so the estimates (5.5) of [Ca], p.14 hold.

Then the right hand side of (3) is $\leq$ $\left(\text{set } p_\varepsilon = \frac{\hat{x}-\hat{y}}{\varepsilon}\right)$

$$H(\hat{y}, p_\varepsilon - \delta\hat{y}) \pm H(\hat{y}, p_\varepsilon) \pm H(\hat{x}, p_\varepsilon) - H(\hat{x}, p_\varepsilon - \delta\hat{x})$$

$$\leq \rho(|p_\varepsilon|, |\delta\hat{y}|) + \omega(|\hat{x}-\hat{y}|(1+|p_\varepsilon|)) + \rho(|p_\varepsilon|, |\delta\hat{x}|)$$

$$\leq 2\rho\left(\sqrt{\frac{C_{\varepsilon,\delta}}{\varepsilon}}, 2\sqrt{\delta}\sqrt{C_1}\right) + \omega\left(|\hat{x}-\hat{y}| + \frac{|\hat{x}-\hat{y}|^2}{\varepsilon}\right)$$

For $\varepsilon > 0$ we let $\delta \to 0+$ and get

$$0 < C \leq \omega\left(\limsup_{\delta\to 0}\left(\sqrt{\varepsilon}\sqrt{C_{\varepsilon,\delta}} + C_{\varepsilon,\delta}\right)\right) \to 0 \quad \text{as } \varepsilon \to 0+,$$

a CONTRADICTION.

Then $\hat{t}$ or $\hat{s} = 0$, say $\hat{t} = 0$.

Set $A := \sup(u-v)|_{t=0}$ and assume by contradiction $\exists \eta > 0$, $\tilde{t}, \tilde{x}$ : $(u-v)(\tilde{t}, \tilde{x}) = A + \eta$.

Choose $\delta$ : $2\delta|\tilde{x}| < \frac{\eta}{2}$. Then

$$A + \frac{\eta}{2} \leq A + \eta - 2\delta|\tilde{x}| = \Phi(\tilde{t}, \tilde{t}, \tilde{x}, \tilde{x}) \leq \Phi(0, \hat{s}, \hat{x}, \hat{y})$$

$$\leq u(0,\hat{x}) - v(\hat{s},\hat{y}) \pm v(0,\hat{x}) \leq A + \omega_v(\hat{s} + |\hat{x}-\hat{y}|)$$

where $\omega_v$ is the modulus of continuity of $v$ at time $t=0$.

Letting $\varepsilon \searrow 0$ $\hat{s} \to 0$, $|\hat{x}-\hat{y}| \to 0$ and we get a contradiction ∎

<u>Example</u> of $H$ satisfying the structure assumptions on $H$:

$H(x,p) = H_1(p) + f(x)|p| + \ell(x)$ $\quad H_1 \in C(\mathbb{R}^n)$, $f$ bounded and Lip, $\ell \in UC(\mathbb{R}^n)$.

Note that in the Proof of the Example we had to assume the (uniform) continuity of $u$ and $v$ as $t \to 0$. To avoid this assumption we'll give a different proof supposing that $H$ is uniformly continuous with respect to $p$. The idea is to prove first the C.P. on bounded cylinders and then use it for subsolutions penalized at infinity.

<u>Theorem 1</u>: Assume $H : \bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ continuous and for a mod. $\omega$

$$(RH) \quad H\left(y, \frac{x-y}{\varepsilon}\right) - H\left(x, \frac{x-y}{\varepsilon}\right) \le \omega\left(|x-y|\left(1 + |x-y|/\varepsilon\right)\right) \qquad \forall x, y \in \bar{\Omega},\ \varepsilon > 0.$$

Let $\Omega \subseteq \mathbb{R}^n$ bounded, $u, v : [0, T[ \times \bar{\Omega} \to \mathbb{R}$ bounded, $u$ U.S.C. subsol, $v$ u.s.c. supersol of

$$(HJ) \qquad u_t + H(x, D_x u) = 0 \quad \text{in } ]0, T[ \times \Omega,$$

$$u \le v \quad \text{on } ]0, T[ \times \partial\Omega.$$

Then
$$\sup_{[0,T[\times\bar{\Omega}} (u - v) \le \sup_{x \in \bar{\Omega}} \left(u(0, x) - v(0, x)\right). \quad (☆)$$

