Differential games with long-time-average cost

Martino Bardi

Department of Pure and Applied Mathematics
University of Padua, Italy

Based on

- O. Alvarez, M.B.: Ergodic problems in differential games, Ann. Internat. Soc. Dynam. Games, 9, Birkhäuser, Boston, 2007.
- M.B.: On differential games with long-time-average cost, Ann. Internat. Soc. Dynam. Games, to appear.
- O. Alvarez, M.B.: Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equations, Mem. Amer. Math. Soc. to appear

Based on

- O. Alvarez, M.B.: Ergodic problems in differential games, Ann. Internat. Soc. Dynam. Games, 9, Birkhäuser, Boston, 2007.
- M.B.: On differential games with long-time-average cost, Ann. Internat. Soc. Dynam. Games, to appear.
- O. Alvarez, M.B.: Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equations, Mem. Amer. Math. Soc. to appear

Based on

- O. Alvarez, M.B.: Ergodic problems in differential games, Ann. Internat. Soc. Dynam. Games, 9, Birkhäuser, Boston, 2007.
- M.B.: On differential games with long-time-average cost, Ann. Internat. Soc. Dynam. Games, to appear.
- O. Alvarez, M.B.: Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equations, Mem. Amer. Math. Soc. to appear

Plan

- LTAC 0-sum differential games
- Ergodic games and the Hamilton-Jacobi-Isaacs PDE
- Sufficient conditions for ergodicity: system with noise
- Controllability conditions
- Combined conditions on the system and the cost
- Perspectives

Plan

- LTAC 0-sum differential games
- Ergodic games and the Hamilton-Jacobi-Isaacs PDE
- Sufficient conditions for ergodicity: system with noise
- Controllability conditions
- Combined conditions on the system and the cost
- Perspectives

Plan

- LTAC 0-sum differential games
- Ergodic games and the Hamilton-Jacobi-Isaacs PDE
- Sufficient conditions for ergodicity: system with noise
- Controllability conditions
- Combined conditions on the system and the cost
- Perspectives

Plan

- LTAC 0-sum differential games
- Ergodic games and the Hamilton-Jacobi-Isaacs PDE
- Sufficient conditions for ergodicity: system with noise
- Controllability conditions
- Combined conditions on the system and the cost
- Perspectives

Plan

- LTAC 0-sum differential games
- Ergodic games and the Hamilton-Jacobi-Isaacs PDE
- Sufficient conditions for ergodicity: system with noise
- Controllability conditions
- Combined conditions on the system and the cost
- Perspectives

Plan

- LTAC 0-sum differential games
- Ergodic games and the Hamilton-Jacobi-Isaacs PDE
- Sufficient conditions for ergodicity: system with noise
- Controllability conditions
- Combined conditions on the system and the cost
- Perspectives

LTAC differential games

We are given a (nonlinear) system with two controls

$$(S) \qquad \begin{aligned} &\dot{y}(t) = f(y(t), a(t), b(t)), \qquad a(t) \in A,\ b(t) \in B, \\ &y(0) = x \in \mathbf{R}^m \end{aligned}$$

with A, B compact sets, and the long time average cost functional

$$J(x, a(\cdot), b(\cdot)) := \lim_{T\to\infty} \frac{1}{T} \int_0^T l(y_x(t), a(t), b(t))\, dt,$$

if the limit exists, where $y_x(\cdot)$ the trajectory starting at x.
Player 1 governing a_s wants to MINIMIZE J,
Player 2 governing b_s wants to MAXIMIZE J.

LTAC differential games

We are given a (nonlinear) system with two controls

$$
\text{(S)} \qquad \begin{aligned} &\dot{y}(t) = f(y(t), a(t), b(t)), \qquad a(t) \in A,\ b(t) \in B, \\ &y(0) = x \in \mathbf{R}^m \end{aligned}
$$

with A, B compact sets, and the long time average cost functional

$$
J(x, a(\cdot), b(\cdot)) := \lim_{T\to\infty} \frac{1}{T} \int_0^T l(y_x(t), a(t), b(t))\, dt,
$$

if the limit exists, where $y_x(\cdot)$ the trajectory starting at x.

