{cases}$ EFF. BVP

for some $\bar{F}$?

Basic classical model: PERIODIC MEDIA: $\exists T > 0$:

$$F(y, x, r, p, X) = F(y + kT, r, p, X) \quad \forall k \in \mathbb{Z}^n \ \forall y, x, r, p, X \dots$$

After rescaling: $T = 1$, all data $\mathbb{Z}^n$-periodic in $y = \frac{x}{\varepsilon}$.

FIRST EXAMPLES:

LINEAR ODEs
- $u^\varepsilon_x = \ell\left(\frac{x}{\varepsilon}\right)$ stationary transport
- $-\left(a\left(\frac{x}{\varepsilon}\right) u^\varepsilon_x\right)_x = f_1(x) + f_2\left(\frac{x}{\varepsilon}\right)$ $a > 0$ elliptic div. form
- $-a\left(\frac{x}{\varepsilon}\right) u^\varepsilon_{xx} + b\left(\frac{x}{\varepsilon}\right) u^\varepsilon_x = f\left(\frac{x}{\varepsilon}\right)$ $a > 0$ elliptic non div. form.

N.B. f 1-periodic $f\left(\frac{x}{\varepsilon}\right) \to \bar{f}$ pointwise $\iff f \equiv$ const.

Ex. 1
$$u^\varepsilon(x) - u^\varepsilon(0) = \int_0^x \ell\left(\frac{\xi}{\varepsilon}\right) d\xi \underset{y = \frac{\xi}{\varepsilon}}{=} \varepsilon \int_0^{\frac{x}{\varepsilon}} \ell(y)\, dy =$$

$$= x \, \frac{\varepsilon}{x} \int_0^{\frac{x}{\varepsilon}} \ell(y)\, dy \longrightarrow x \langle \ell \rangle$$

where $\langle \ell \rangle = \int_0^1 \ell(y)\, dy$ because $\frac{1}{T}\int_0^T \ell\, dy \to \langle \ell \rangle$ $\forall \ell$ periodic