Elliptic Hamilton-Jacobi-Bellman equations degenerating at the boundary and applications to stochastic control

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H-J-Bellman operator

$$F[u] := \sup_{\alpha \in A} \left\{ -\operatorname{tr}\left(a(x,\alpha) D^2 u(x) \right) - b(x,\alpha) \cdot Du(x) \right\},$$

in $\Omega \subseteq \mathbb{R}^n$ bounded connected open set with C^2 boundary,
$A \subseteq \mathbb{R}^m$ closed, $a \equiv \sigma\sigma^T \geq 0$

$$b : \overline{\Omega} \times A \to \mathbb{R}^n, \qquad \sigma : \overline{\Omega} \times A \to \mathbf{M}_{n \times r}, \quad r \geq 1$$

Associated controlled diffusion process

$$\begin{cases} dX_t^{\alpha.} = b(X_t^{\alpha.}, \alpha_t) dt + \sqrt{2} \sigma(X_t^{\alpha.}, \alpha_t) dW_t \\ X_0^{\alpha.} = x \in \overline{\Omega}, \end{cases}$$

W_t $r-$dimensional Brownian motion.

Outline

- Solutions of $F[u] \leq 0$ or $F[u] \geq 0$ in Ω with an "infinite boundary condition" are constant.
- Application 1 - Stochastic control with exit time:

$$v(x) := \inf_{\alpha_\cdot \in \mathcal{A}} \mathbb{E}\left[\phi(X^{\alpha_\cdot}_{\tau^{\alpha_\cdot}_x})\right] = \text{ explicit constant,}$$

where $\tau^{\alpha_\cdot}_x$ is the first exit time of the process $X^{\alpha_\cdot}_t$ from Ω.

- Application 2 - Ergodic HJB equation:

$$F[\chi] - l(x) = c \text{ in } \Omega, \quad + \text{"infinite boundary condition" on } \partial\Omega$$

in the unknowns (c, χ) has a unique solution (χ up to additive constants).

Standing assumptions

- (H) Hölder continuity of data

$$\begin{cases} |b(x,\alpha) - b(y,\alpha)| \le C|x-y|^{\eta}, & 0 < \eta \le 1 \\ |\sigma(x,\alpha) - \sigma(y,\alpha)| \le B|x-y|^{\beta}, & \frac{1}{2} < \beta \le 1 \end{cases}$$

- (SMP) Strong Maximum Principle:

  $u \in USC(\Omega)$ solving $F[u] \le 0$ in Ω and $\exists\, x_o$ such that $u(x_o) = \max_\Omega u \implies u$ is constant.

Sufficient conditions for (SMP) [M.B. - F. Da Lio 2001-3]:
$\forall\, x \quad \exists\, \alpha_x \in A$ such that

- either $a(x, \alpha_x) > 0$ (strict ellipticity for some control)
- or the operator $\operatorname{tr}\left(a(x,\alpha_x) D^2 \cdot\right)$ is Hörmander hypoelliptic.

Standing assumptions continued

Define

$$d(x) := \text{dist}(x, \mathbb{R}^n \backslash \Omega) - \text{dist}(x, \overline{\Omega})$$

- (B1) $\exists\, \delta, k > 0,\ \gamma < 2\beta - 1$ such that $\forall \overline{x} \in \partial\Omega,\ \exists\, \overline{\alpha} \in A$:

$$\begin{cases} \sigma^T(\overline{x}, \overline{\alpha}) Dd(\overline{x}) = 0, & (a) \\ b(x, \overline{\alpha}) \cdot Dd(x) + \text{tr}(a(x, \overline{\alpha}) D^2 d(x)) \geq k\, d^\gamma(x),\ \forall x \in \Omega \cap B_\delta(\overline{x}) & (b) \end{cases}$$

Meaning: (a): diffusion with control $\overline{\alpha}$ degenerates in the direction $n(\overline{x}) = Dd(\overline{x})$, but
(b) $b(x, \overline{\alpha})$ points inward Ω in a neighborhood of $\overline{x}$.

Sufficient condition for (b): (a) +
$b(\overline{x}, \overline{\alpha}) \cdot Dd(\overline{x}) + \text{tr}(a(\overline{x}, \overline{\alpha}) D^2 d(\overline{x})) \geq k > 0$.

N.B.: (B1) $\Longrightarrow$ Ω viable, or weakly invariant.