<u>Theorem 2</u> Ass. $F : \bar{\Omega} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ cont., degenerate elliptic, and

$$(R) \quad F\left(y, \frac{x-y}{\varepsilon}, Y\right) - F\left(x, \frac{x-y}{\varepsilon}, X\right) \le \omega\left(|x-y|\left(1 + |x-y|^2/\varepsilon\right)\right) \qquad \forall x, y\ \forall \varepsilon > 0$$

$$\forall X, Y : \quad -\frac{3}{\varepsilon} I \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Let $\Omega, u, v$ as above for the PDE

$$u_t + F(x, D_x u, D^2_{xx} u) = 0 \quad \text{in } ]0, T[ \times \Omega.$$

Then we have the same conclusion (☆).

The proof of Thm. 2 is in [Cardaliaguet, CIME L.N.], sect. 12.

Proof of Thm. 1 Take $\tilde{u}$ as on p. 2 and $T_\varepsilon$ such that $\tilde{u} \le v$ on $[T_\varepsilon, T[ \times \Omega$. Then $\forall \tilde{T} \in [T_\varepsilon, T[$ $\tilde{u}$ is bounded in $[0,\tilde{T}[ \times \bar{\Omega}$, subsolution of (HJ), $\tilde{u} \le v$ on $\partial\Omega$ and for $t = \tilde{T}$. Our goal is to prove

$$\sup_{[0,\tilde{T}[\times\bar{\Omega}} (\tilde{u} - v) \le \sup_{\bar{\Omega}} (\tilde{u} - v)^+_{|t=0} =: A \quad \forall \tilde{T} < T$$

We use an argument similar to the Dirichlet problem.

Let us drop $\sim$ in $\tilde{u}$ and $\tilde{T}$ for notational simplicity.

Take
$$\Phi(t,s,x,y) = u(t,x) - v(s,y) - \frac{|x-y|^2 + |t-s|^2}{2\varepsilon}, \quad \varepsilon > 0.$$

Assume by contradiction $\exists\, (\tilde{t},\tilde{x}) : (u-v)(\tilde{t},\tilde{x}) > A$. Then

$$0 \le A < \Phi(\tilde{t},\tilde{t},\tilde{x},\tilde{x}) \le \max \Phi(t,s,x,y) =: \Phi(\hat{t},\hat{s},\hat{x},\hat{y})$$

Assume first that either $\hat{t}$ or $\hat{s} = 0$, say $\hat{t} = 0$, for some $\varepsilon_j \to 0$.

By compactness of $\bar{\Omega}$ $\exists$ subsequence s.t. $\hat{x}_j \to \bar{x}$.

Recall the usual estimates

$$\frac{|\hat{t}-\hat{s}|^2}{\varepsilon}, \frac{|\hat{x}-\hat{y}|^2}{\varepsilon} \le o(1) \quad \text{as } \varepsilon \to 0.$$

Then $\hat{s} \to 0$ and $\hat{y} \to \bar{x}$ as $\varepsilon \to 0$. Then for some $\eta > 0$

$$A + \eta \le \limsup_{\varepsilon_j \to 0} \left(u(0,\hat{x}) - v(\hat{s},\hat{y})\right) \le (u-v)(0,\bar{x}) \le A,$$

a contradiction. The cases $\hat{t}$ or $\hat{s} \to T$ and $\hat{x}$ or $\hat{y} \to \partial\Omega$ are similar (with $A$ replaced by $0$ in the r.h.s.)

