Then $\underbrace{u^\varepsilon(x) \to u^0 + x\langle \ell\rangle \equiv u(x)}_{u^\varepsilon(0) = u_0 \ \forall \varepsilon \Rightarrow}$ unique soln. of

$$\begin{cases} u_x = \langle \ell \rangle & \text{Eff. eq.} \\ u(0) = u_0 . \end{cases} \qquad \blacksquare$$

→ **Ex. 2** HW ←

$\langle \ell \rangle$ is $\lim_{\varepsilon\to 0} \ell(\frac{x}{\varepsilon})$ in the following sense

Lemma $f \in C(\mathbb{R})$, 1-periodic $\Rightarrow$ $f(\frac{\cdot}{\varepsilon}) \xrightarrow[\varepsilon\to 0]{} \langle f \rangle$ in $\mathcal{D}'([0,1])$

i.e. $\int_0^1 f(\frac{x}{\varepsilon})\varphi(x)\,dx \to \int_0^1 \langle f\rangle \varphi(x)\,dx = \langle f\rangle\langle \varphi\rangle \quad \forall \varphi \in C^1([0,1])$.

Pf. $f(x) = \langle f \rangle + g(x)$, g 1-periodic with $\langle g \rangle = 0$.

Goal $\int_0^1 g(\frac{x}{\varepsilon})\varphi(x)\,dx \to 0$ as $\varepsilon \to 0$. $\forall \varphi \in C^1$

By parts. $G(x) := \int_0^x g(t)\,dt$, 1-periodic $(G(1) = \langle g \rangle = 0 = G(0))$

$\frac{d}{dx}\varepsilon G(\frac{x}{\varepsilon}) = G'(\frac{x}{\varepsilon}) = g(\frac{x}{\varepsilon})$. Then

$$\int_0^1 g\left(\frac{x}{\varepsilon}\right)\varphi(x)\,dx = \varepsilon G\left(\frac{x}{\varepsilon}\right)\varphi(x)\Big|_0^1 - \varepsilon\int_0^1 \underbrace{G\left(\frac{x}{\varepsilon}\right)\varphi'(x)}_{\le C}\,dx$$

$$= \underbrace{\varepsilon G\left(\frac{1}{\varepsilon}\right)}_{\varepsilon\int_0^{1/\varepsilon} g(t)\,dt \to \langle g\rangle = 0}\varphi(1) - \varepsilon \underbrace{G(0)}_{0}\varphi(0) - O(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0 \qquad \blacksquare$$

Cor. $\int_0^1 f(\frac{x}{\varepsilon})\varphi(x)\,dx \to \langle f\rangle\langle\varphi\rangle \quad \forall \varphi \in C([0,1])$

Pf. $\exists \varphi_n \in C^1 : \varphi_n \rightrightarrows \varphi$

$$\int_0^1 f\left(\frac{x}{\varepsilon}\right)\varphi(x)\,dx = \underbrace{\int_0^1 f\left(\frac{x}{\varepsilon}\right)\varphi_n(x)\,dx}_{\downarrow \ \langle f\rangle\langle\varphi_n\rangle \xrightarrow[n]{} \langle f\rangle\langle\varphi\rangle} + \underbrace{\int_0^1 f(\varphi-\varphi_n)}_{\downarrow\ 0}$$

$\blacksquare$

2 bis

Rmk $\int_0^T g\left(\frac{x}{\varepsilon}\right)\varphi(x)\,dx = \varepsilon G\left(\frac{T}{\varepsilon}\right)\varphi(T) - \varepsilon G(0)\varphi(0) - O(\varepsilon)$

$$= T\,\frac{\varepsilon}{T}\int_0^{\frac{T}{\varepsilon}} g(x)\,dx + O(\varepsilon) \to T\langle g\rangle = 0$$

$$\int_0^T f\left(\frac{x}{\varepsilon}\right)\varphi(x)\,dx = \int_0^T \langle f\rangle \varphi\,dx + \int_0^T g\left(\frac{x}{\varepsilon}\right)\varphi(x)\,dx$$

$$\to \langle f\rangle \int_0^T \varphi(x)\,dx \qquad \forall \varphi\in C(\mathbb{R}) \quad \forall T>0$$