A Liouville-type property 1: subsolutions

Theorem (1)

(H), (SMP), (B1), $u \in USC(\Omega)$ *viscosity subsolution* $F[u] \le 0$ *in* Ω*, and*

(BC1)
$$\limsup_{x \to \partial\Omega} \frac{u(x)}{-\log d(x)} \le 0,$$

$\Longrightarrow$ *u is constant.*

Lemma (Lyapunov-like function)

$\forall\, M \ge 0, \exists\, \delta > 0$ *such that*

$$F[-\log(d(x))] > M, \qquad \forall\, x \in \Omega_\delta,$$

where $\Omega_\delta := \{x \in \Omega \mid d(x) < \delta\}$.

Proof of the Lyapunov Lemma

Choose $\overline{x} \in \partial\Omega : d(x) = |x - \overline{x}|$ and $Dd(x) = Dd(\overline{x})$,

and $\overline{\alpha} \in A$ such that $\sigma^T(\overline{x}, \overline{\alpha}) Dd(\overline{x}) = 0$ by (B1).

For $w(x) := -\log(d(x))$,

$$Dw = -Dd/d, \quad D^2 w = -D^2 d/d + Dd \otimes Dd/d^2, \quad \Longrightarrow$$

$$F[w](x) =$$

$$\sup_{\alpha \in A} \left(b(x,\alpha) \cdot \frac{Dd(x)}{d(x)} + \operatorname{tr}\left(a(x,\alpha) \frac{D^2 d(x)}{d(x)} \right) - \frac{1}{d^2(x)} |\sigma^T(x,\alpha) Dd(x)|^2 \right)$$

$$\geq \frac{1}{d(x)} \left(b(x,\overline{\alpha}) \cdot Dd(x) + \operatorname{tr}(a(x,\overline{\alpha}) D^2 d(x)) \right) - \frac{1}{d^2(x)} |\sigma^T(x,\overline{\alpha}) Dd(x)|^2$$

$$\geq k d^{\gamma-1}(x) - B^2 d^{2\beta-2}(x) \geq M$$

by (H) and $\gamma < 2\beta - 1$, for d small enough.

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Proof of the Theorem

Set

$$U_\varepsilon(x) := u(x) - \max_{\Omega\setminus\Omega_\delta} u + \varepsilon \log d(x)$$

Then

- $U_\varepsilon(x) = \varepsilon \log d(x) < 0 \quad$ if $\quad d(x) = \delta < 1$
- $\lim_{x\to\partial\Omega} U_\varepsilon(x) = -\infty \quad$ by (BC1).

$\Longrightarrow \quad U_\varepsilon < 0 \quad$ in Ω_δ for δ small, because, if not,

$u - \varepsilon w$ has a max at $x_o \in \Omega_\delta$, but

$F[\varepsilon w](x_o) = \varepsilon F[w] > 0$ by the Lemma, contradiction with $F[u] \leq 0$.

$\Longrightarrow \quad u(x) \leq \max_{\Omega\setminus\Omega_\delta} u - \varepsilon \log d(x)$ in Ω_δ $\forall \varepsilon > 0$

$\Longrightarrow \quad \max_\Omega u \leq \max_{\Omega\setminus\Omega_\delta} u.$

(SMP) $\quad \Longrightarrow \quad u \equiv$ constant.

A Liouville-type property 2: supersolutions

- (B2) $\exists\, \delta, k > 0,\ \gamma < 2\beta - 1$ such that $\forall \overline{x} \in \partial\Omega,\ \forall \alpha \in A$
$$\begin{cases} \sigma^T(\overline{x}, \alpha) Dd(\overline{x}) = 0, \\ b(x,\alpha) \cdot Dd(x) + \mathrm{tr}(a(x,\alpha) D^2 d(x)) \ge k\, d^\gamma(x),\ \forall x \in \Omega_\delta \end{cases}$$
- Strong Minimum Principle:
$u \in LSC(\Omega)$ solving $F[u] \ge 0$ in Ω and $\exists\, x_o$ such that
$u(x_o) = \min_\Omega u \quad \Longrightarrow \quad u$ is constant.

Sufficient conditions [M.B. - F. Da Lio]: $a(x,\alpha) > 0\ \forall\, x \in \Omega, \alpha \in A$.
N.B.: (B2) $\Longrightarrow$ Ω invariant.