Player 1 governing a_s wants to MINIMIZE J,

Player 2 governing b_s wants to MAXIMIZE J.

LTAC differential games

We are given a (nonlinear) system with two controls

$$(S) \qquad \begin{aligned} &\dot{y}(t) = f(y(t), a(t), b(t)), \qquad a(t) \in A,\ b(t) \in B, \\ &y(0) = x \in \mathbf{R}^m \end{aligned}$$

with A, B compact sets, and the long time average cost functional

$$J(x, a(\cdot), b(\cdot)) := \lim_{T \to \infty} \frac{1}{T} \int_0^T l(y_x(t), a(t), b(t))\, dt,$$

if the limit exists, where $y_x(\cdot)$ the trajectory starting at x.
Player 1 governing a_s wants to MINIMIZE J,
Player 2 governing b_s wants to MAXIMIZE J.

Define the upper and the lower cost functionals

$$J^{\infty}(x, a(\cdot), b(\cdot)) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T l(y_x(t), a(t), b(t))\, dt,$$

$$J_{\infty}(x, a(\cdot), b(\cdot)) := \liminf_{T\to\infty} \frac{1}{T}\int_0^T l(y_x(t), a(t), b(t))\, dt.$$

the admissible open-loop controls for the players

$$\mathcal{A} := \{a : [0, +\infty) \to A \text{ measurable}\}$$

$$\mathcal{B} := \{b : [0, +\infty) \to B \text{ measurable}\}$$

and the nonanticipating strategies for the first and the second player, respectively,

$$\Gamma := \{\alpha : \mathcal{B} \to \mathcal{A} \,|\, b(s) = \tilde{b}(s)\, \forall s \le t \;\Rightarrow\; \alpha[b](s) = \alpha[\tilde{b}](s)\, \forall s \le t\}$$

$$\Delta := \{\beta : \mathcal{A} \to \mathcal{B} \,|\, a(s) = \tilde{a}(s)\, \forall s \le t \;\Rightarrow\; \beta[a](s) = \beta[\tilde{a}](s)\, \forall s \le t\}$$

Definition of value

The upper and the lower value are

$$u - \mathrm{val}\, J^{\infty}(x) := \sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} J^{\infty}(x, a(\cdot), \beta[a](\cdot))$$

$$l - \mathrm{val}\, J_{\infty}(x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} J_{\infty}(x, \alpha[b](\cdot), b(\cdot))$$

The game has a value if

$$u - \mathrm{val}\, J^{\infty}(x) = l - \mathrm{val}\, J_{\infty}(x).$$

I'll give conditions under which the value exists and it is A CONSTANT (i.e., it does not depend on the initial position of the system)

Link with the finite horizon problem

Consider the lower value of the finite horizon problem

$$v(t,x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds,$$

Question:

$$l - \mathrm{val}\, J_\infty(x) = \liminf_{t\to+\infty} \frac{v(t,x)}{t} \ ?$$

i.e., can exchange $\liminf_{t\to+\infty}$ with $\inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}}$?

Link with the finite horizon problem

Consider the lower value of the finite horizon problem

$$v(t,x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds,$$

Question:

$$l - \mathrm{val}\, J_\infty(x) = \liminf_{t\to+\infty} \frac{v(t,x)}{t} \ ?$$

i.e., can exchange $\liminf_{t\to+\infty}$ with $\inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}}$?

Ergodicity

Theorem

If the lower game is ERGODIC, i.e.,

$$\lim_{t \to +\infty} \frac{v(t,x)}{t} = \lambda \text{ constant, locally uniformly,}$$

then

$$l - \mathrm{val}\, J_\infty(x) = \lambda.$$

A symmetric result holds for the upper game.

For a single player this problem is called ergodic control.
The name comes from classical ergodic theory.

Ergodicity

Theorem

If the lower game is ERGODIC, i.e.,

$$\lim_{t \to +\infty} \frac{v(t,x)}{t} = \lambda \text{ constant, locally uniformly,}$$

then

$$l - \mathrm{val}\, J_\infty(x) = \lambda.$$

A symmetric result holds for the upper game.

For a single player this problem is called ergodic control.
The name comes from classical ergodic theory.

