UNIQUENESS and EXISTENCE for CAUCHY PROBLEMS

We are interested in Comparison Principles for:

$$\begin{cases} u_t + H(x, D_x u) = 0 & [0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) \end{cases} \qquad (1)$$

and

$$\begin{cases} u_t + F(x, D_x u, D^2_{xx} u) = 0 & (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^n \end{cases} \qquad (2)$$

with $F(x, p, X + M) \le F(x, p, X) \quad \forall M \ge 0$

and the other structure conditions on H and F seen so far for stationary equations. We could also easily add a CONTINUOUS dependence on t of H & F, and a nondecreasing dependence on u (or even some controlled decreasing dependence ...).

The C.P. will take the form:

u, v bounded in $[0,T) \times \mathbb{R}^n$, u U.S.C. subsol, v L.S.C supersol of (1) or (2), then

$$\sup_{[0,T) \times \mathbb{R}^n} (u - v) = \sup_{x \in \mathbb{R}^n} \left(u(0,x) - v(0,x) \right)^+. \qquad (3)$$

MAIN IDEA of PROOF:

for $c, \varepsilon > 0$ def.

$$\tilde{u}(t,x) = u(t,x) - ct - \frac{\varepsilon}{T-t}$$

Then $\tilde{u}_t = u_t - c - \dfrac{\varepsilon}{(T-t)^2}$ $\quad \tilde{u}_{x_i} = u_{x_i}$ $\quad D^2\tilde{u} = D^2 u$

Then $\tilde{u}_t + F(x, D_x\tilde{u}, D^2_x\tilde{u}) \leq -c - \dfrac{\varepsilon}{(T-t)^2} \leq -c \quad \forall \varepsilon > 0$

and $\lim_{t \to T-} \tilde{u}(t,x) \equiv -\infty$ unif. in x & c.

Then one uses the arguments of the C.P. for Dirichlet problem in unbounded domains to get

$$\sup_{[0,T[\times\mathbb{R}^n} (\tilde{u} - v) = \sup_{\mathbb{R}^N} (\tilde{u}_{c,\varepsilon}(0,x) - v(x))^+$$

$$= \sup_{\mathbb{R}^N} \left(u(0,x) - \frac{\varepsilon}{T} - v(0,x)\right)^+ \xrightarrow[\varepsilon \to 0]{} \sup (u-v)\Big|_{t=0}$$

<u>Note that</u>
- The lack of term λu in the PDE is replaced u being STRICT SUBsolution.
- $(\tilde{u} - v) \to -\infty$ at $t = T$, so the $\sup(\tilde{u} - v)$ cannot be attained at $t = T$ (where no infos on u & v are known)

Example. (HW)
Proof of C.P. under the assumptions:

- $|H(x,p) - H(y,p)| \le \omega(|x-y|(1+|p|))$, $\lim_{r\to 0+} \omega(r) = 0$
- $|H(x,p+q) - H(x,p)| \le \rho(R,r)$ $\forall\, x\in\mathbb{R}^n$, $|p|\le R$, $|q|<r$ and $\forall R>0$ $\lim_{r\to 0+} \rho(R,r) = 0$
- u and v attain continuously the initial data.

Pf.: As in Sect. 5 of [Crandall] consider (we write $u = \tilde{u}$ ...)

$$\Phi(t,s,x,y) = u(t,x) - v(s,y) - \frac{|x-y|^2 + |t-s|^2}{2\varepsilon} - \frac{\delta}{2}(|x|^2 + |y|^2)$$

Then $u-v$ bdd. above, $u-v$ u.s.c., $\lim_{t\to T}(u-v) = -\infty$ implies that Φ has a max at $(\hat t, \hat s, \hat x, \hat y)$. Case 1: $\hat t, \hat s > 0$. Then

$$\frac{\hat t - \hat s}{\varepsilon} + H\left(\hat x, \frac{\hat x - \hat y}{\varepsilon} + \delta \hat x\right) \le -c$$

$$\frac{\hat t - \hat s}{\varepsilon} + H\left(\hat y, \frac{\hat x - \hat y}{\varepsilon} - \delta \hat y\right) \ge 0$$

So
$$0 < c \le H\left(\hat y, \frac{\hat x - \hat y}{\varepsilon} - \delta \hat y\right) - H\left(\hat x, \frac{\hat x - \hat y}{\varepsilon} - \delta \hat x\right). \quad (3)$$

As in [Cr.]

$$\frac{|\hat x - \hat y|^2}{\varepsilon} \le c_{\varepsilon,\delta} \quad \left(\text{and also } \frac{|\hat t - \hat s|^2}{\varepsilon} \dots\right)$$

with $\lim_{\varepsilon} \limsup_{\delta} c_{\varepsilon,\delta} = 0$.

Also $u(t,0) - v(t,0) = \Phi(t,t,0,0) \le \Phi(\hat t, \hat s, \hat x, \hat y)$ implies

$$\frac{\delta}{2}(|\hat x|^2 + |\hat y|^2) \le v(t,0) - u(t,0) + u(\hat t, \hat x) - v(\hat s, \hat y) - \frac{|\hat x - \hat y|^2 + |\hat t - \hat s|^2}{2\varepsilon} \le C_1$$