P.2

1 **Ex. 2 HW**

$$\begin{cases} -(a u^\varepsilon_x)_x = f\left(\frac{x}{\varepsilon}\right) & \text{in } (0,1) \quad a>0 \\ u^\varepsilon(0) = u^\varepsilon(1) = 0 \end{cases}$$

$$a u^\varepsilon_x = -\int_0^x f\left(\frac{\varrho}{\varepsilon}\right) d\varrho + c^\varepsilon \qquad u^\varepsilon(x) = -\frac{1}{a}\int_0^x \left(\int_0^\varrho f\left(\frac{\eta}{\varepsilon}\right) d\eta + c^\varepsilon\right) d\varrho$$

$$u^\varepsilon(1) = -\frac{1}{a}\left(\int_0^1 \int_0^\varrho f\left(\frac{\eta}{\varepsilon}\right) d\eta \, d\varrho + c^\varepsilon\right) = 0 \iff c^\varepsilon = -\int_0^1\int_0^\varrho f\left(\frac{\eta}{\varepsilon}\right) d\eta \, d\varrho.$$

$$\int_0^x \int_0^\varrho f\left(\frac{\eta}{\varepsilon}\right) d\eta = \int_0^x \underbrace{\varepsilon \int_0^{\frac{\varrho}{\varepsilon}} f(z)\, dz}_{\downarrow \; \varrho \langle f \rangle} \, d\varrho \longrightarrow \frac{x^2}{2}\langle f \rangle$$

$$\frac{\eta}{\varepsilon} = z$$

$$c^\varepsilon \to -\frac{1}{2}\langle f \rangle \qquad \Rightarrow \quad u^\varepsilon(x) \to -\frac{\langle f \rangle}{2a} x^2 + \frac{\langle f \rangle}{2a} x = \frac{\langle f \rangle}{2a}(x - x^2)$$

$$=: u(x)$$

$$u_x = \frac{\langle f \rangle}{2a}(1-2x) \qquad (a u_x)_x = \frac{\langle f \rangle}{2}(-2) = -\langle f \rangle.$$

$$\Rightarrow \quad \begin{cases} -(a u_x)_x = \langle f \rangle \\ u(0) = u(1) = 0. \end{cases}$$

∎

Ex. 3

$$\begin{cases} -\left(a\left(\frac{x}{\varepsilon}\right) u^\varepsilon_x\right)_x = f(x) & \text{in } (0,1) \\ u^\varepsilon(0) = u^\varepsilon(1) = 0 \end{cases}$$

$$a u^\varepsilon_x = -\left(\int_0^x f(\eta)\, d\eta + c^\varepsilon\right), \quad u^\varepsilon = \underbrace{-\int_0^x a^{-1}\left(\frac{\varrho}{\varepsilon}\right)\left(\int_0^\varrho f(\eta)\, d\eta + c^\varepsilon\right) d\varrho}$$

Rmk $\downarrow$

$$\langle a^{-1} \rangle \int_0^x \int_0^\varrho f(\eta)\, d\eta \, d\varrho$$

$$c^\varepsilon \int_0^1 a^{-1}\left(\frac{\varrho}{\varepsilon}\right) d\varrho = -\int_0^1 a^{-1}\left(\frac{\varrho}{\varepsilon}\right) \int_0^\varrho f(\eta)\, d\eta \, d\varrho$$

$$\Rightarrow \quad c^\varepsilon \longrightarrow -\frac{\cancel{\langle a^{-1} \rangle} \int_0^1 \int_0^\varrho f(\eta)\, d\eta\, d\varrho}{\cancel{\langle a^{-1} \rangle}} =: \bar{c}$$

Rmk

$$u^\varepsilon(x) \longrightarrow -\langle a^{-1}\rangle \int_0^x \int_0^\rho f(\eta)\,d\eta\,d\rho + \langle a^{-1}\rangle x \underbrace{\int_0^1\int_0^\rho f(\eta)\,d\eta\,d\rho}_{\bar{C}} = u(x)$$

$$u_x = -\langle a^{-1}\rangle \int_0^x f(\eta)\,d\eta + \langle a^{-1}\rangle \int_0^1\int_0^\rho f\,d\eta\,d\rho$$

$$-\left(\frac{u_x}{\langle a^{-1}\rangle}\right)_x = f(x) \quad \Rightarrow \quad -(\bar{a}\,u_x)_x = f(x) \qquad \text{EFF. EQ.}$$

$$\text{in}\quad \bar{a} = \frac{1}{\int_0^1 \frac{1}{a(y)}\,dy} \qquad \text{HARMONIC MEAN of } a(\cdot).$$

∎

N.B. In many space dim. explicit soln. is impossible.

Tools: elliptic estimates, weak convergence in L^p & Sobolev spaces etc. : see [BLP, Ch1] [JKO]

OR

Variational methods (for eqs. that are Euler-Lagrange of some functional ...) [BDF].