Ergodicity

Theorem

If the lower game is ERGODIC, i.e.,

$$\lim_{t \to +\infty} \frac{v(t,x)}{t} = \lambda \text{ constant, locally uniformly,}$$

then

$$l - \mathrm{val}\, J_\infty(x) = \lambda.$$

A symmetric result holds for the upper game.

For a single player this problem is called ergodic control.
The name comes from classical ergodic theory.

Ergodicity

Theorem

If the lower game is ERGODIC, i.e.,

$$\lim_{t\to+\infty} \frac{v(t,x)}{t} = \lambda \ \text{constant, locally uniformly,}$$

then

$$l - \mathrm{val}\, J_\infty(x) = \lambda.$$

A symmetric result holds for the upper game.

For a single player this problem is called ergodic control.
The name comes from classical ergodic theory.

A dynamical system

$$\dot{y}(t) = f(y(t)), \qquad y(0) = x,$$

is ergodic with respect to an invariant measure μ if $\forall$ measurable l

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t l(y_x(s))\, ds = \int l\, d\mu \quad \text{for a.e. x.}$$

The convergence is loc. uniform $\Longleftrightarrow$ the system has a unique invariant measure.

A dynamical system

$$\dot{y}(t) = f(y(t)), \qquad y(0) = x,$$

is ergodic with respect to an invariant measure μ if $\forall$ measurable l

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t l(y_x(s))\, ds = \int l\, d\mu \quad \text{for a.e. x.}$$

The convergence is loc. uniform $\iff$ the system has a unique invariant measure.

Existence of value

Corollary

If the lower game is ergodic and the Isaacs condition

$$\min_{b\in B}\max_{a\in A}\{-f(y,a,b)\cdot p - l(y,a,b)\} = \max_{a\in A}\min_{b\in B}\{-f(y,a,b)\cdot p - l(y,a,b)\},$$

holds $\forall y, p \in \mathbf{R}^m$, then the LTAC game has a value.

Proof: under the Isaacs condition the lower and upper value of the finite horizon game coincide.
By the ergodic assumption they converge to a constant λ, which must be $l - \mathrm{val}\, J_\infty(x)$ and $u - \mathrm{val}\, J^\infty(x)$, so these values coincide.

Existence of value

Corollary

If the lower game is ergodic and the Isaacs condition

$$\min_{b\in B}\max_{a\in A}\{-f(y,a,b)\cdot p - l(y,a,b)\} = \max_{a\in A}\min_{b\in B}\{-f(y,a,b)\cdot p - l(y,a,b)\},$$

holds $\forall y, p \in \mathbf{R}^m$, then the LTAC game has a value.

Proof: under the Isaacs condition the lower and upper value of the finite horizon game coincide.
By the ergodic assumption they converge to a constant λ, which must be $l - \mathrm{val}\, J_\infty(x)$ and $u - \mathrm{val}\, J^\infty(x)$, so these values coincide.

The infinite horizon small discount problem

Lower value of the infinite horizon problem with discount rate $\delta > 0$ is

$$w_\delta(x) := \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} \int_0^\infty e^{-\delta s} l(y_x(s), \alpha[b](s), b(s))\, ds,$$

Question: what is

$$\lim_{\delta \to 0} \delta w_\delta(x)\ ?$$

Should be related to $\lim_{t \to +\infty} v(t,x)/t$ by the Abel-Tauber theorem

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \phi(s)\, ds = \lim_{\delta \to 0} \delta \int_0^\infty e^{-\delta s} \phi(s)\, ds$$

whenever one of the two limits exists.

The infinite horizon small discount problem

Lower value of the infinite horizon problem with discount rate $\delta > 0$ is

$$w_\delta(x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \int_0^\infty e^{-\delta s} l(y_x(s), \alpha[b](s), b(s))\, ds,$$

Question: what is

$$\lim_{\delta\to 0} \delta w_\delta(x) \; ?$$

Should be related to $\lim_{t\to+\infty} v(t,x)/t$ by the Abel-Tauber theorem

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \phi(s)\, ds = \lim_{\delta\to 0} \delta \int_0^\infty e^{-\delta s}\phi(s)\, ds$$

whenever one of the two limits exists.