Theorem (1bis)

(H), (SMinP), (B2), $u \in LSC(\Omega)$ visco. supersol. $F[u] \ge 0$ in Ω, and

(BC2)
$$\limsup_{x \to \partial\Omega} \frac{u(x)}{-\log d(x)} \ge 0,$$

$\Longrightarrow$ *u is constant.*

Application 1: control with exit time

Assume

(L) $\eta = \beta = 1$ in (H), i.e., b, σ Lipschitz uniformly in α.

$\Longrightarrow \quad \forall \alpha_{\cdot} \in \mathcal{A}$ admissible control, exists a unique trajectory of the control diffusion process

$$\begin{cases} dX_t^{\alpha_\cdot} = b(X_t^{\alpha_\cdot}, \alpha_t)dt + \sqrt{2}\sigma(X_t^{\alpha_\cdot}, \alpha_t)dW_t \\ X_0^{\alpha_\cdot} = x \in \overline{\Omega}, \end{cases}$$

Define

$$\tau_x^{\alpha_\cdot} := \inf\{t \geq 0 \mid X_t^{\alpha_\cdot} \notin \Omega\} \in [0, +\infty]$$

$$G(x, \alpha) := \begin{cases} \phi(X_{\tau_x^{\alpha_\cdot}}^{\alpha_\cdot}) & \tau_x^{\alpha_\cdot} < +\infty \\ 0 & \tau_x^{\alpha_\cdot} = +\infty, \end{cases} \qquad v(x) := \inf_{\alpha_\cdot \in \mathcal{A}} \mathbb{E}\left[G(x, \alpha)\right]$$

By a Dynamic Programming Principle we expect $F[v] = 0$ in Ω and then $v \equiv$ constant under the assumptions of Thm. 1.

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Theorem (2)

Assume (L), (SMP), (B1), and

- *A compact and* $\{(b(x,\alpha), a(x,\alpha)) \mid \alpha \in A\}$ *convex* $\forall\ x \in \overline{\Omega}$
- $\exists\ \underline{x} \in \partial\Omega,\ \underline{\alpha} \in A$ *such that* $\phi(\underline{x}) = \min \phi$ *and*

$$\text{(ND)} \qquad \begin{cases} \textit{either} \quad \sigma^T(\underline{x},\underline{\alpha}) Dd(\underline{x}) \neq 0, & (a) \\ \textit{or} \quad b(\underline{x},\underline{\alpha}) \cdot Dd(\underline{x}) + tr(a(\underline{x},\underline{\alpha}) D^2 d(\underline{x})) < 0. & (b) \end{cases}$$

Then

$$v(x) = \min\{\min \phi, 0\} \qquad \forall x \in \Omega.$$

Rmk.: (ND) (a) is a non-degeneracy condition of the diffusion at $\underline{x} \in \partial\Omega$,
(b) says that $b(\underline{x}, \underline{a})$ points outward Ω at $\underline{x}$:

the control $\underline{\alpha}$ "does the opposite" than $\overline{\alpha}$ in condition (B1).

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Application 2: ergodic HJB equation

Further data: running cost $l : \mathbb{R}^n \times A \to \mathbb{R}$ satisfying

(H') $\quad |l(x,\alpha) - l(y,\alpha)| \le B|x-y|^\gamma$.

Ergodic H-J-Bellman equation:

(EBE) $$\sup_{\alpha \in A} \left\{ -\operatorname{tr}\left(a(x,\alpha)D^2u(x)\right) - b(x,\alpha)\cdot Du(x) - l(x,\alpha) \right\} = c$$

in Ω + boundary conditions, in the unknowns (c, u).

Motivations: ergodic stochastic control (with state constraints),
homogenization, singular perturbations,
weak KAM theory,
Mean Field Games,....

Notation: $H(x, Du, D^2u) :=$ left-hand side of (EBE).

