Control system (DETERMINISTIC)

$$(S)\quad \begin{cases} y' = f(y,\alpha), & s \ge t \\ y(t) = x \end{cases}$$

$f: \mathbb{R}^n \times A \to \mathbb{R}^n$ $\quad f(\mathbb{R}^n \times A)$ bounded

$|f(y,a) - f(z,a)| \le L|z-y| \quad \forall z, y \in \mathbb{R}^n,\ a \in A$

$\alpha \in \mathcal{A} = \{$admissible control fns$\}$ measurable or piecewise continuous ...

$\exists!$ sol. $y_x(s) = y_x(s;\alpha)$ to (S)

Cost functional
$$J(t,x,\alpha) := \int_t^T \ell(y_x(s;\alpha),\alpha(s))\, ds + g(y_x(T;\alpha))$$

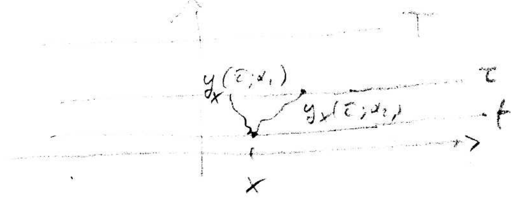
$\ell: \mathbb{R}^n \times A \to \mathbb{R}^n$ bounded and $|\ell(y,a) - \ell(z,a)| \le \omega_\ell(|y-z|)$
$\forall y, z \in \mathbb{R}^n,\ a \in A$.

Value function
$$V(t,x) := \inf_{\alpha \in \mathcal{A}} J(t,x,\alpha)$$

Dynamic Programming Principle

$$V(t,x) = \inf_{\alpha|_{[t,\tau]}} \left\{ \int_t^\tau \ell(y_x(s),\alpha(s))\, ds + V(\tau, y_x(\tau;\alpha)) \right\} \quad \forall t < \tau < T$$

Proof Easy, see [BCD] ∎



Thm If $V \in C^1$ then it solves

$$(HJB)\qquad -V_t + \sup_{a \in A} \{-D_x V \cdot f(x,a) - \ell(x,a)\} = 0 \quad \text{in }]-\infty, T[\times \mathbb{R}^n,$$

If $V \in C(]-\infty,T] \times \mathbb{R}^n)$ then it is a viscosity solution of (HJB).

Proof. "$\le$": $\alpha(s) \equiv a$ const. DPP $\Rightarrow$

$$v(t,x) - v(\tau, y_x(\tau)) \le \int_t^\tau \ell(y_x(s), a)\, ds \qquad (1)$$

$$y_x(s) = x + O(s-t) \qquad \ell(y_x(s), a) = \ell(x,a) + \omega(O(s-t)) \quad s \to t$$

$$\Rightarrow \quad \int_t^\tau \ell(y_x(s), a)\, ds = (\tau - t)(\ell(x,a) + o(1)) \quad s \to t$$

$$v(\tau, y) = v(t,x) + v_t(t,x)(\tau - t) + D_x v(t,x)\cdot(y-x) + o(|\tau - t| + |y-x|)$$

$$v(\tau, y_x(\tau)) = v(t,x) + v_t(t,x)(\tau - t) + D_x v(t,x)\cdot f(x,a)(\tau - t) + o(\tau - t)$$

because $y_x(\tau) - x = \int_t^\tau f(y_x(s), a)\, ds = \int_t^\tau (f(x,a) + O(s-t))\, ds$

$$= (\tau - t)(f(x,a) + o(1))$$

Divide (1) by $\tau - t$ and let $\tau \to t$:

$$-v_t(t,x) - D_x v(t,x)\cdot f(x,a) \le \ell(x,a)$$

$$\Leftrightarrow \quad -v_t + H(x, D_x v) \le 0 \quad , \quad \boxed{H(x,p) := \sup_{a\in A}\{-f(x,a)\cdot p - \ell(x,a)\}}$$

If $v \notin C^1$, let $\phi \in C^1$: $v - \phi$ max at (t,x). Then

$$\phi(t,x) - \phi(\tau, y_x(\tau)) \le v(t,x) - v(\tau, y_x(\tau)) \overset{(1)}{\le} \int_t^\tau \ell(y_x(s), a)\, ds .$$

Then the preceding argument gives

$$-\phi_t + H(x, D_x\phi) \le 0 . \qquad \square \text{ "}\le\text{"}$$

"$\ge$" By DPP $\forall \varepsilon > 0\ \exists\ \bar\alpha \nexists\ \bar y_x(s) = y_x(s; \bar\alpha)$:

$$v(t,x) - v(\tau, \bar y_x(\tau)) \ge \int_t^\tau \ell(\bar y_x(s), \bar\alpha(s))\, ds ,$$

Then

$$-v_t(t,x)(\tau-t) - D_x v(t,x)\cdot\int_t^\tau f(x,\bar\alpha(s))\,ds - \int_t^\tau \ell(x,\bar\alpha(s))\,ds \geq o(\tau-t)$$

$\Rightarrow$

$$-v_t(t,x) + \sup_a\{-D_x v(t,x)\cdot f(x,a) - \ell(x,a)\} \geq o(1) \quad \text{as } \tau\to t$$

$\Rightarrow$ $\quad -v_t + H(x, D_x v) \geq 0.$

If $v \notin C^1$, $\phi \in C^1$: $v-\phi$ <u>min</u> at (t,x) $\Rightarrow$

$$\phi(t,x) - \phi(\tau,\bar y_x(\tau)) \geq v(t,x) - v(\tau,\bar y_x(\tau)) \geq \int_t^\tau \ell(\bar y_x(s),\bar\alpha(s))\,ds$$

and the argument above gives

$$-\phi_t + H(x, D_x v) \geq 0. \quad \blacksquare$$

<u>Exercises</u> (HW) [BCD, Ch. III]

1. Under the preceding assumptions and $g \in BUC(\mathbb{R}^n)$ the value function v is bounded and continuous in $[t_1, T]\times\mathbb{R}^n$ for all $t_1 < T$.