The infinite horizon small discount problem

Lower value of the infinite horizon problem with discount rate $\delta > 0$ is

$$w_\delta(x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \int_0^\infty e^{-\delta s} l(y_x(s), \alpha[b](s), b(s))\, ds,$$

Question: what is

$$\lim_{\delta\to 0} \delta w_\delta(x) \ ?$$

Should be related to $\lim_{t\to+\infty} v(t,x)/t$ by the Abel-Tauber theorem

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \phi(s)\, ds = \lim_{\delta\to 0} \delta \int_0^\infty e^{-\delta s}\phi(s)\, ds$$

whenever one of the two limits exists.

Abelian-Tauberian theorem for games

Theorem [O. Alvarez - M.B. 2003]

For compact state space the lower (finite horizon) game is ergodic, i.e.,

$$\lim_{t\to+\infty} v(t,x)/t = \lambda \ \text{ uniformly,}$$

$$\iff \qquad \lim_{\delta\to 0} \delta w_\delta(x) \ = \ \text{constant, uniformly}$$

and then the constant is the same; λ is also the unique constant s. t.

$$\lambda + H(x, D\chi) = 0 \quad \text{in } \mathbf{R}^m$$

has a (possibly discontinuous) viscosity solution χ.

The proof uses the Isaacs PDEs satisfied by the values where

$$H(x,p) = \min_{b\in B} \max_{a\in A} \{-f(y,a,b)\cdot p - l(y,a,b)\}.$$

The H-J-Isaacs PDE of ergodic games

- $$\lambda + H(x, D\chi) = 0 \quad \text{in } \mathbf{R}^m$$
  is a nonlinear additive-eigenvalue problem with unknowns $\lambda \in \mathbf{R}$ and $\chi : \mathbf{R}^m \to \mathbf{R}$
- for H convex in p (e.g., just one player) λ is also known as Mañe critical value of H in the theory of Hamiltonian systems, very important in the weak KAM theory (Mather, Fathi and others)
- If the system is disturbed by a white noise

  $$dy(t) = f(y(t), a(t), b(t))\, dt + \sqrt{2}\, dW_t$$

  the PDE becomes $\lambda - \Delta\chi + H(x, D\chi) = 0 \quad \text{in } \mathbf{R}^m$
- systems of N such equations arise in the theory of Nash equilibria for N-player games, and in the limit as $N \to \infty$, i.e., in Mean Field Games.

The H-J-Isaacs PDE of ergodic games

- $$\lambda + H(x, D\chi) = 0 \quad \text{in } \mathbf{R}^m$$
  is a nonlinear additive-eigenvalue problem with unknowns $\lambda \in \mathbf{R}$ and $\chi : \mathbf{R}^m \to \mathbf{R}$
- for H convex in p (e.g., just one player) λ is also known as Mañe critical value of H in the theory of Hamiltonian systems, very important in the weak KAM theory (Mather, Fathi and others)
- If the system is disturbed by a white noise
  $$dy(t) = f(y(t), a(t), b(t))\, dt + \sqrt{2}\, dW_t$$
  the PDE becomes $\lambda - \Delta\chi + H(x, D\chi) = 0 \quad \text{in } \mathbf{R}^m$
- systems of N such equations arise in the theory of Nash equilibria for N-player games, and in the limit as $N \to \infty$, i.e., in Mean Field Games.

The H-J-Isaacs PDE of ergodic games

- $$\lambda + H(x, D\chi) = 0 \quad \text{in } \mathbf{R}^m$$
  is a nonlinear additive-eigenvalue problem with unknowns $\lambda \in \mathbf{R}$ and $\chi : \mathbf{R}^m \to \mathbf{R}$
- for H convex in p (e.g., just one player) λ is also known as Mañe critical value of H in the theory of Hamiltonian systems, very important in the weak KAM theory (Mather, Fathi and others)
- If the system is disturbed by a white noise

  $$dy(t) = f(y(t), a(t), b(t))\, dt + \sqrt{2}\, dW_t$$

  the PDE becomes $\quad \lambda - \Delta\chi + H(x, D\chi) = 0 \quad \text{in } \mathbf{R}^m$
- systems of N such equations arise in the theory of Nash equilibria for N-player games, and in the limit as $N \to \infty$, i.e., in Mean Field Games.