Nondivergence form operators

$$(E) \qquad -a\left(\tfrac{x}{\varepsilon}\right) u^\varepsilon_{xx} + b\left(\tfrac{x}{\varepsilon}\right) u^\varepsilon_x = f\left(\tfrac{x}{\varepsilon}\right). \qquad \text{in } (a,b)$$

Meth. 1 As. expansion

$$(AE) \qquad u^\varepsilon(x) = \bar{u}(x) + \varepsilon v\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon^2 w\left(x, \tfrac{x}{\varepsilon}\right) + \text{h.o.t.} \qquad \frac{x}{\varepsilon} = y$$

$$u^\varepsilon_x = \bar{u}_x + \varepsilon v_x + v_y + \varepsilon^2 w_x + \varepsilon w_y + \text{h.o.t.}$$

$$u^\varepsilon_{xx} = \bar{u}_{xx} + \varepsilon v_{xx} + \frac{1}{\varepsilon} v_{yy} + \varepsilon^2 w_{xx} + w_{yy} + \text{h.o.t.}$$

Look for v, w periodic in y, plug into (E) and set $= 0$ the coeff. of $\frac{1}{\varepsilon}$ and of ε^0 (then of $\varepsilon, \varepsilon^2, \dots$).

$\frac{1}{\varepsilon}$: $\quad -a(\frac{x}{\varepsilon})\, v_{yy}(x, \frac{x}{\varepsilon}) = 0$

Freeze x and consider $\quad -a(y)\, v_{yy}(y) = 0$

$\Rightarrow v_y = \text{const.} \quad \Rightarrow v(1) = v(0) \quad \Rightarrow v = v(x)$ only.

ε: with frozen x

$$-a(y)\,\bar{u}_{xx}(x) - a\, w_{yy} + b\,\bar{u}_x(x) + b\, \overset{=0}{v_y} = f(y)$$

$$-a(y) w_{yy} = g(y) \qquad g(y) := f(y) + a(y)\,\bar{u}_{xx} - b(y)\,\bar{u}_x$$

Does w exist, with B.C. w 1-periodic?

$$w_{yy} = -\frac{g}{a}(y) \qquad w_y(y) = -\int_0^y \frac{g}{a}(\eta)\,d\eta + c$$

this is periodic $\Leftrightarrow \int_0^1 \frac{g}{a}(\eta)\,d\eta = 0$

Then $\quad w(y) = -\int_0^y \left(\int_0^\xi \frac{g}{a}(\eta)\,d\eta + c \right) d\xi + d$

that is periodic $\Leftrightarrow \int_0^1 \int_0^\xi \frac{g}{a}(\eta)\,d\eta\, d\xi = -c \int_0^1 d\xi = -c$ which determines c

and w is unique modulo translations ($+d$).

The condition for solvability is

$$\int_0^1 g(y)\, m(y)\,dy = 0 \quad , \; m(y) = \frac{1}{a(y)} > 0$$

N.B. $\quad (a(y)\, m(y))_{yy} = 0 \quad$ & m is periodic and $m \not\equiv 0$,

can be normalized (dividing by $\int_0^1 m$) so that $\int_0^1 m(y)\,dy = 1$

Then $\quad m(y) = \dfrac{1}{\langle a^{-1} \rangle\, a(y)} \quad$ is (the density of) a MEASURE,

called INvariant measure for the op. $-a(y) D_{xx}$.

CONCLUSION the CELL PROBLEM

$$\begin{cases} -a(y)\, w_{yy} = f(y) + a(y)\bar u_{xx} - b(y)\bar u_x & \text{in } (0,1) \\ w \quad \text{periodic} \end{cases}$$

has a soln $\iff \displaystyle\int_0^1 (f + a\bar u_{xx} - b\bar u_x)\, m(y)\,dy = 0$

i.e. $\quad -\langle a \rangle_m \bar u_{xx} + \langle b \rangle_m \bar u_x = \langle f \rangle_m . \qquad$ (EE)

So, if $u^\varepsilon \to \bar u$ with an As. exp. (AE) the EFFECTIVE PDE for $\bar u$ must be (EE).

N.B. If $a(y) \equiv a$ constant, then $m(y) \equiv 1$.

$\langle\ \rangle_m = \langle\ \rangle$. On the other hand m const. $\Rightarrow a_{yy} = 0$

& periodicity $\Rightarrow a \equiv$ const.