We are left with the case $\hat{s} > 0, \hat{t} > 0$ $\forall \varepsilon$ small. From (HJ) we get

$$\frac{\hat t-\hat s}{\varepsilon}+H\left(\hat x,\frac{\hat x-\hat y}{\varepsilon}\right)\le -c<0\le \frac{\hat t-\hat s}{\varepsilon}+H\left(\hat y,\frac{\hat x-\hat y}{\varepsilon}\right).$$

Then
$$0<c\le H\left(\hat y,\frac{\hat x-\hat y}{\varepsilon}\right)-H\left(\hat x,\frac{\hat x-\hat y}{\varepsilon}\right)\le \omega\left(|\hat x-\hat y|+\frac{|\hat x-\hat y|^2}{\varepsilon}\right)\to 0 \quad \text{as } \varepsilon\to 0$$

a contradiction. ∎

Next we prove a Comparison Principle on $[0,T[\times\mathbb{R}^n$ in a unified way for 1st and 2nd order equations under the assumption of uniform continuity in $Du$ & $D^2u$: for a mod. $\omega$

(R2) $\quad |F(x,p,X)-F(x,q,Y)|\le \omega(|p-q|+\|X-Y\|) \quad \forall p,q,X,Y,$

where $\|\cdot\|$ is a matrix norm.

<u>Theorem 3</u>. Assume $F$ as in Thm. 2, & satisfying (R2), $u,v$ as on p. 1, sub- and supersol. of (2) on $(0,T)\times\mathbb{R}^n$. Then the Comparison Principle (3) holds.

<u>Proof</u>. See the proof of Prop 1, p. 28-9 of [AB 03]. ∎

Next we turn to the existence of the Cauchy problem and treat it in a unified way for 1st and 2nd order under the additional assumption

(B) $\quad |F(x,0,0)|\le M \quad \forall x\in\mathbb{R}^n.$

<u>Theorem 4</u> Under the assumptions of Thm. 3 and (B), for any $u_0\in BUC(\mathbb{R}^n)$ there exists a unique $u\in C([0,T]\times\mathbb{R}^n)$ viscosity solution of the Cauchy problem (2) + $u(0,x)=u_0(x)$ $\forall x$.

Proof See the second part of the proof of Prop. 1, p. 29, [AB03]. ∎

Corollary Take $F$ and $u_0$ as in Thm. 4. For any $\varepsilon > 0$ there $\exists$ a unique continuous solution $u^\varepsilon$ of

$$\begin{cases} u^\varepsilon_t + F(\frac{x}{\varepsilon}, Du^\varepsilon, D^2u^\varepsilon) = 0 & \text{in } (0,T)\times\mathbb{R}^n, \\ u^\varepsilon(x,0) = u_0(x), \end{cases}$$

and the family $\{u_\varepsilon\}$ is EQUIBOUNDED; more precisely

$$|u^\varepsilon(t,x)| \le \sup_{\mathbb{R}^n}|u_0| + Mt \qquad \forall t, x.$$

Proof. $V(t,x) = \sup|u_0| + Mt$ solves $V_t + F(\frac{x}{\varepsilon}, DV, D^2V) = M + F(\frac{x}{\varepsilon}, 0, 0) \ge 0$, $V(0,x) \ge u_0(x)$, so the C.P. gives $u^\varepsilon \le V$ $\forall t, x$. Similarly $-V$ is a subsolution, so $u^\varepsilon \ge -V$. ∎

Remark For first order equations the assumption (R) becomes (RH). The three assumptions (RH)(R2)(B) are satisfied by

$$H(x,p) = f(x)|p| + \ell(x) \quad \text{if } f \text{ is bounded \& Lip and } \ell \in BUC(\mathbb{R}^n).$$

It is also satisfied by Bellman's Hamiltonians

$$H(x,p) = \sup_{a\in A}\{-f(x,a)\cdot p - \ell(x,a)\}$$

if $f$ & $\ell$ are bounded, $|f(x,a) - f(y,a)| \le L|x-y|$ $\forall x, y, a$;

$|\ell(x,a) - \ell(y,a)| \le \omega(|x-y|)$ $\forall x, y, a$.

Under similar conditions also the second order Bellman operators of stochastic control satisfy (R)(R2)(B).

On the other hand, if $\limsup_{|p|\to+\infty} \frac{H(x,p)}{|p|} = +\infty$ (R2) cannot hold.