2. $H(x,p)$ satisfies

$$|H(x,p) - H(y,p)| \leq L|x-y||p| + \omega_\ell(|x-y|),$$

$$|H(x,p) - H(x,q)| \leq M|p-q|.$$

3. $u(t,x) = v(T-t,x)$ is the unique v-solution of

$$u_t + H(x, D_x u) = 0 \text{ in } (0,+\infty)\times\mathbb{R}^n, \quad u(0,x) = g(x),$$

and v is the unique v-solution of

$$-v_t + H(x, D_x v) = 0 \text{ in } (-\infty, T)\times\mathbb{R}^n, \quad v(T,x) = g(x). \quad \square$$

STOCHASTIC CONTROL

$$\begin{cases} dY_s = f(Y_s, \alpha_s)\,ds + \sigma(Y_s, \alpha_s)\,dB_s \\ Y_t = x \end{cases}$$

$\alpha_\cdot$ is admissible if it's progressively measurable with respect to the filtration associated to the Brownian B_s ... see the more precise setting in [FS].

Value function

$$v(t,x) = \inf_{\alpha_\cdot} E\left[\int_t^T \ell(Y_s, \alpha_s)\,ds + g(Y_T)\right].$$

Associated H-J-B equation is of SECOND ORDER:

(SHJB) $\quad -v_t + \mathcal{H}(x, D_x v, D^2_{xx} v) = 0 \quad$ in $]-\infty, T[\times \mathbb{R}^n$,

where

$$\mathcal{H}(x, p, X) = \sup_{a \in A} \left\{ -\operatorname{tr}\left(\frac{\sigma\sigma^T}{2}(x,a)\, X\right) - f(x,a)\cdot p - \ell(x,a) \right\}.$$

The PDE is (degenerate) parabolic because $\sigma\sigma^T \geq 0$.

Ex. If $\sigma \equiv I$ the HJB is

$$-v_t + H(x, D_x v) = \Delta_{xx} v. \quad ∎$$

Here the DPP and the rigorous derivation of the HJB equation are much more technical.

Notation: $w \in C^2$, $\mathcal{L}^a w := \operatorname{tr}\left(\frac{\sigma\sigma^T(x,a)}{2} D^2 w\right) + f(x,a)\cdot Dw$

Verification Theorem: Assume A compact, $w \in C^2$ a solution of (SHJB), i.e.,

$$\min_{a \in A} \{ w_t + \mathcal{L}^a w + \ell(x,a) \} = 0$$

and (I.C.) $w(T,x) = g(x)$.

Then
(i) $w(t,x) \le E[J(t,x,\alpha_\cdot)]$ $\forall$ admissible α.

(ii) if $\bar{\alpha}(s) \in \arg\min_a \{ \mathcal{L}^a w(s,\bar{Y}_s) + \ell(\bar{Y}_s, a) \}$ a.a. s

$$\dot{\bar{Y}}_s = f(\bar{Y}_s, \bar{\alpha}_s) ds + \sigma(\bar{Y}_s, \bar{\alpha}_s) dB_s ,$$

then $w(t,x) = E[J(t,x,\bar{\alpha}_\cdot)]$, i.e., $\bar{\alpha}_\cdot$ is OPTIMAL.

Proof. (i) Ito's rule: $\forall$ admissible $\alpha_\cdot$ & traj. $Y_\cdot$

$$w(\tau, Y_\tau) - w(t,x) = \int_t^\tau (w_t + L^{\alpha_s} w)(s, Y_s)\, ds + \int_t^\tau (D_x w \cdot \sigma)(s, Y_s)\, dW_s .$$

Take E:

$$w(t,x) = E\left[-\int_t^\tau (w_t + L^{\alpha_s} w)(s, Y_s)\, ds + w(\tau, Y_\tau) \right]$$

Dynkin formula. Use (SHJB) for $\tau = T$

$$w(t,x) \le E\left[\int_t^T \ell(Y_s, \alpha_s)\, ds + w(T, \bar{Y}_T) \right] = E\, J(t,x,\alpha_\cdot)$$

by (I.C.)

(ii) For $\bar{\alpha}_s, \bar{Y}_s$ is $=$. ∎

Exercises (HW)

1. Under the preceding assumptions on f and ℓ, and for $|\sigma(y,a)-\sigma(z,a)| \le L|y-z|$, $\sigma(\mathbb{R}^n \times A)$ bounded $\mathcal{H}$ satisfies the conditions for $\exists$ and uniqueness of the Cauchy problem (visco. sols.) for (SHJB).

2. If v solves (SHJB) then $u(t,x) = v(T-t,x)$ solves $u_t + \mathcal{H}(x, D_x u, D^2_{xx} u) = 0$ in $]0,\infty[\times\mathbb{R}^n$.

References:

[BCD] M. Bardi, I. Capuzzo-Dolcetta, book, Birkhäuser 1997.

[FS] W. Fleming, H.M. Soner, book, Springer 1993, 2nd Ed. 2006