The n-dim case

$$(E_n) \qquad -\sum_{i,j}^{n} a_{ij}\left(\tfrac{x}{\varepsilon}\right) u^\varepsilon_{x_i x_j} + b\left(\tfrac{x}{\varepsilon}\right)\cdot \nabla u^\varepsilon = f\left(\tfrac{x}{\varepsilon}\right) \quad \text{in } \Omega \subseteq \mathbb{R}^n$$

Assume (AE) as before and arrive at

$$(CP) \begin{cases} -\sum a_{ij}(y) w_{y_i y_j} = f(y) + \sum a_{ij}(y)\, \bar{u}_{x_i x_j} - b(y)\cdot \nabla \bar{u} =: g & \text{in } (0,1)^n \\ w \ \mathbb{Z}^n\text{-periodic} \end{cases}$$

Assume $\exists$ INVARIANT MEASURE m

$$\begin{cases} \sum (a_{ij}(y) m)_{y_i y_j} = 0 \quad \text{in } (0,1)^n \\ m > 0, \quad \int_{(0,1)^n} m = 1, \quad m\ \mathbb{Z}^n \text{ periodic} \end{cases}$$

Multiply (+) by m & $\int_{(0,1)^n}$

$$\int_{(0,1)^n} g(y) m(y)\, dy = -\int (a_{ij} w_{y_i y_j}) m\, dy \overset{\text{by parts \& use periodicity}}{=}$$

$$= \int_{(0,1)^n} w_{y_i} (a_{ij} m)_{y_j}\, dy \overset{\text{again}}{=} -\int_{(0,1)^n} w \underbrace{(a_{ij} m)_{y_i y_j}}_{=0} dy$$

$$= 0$$

Then (CP) is solvable ONLY IF

$$\int_{(0,1)^n} \left(f + \sum a_{ij} \bar{u}_{x_i x_j} + b\cdot \nabla \bar{u}\right) m = 0 ,$$

and the EFFECTIVE PDE is

$$-\sum_{i,j} \langle a_{ij} \rangle_m \bar{u}_{x_i x_j} + \langle b \rangle_m \cdot \nabla \bar{u} = \langle f \rangle_m \quad \text{in } \Omega.$$

All this can be made rigorous by probabilistic methods (ergodic properties of diffusions) & analytic methods (Fredholm Alternative to show that (CP) is solvable $\iff \int g\, m = 0$) and convergence of $u^\varepsilon \to \bar{u}$ can be proved. The solution w of (CP) is called the FIRST CORRECTOR and sometime also the exp.

$$u^\varepsilon = \bar{u} + \varepsilon w + o(\varepsilon), \quad \varepsilon \to 0$$

can be proved. See Ch. 3 of [BLP].

HW 1. $\begin{cases} u^\varepsilon_t + b \cdot u^\varepsilon_x = f\left(\frac{x}{\varepsilon}\right) &]0,T[\times \mathbb{R} \\ u^\varepsilon(0,x) = u_0(x) \end{cases}$

2. $\begin{cases} u^\varepsilon_t + b\left(\frac{x}{\varepsilon}\right) u^\varepsilon_x = 0 \\ u^\varepsilon(0,x) = u_0(x) \end{cases}$

The ADDITIVE EIGENVALUE PB:

Once we know that $V_y = 0$ in (AE), we plug it in the (E) and get, $\frac{x}{\varepsilon} = y$, $\bar{u}_{xx} = M$, $\bar{u}_x = p$

$$-a\left(\tfrac{x}{\varepsilon}\right) M + b\left(\tfrac{x}{\varepsilon}\right) p - a\left(\tfrac{x}{\varepsilon}\right) w_{yy} - f\left(\tfrac{x}{\varepsilon}\right) = o(1) \text{ as } \varepsilon \to 0$$

Consider $o(1)$ negligible. Then

Can eliminate $\frac{x}{\varepsilon}$ & keep only $M = \bar{u}_{xx}$ & $p = \bar{u}_x$ if w solves

$$\text{l.h.s.} = \lambda(M,p) \qquad M, p \text{ parameters}$$

$$(CP') \qquad -a(y)\, w_{yy} = g(y, M, p) + \lambda \qquad \text{for } y \in (0,1).$$

$$g = a(y) M - b(y) p + f(y)$$

Can view this as an ADDITIVE E.VALUE (λ) PROBLEM parametrized by M, p. We have already shown that $\exists!\ \lambda$ s.t. (CP') has a periodic soln. and it is s.t.

$$\langle g(\cdot, M, p) + \lambda \rangle = 0 \qquad \text{i.e.}$$

$$\lambda = -\langle a \rangle_m M + \langle b \rangle_m p - \langle f \rangle_m .$$

With this choice of w the eq. for $\bar{u}$ becomes

$$\lambda(\bar{u}_{xx}, \bar{u}_x) = 0$$

i.e. (EE).

This point of view works for FULLY NONLINEAR PDEs and allows to get the effective operator.

REFERENCES:

[BLP] Bensoussan, J.L.Lions, Papanicolaou, book 1978.

[JKO] Jikov, Kozlov, Oleinik, book.

[BDF] Braides, De Franceschi, book.