The H-J-Isaacs PDE of ergodic games

- $$\lambda + H(x, D\chi) = 0 \quad \text{in } \mathbf{R}^m$$
  is a nonlinear additive-eigenvalue problem with unknowns $\lambda \in \mathbf{R}$ and $\chi : \mathbf{R}^m \to \mathbf{R}$
- for H convex in p (e.g., just one player) λ is also known as Mañe critical value of H in the theory of Hamiltonian systems, very important in the weak KAM theory (Mather, Fathi and others)
- If the system is disturbed by a white noise

  $$dy(t) = f(y(t), a(t), b(t))\, dt + \sqrt{2}\, dW_t$$

  the PDE becomes $\lambda - \Delta\chi + H(x, D\chi) = 0 \quad \text{in } \mathbf{R}^m$
- systems of N such equations arise in the theory of Nash equilibria for N-player games, and in the limit as $N \to \infty$, i.e., in Mean Field Games.

Sufficient conditions for ergodicity

In view of the previous results we concentrate on proving the ergodicity of the lower game, i.e.

$$\lim_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \frac{1}{t}\int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds = \lambda \text{ constant, loc. uniformly,}$$

because the existence of the LTAC value follows from it.

For simplicity we assume from now on that all data are $\mathbb{Z}^m$ periodic in y, i.e. the state space is the torus $\mathbb{T}^M := \mathbf{R}^m/\mathbb{Z}^m$. I show 4 cases

- systems with nondegenerate noise
- controllability by one player
- separate controllability
- combined conditions on the system and the cost.

Sufficient conditions for ergodicity

In view of the previous results we concentrate on proving the ergodicity of the lower game, i.e.

$$\lim_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \frac{1}{t}\int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds = \lambda \ \text{ constant, loc. uniformly,}$$

because the existence of the LTAC value follows from it.
For simplicity we assume from now on that all data are $\mathbb{Z}^m$ periodic in y, i.e. the state space is the torus $\mathbb{T}^M := \mathbf{R}^m/\mathbb{Z}^m$. I show 4 cases

- systems with nondegenerate noise
- controllability by one player
- separate controllability
- combined conditions on the system and the cost.

Sufficient conditions for ergodicity

In view of the previous results we concentrate on proving the ergodicity of the lower game, i.e.

$$\lim_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \frac{1}{t}\int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds = \lambda \text{ constant, loc. uniformly,}$$

because the existence of the LTAC value follows from it.
For simplicity we assume from now on that all data are $\mathbb{Z}^m$ periodic in y, i.e. the state space is the torus $\mathbb{T}^M := \mathbf{R}^m/\mathbb{Z}^m$. I show 4 cases

- systems with nondegenerate noise
- controllability by one player
- separate controllability
- combined conditions on the system and the cost.

Sufficient conditions for ergodicity

In view of the previous results we concentrate on proving the ergodicity of the lower game, i.e.

$$\lim_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \frac{1}{t}\int_0^t l(y_x(s),\alpha[b](s),b(s))\,ds = \lambda \text{ constant, loc. uniformly,}$$

because the existence of the LTAC value follows from it.
For simplicity we assume from now on that all data are $\mathbb{Z}^m$ periodic in y, i.e. the state space is the torus $\mathbb{T}^M := \mathbf{R}^m/\mathbb{Z}^m$. I show 4 cases

- systems with nondegenerate noise
- controllability by one player
- separate controllability
- combined conditions on the system and the cost.

Sufficient conditions for ergodicity

In view of the previous results we concentrate on proving the ergodicity of the lower game, i.e.

$$\lim_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \frac{1}{t} \int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds = \lambda \text{ constant, loc. uniformly,}$$

because the existence of the LTAC value follows from it.
For simplicity we assume from now on that all data are $\mathbb{Z}^m$ periodic in y, i.e. the state space is the torus $\mathbb{T}^M := \mathbf{R}^m/\mathbb{Z}^m$. I show 4 cases

- systems with nondegenerate noise
- controllability by one player
- separate controllability
- combined conditions on the system and the cost.