Theorem (3)

Assume (H), (H'), (B2) (i.e., $\overline{\Omega}$ invariant), and

$a(x,\alpha) > 0 \quad \forall\, x \in \Omega, \alpha \in A.$ *Then*

- $\exists$ *unique $c \in \mathbb{R}$ such that (EBE) has a visco. sol. χ :*

$$\text{(BC)} \qquad \lim_{x \to \partial\Omega} \frac{\chi(x)}{-\log d(x)} = 0,$$

- $\chi \in C^2(\Omega)$ *and is unique up to addition of constants.*

The proof is constructive: appoximate (EBE) with

$$(\lambda\text{BE}) \qquad \lambda u_\lambda + H(x, Du_\lambda, D^2 u_\lambda) = 0, \qquad x \in \Omega,$$

for $\lambda > 0$, the HJB equation of discounted, infinite horizon stochastic control with state constrained in Ω.
Idea: let $\lambda \to 0+$.

Proposition

Under the assumptions of Thm. 3

1. $\exists$ *unique bounded solution* $u_\lambda \in C^2(\Omega)$ *of* (λ*BE*) *and* $\|u_\lambda\|_\infty \le \|l\|_\infty/\lambda$;
2. $\forall K \subset\subset \Omega$, $\forall \tilde{x} \in \Omega$, *the family* $(u_\lambda - u_\lambda(\tilde{x}))_{\lambda \in (0,1]}$ *is bounded in* $C^{1,\theta}(K)$, *for some* $\theta \in (0,1)$.
3. *for all* $h > 0$, *there exists* $\delta_h > 0$ *such that*

$$h \log(d(x)) + \min_{\Omega \setminus \Omega_{\delta_h}} u_\lambda \le u_\lambda(x) \le -h \log(d(x)) + \max_{\Omega \setminus \Omega_{\delta_h}} u_\lambda .$$

$$\forall \lambda \in (0,1], \ x \in \Omega_{\delta_h} .$$

Main ingredient of the proof: Krylov-Safonov estimates for uniformly elliptic operators.

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Steps of the proof of Thm. 3

- $\lambda u_\lambda(\tilde{x}) \to -c$ as $\lambda \to 0+$
- $u_\lambda - u_\lambda(\tilde{x}) \to \chi$ solution of (EBE), locally uniformly on a subsequence;
- χ satisfies (BC) by point 3 of the Proposition.
- Uniqueness: (c_1, χ_1), (c_2, χ_2) solutions of (EBE), w.l.o.g. $c_1 \geq c_2$.

$$F[\chi_1 - \chi_2] = \sup_{\alpha \in A} \left\{ -\operatorname{tr}(a(x,\alpha)D^2(\chi_1 - \chi_2)) - b(x,\alpha) \cdot D(\chi_1 - \chi_2) \right\}$$

$$\geq \sup_{\alpha \in A} \left\{ -\operatorname{tr}(a(x,\alpha)D^2\chi_1) - b(x,\alpha) \cdot D\chi_1 - l(x,\alpha) \right\}$$

$$+ \inf_{\alpha \in A} \left\{ \operatorname{tr}(a(x,\alpha)D^2\chi_2)) + b(x,\alpha) \cdot D\chi_2 + l(x,\alpha) \right\}$$

$$= c_1 - c_2 \geq 0.$$

By Theorem 1bis $\chi_1 - \chi_2 \equiv$ constant, and then $c_1 = c_2$.

Stochastic control formula for c

Corollary

Assume in addition (L). Then

$$c = \lim_{\lambda \to 0+} \inf_{\alpha_\cdot \in \mathcal{A}} \mathbb{E}\left[\lambda \int_0^\infty e^{-\lambda t} l(X_t^{\alpha_\cdot}, \alpha_t)dt\right],$$

uniformly w.r.t. $x = X_0^{\alpha_\cdot}$,

i.e., $c =$ vanishing discount limit of minimum long-time average cost,

for all initial positions x.

Remark. Uniqueness holds also in the larger class:
for some $\kappa > 0$

$$\lim_{x \to \partial\Omega} \chi(x) d(x)^\kappa = 0.$$

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Developments

D. Castorina, A. Cesaroni and L. Rossi studied the evolutive problem

$$u_t + H(x, Du, D^2u) = 0$$

with initial conditions and (infinite) boundary conditions, and showed

$$u(x,t) - ct \to \chi(x) + k, \quad \text{as } t \to +\infty,$$

where (c, χ) solves (EBE) and $k \in \mathbb{R}$ depends on $u(x, 0)$.

Using that $\chi(x) - ct$ is a solution of the parabolic equation $\forall\, t \in \mathbb{R}$, they also show that χ is bounded, which improves the existence part of Thm. 3.

Some references

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Thanks for your attention!