Sufficient conditions for ergodicity

In view of the previous results we concentrate on proving the ergodicity of the lower game, i.e.

$$\lim_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \frac{1}{t}\int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds = \lambda \text{ constant, loc. uniformly,}$$

because the existence of the LTAC value follows from it.
For simplicity we assume from now on that all data are $\mathbb{Z}^m$ periodic in y, i.e. the state space is the torus $\mathbb{T}^M := \mathbf{R}^m/\mathbb{Z}^m$. I show 4 cases

- systems with nondegenerate noise
- controllability by one player
- separate controllability
- combined conditions on the system and the cost.

Noisy systems

Given a standard m-dimensional Brownian motion W_t take

$$dy(t) = f(y(t), a(t), b(t))\, dt + \sigma\, dW_t, \quad y(0) = x,$$

with σ non-singular matrix and controls adapted to W_t.

Theorem

$$\lim_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \frac{1}{t} E\left[\int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds\right] = \lambda \quad \text{uniformly in } x.$$

The proof is by PDE methods and relies on the Krylov-Safonov estimates for elliptic equations.

It hold also or $\sigma = \sigma(y(t), a(t), b(t))$ if for some $\nu > 0$

$$\sigma\sigma^T(y, a, b) \geq \nu I \quad \forall\, y, a, b.$$

Noisy systems

Given a standard m-dimensional Brownian motion W_t take

$$dy(t) = f(y(t), a(t), b(t))\, dt + \sigma\, dW_t, \quad y(0) = x,$$

with σ non-singular matrix and controls adapted to W_t.

Theorem

$$\lim_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \frac{1}{t} E\left[\int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds\right] = \lambda \quad \text{uniformly in } x.$$

The proof is by PDE methods and relies on the Krylov-Safonov estimates for elliptic equations.

It hold also or $\sigma = \sigma(y(t), a(t), b(t))$ if for some $\nu > 0$

$$\sigma\sigma^T(y, a, b) \geq \nu I \quad \forall\, y, a, b.$$

Noisy systems

Given a standard m-dimensional Brownian motion W_t take

$$dy(t) = f(y(t), a(t), b(t))\, dt + \sigma\, dW_t, \quad y(0) = x,$$

with σ non-singular matrix and controls adapted to W_t.

Theorem

$$\lim_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \frac{1}{t} E\left[\int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds\right] = \lambda \text{ uniformly in } x.$$

The proof is by PDE methods and relies on the Krylov-Safonov estimates for elliptic equations.
It hold also or $\sigma = \sigma(y(t), a(t), b(t))$ if for some $\nu > 0$

$$\sigma\sigma^T(y, a, b) \geq \nu I \quad \forall\, y, a, b.$$

Noisy systems

Given a standard m-dimensional Brownian motion W_t take

$$dy(t) = f(y(t), a(t), b(t))\, dt + \sigma\, dW_t, \quad y(0) = x,$$

with σ non-singular matrix and controls adapted to W_t.

Theorem

$$\lim_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \frac{1}{t} E\left[\int_0^t l(y_x(s), \alpha[b](s), b(s))\, ds\right] = \lambda \text{ uniformly in } x.$$

The proof is by PDE methods and relies on the Krylov-Safonov estimates for elliptic equations.
It hold also or $\sigma = \sigma(y(t), a(t), b(t))$ if for some $\nu > 0$

$$\sigma\sigma^T(y, a, b) \geq \nu I \quad \forall\, y, a, b.$$

Bounded-time controllability by the 1st player

The system is BTC by 1st player if $\exists\, S > 0$ and $\forall\, x, \tilde{x} \in \mathbb{T}^m$ there is a strategy $\alpha \in \Gamma$ such that $\forall b \in \mathcal{B}$

$$y_x(\tilde{t}) = \tilde{x} \quad \text{for some } \tilde{t} \leq S.$$

Theorem

System BTC by 1st player $\Rightarrow$ the lower game is ergodic.

It is easy to give examples of such systems, but they are very unbalanced.

Bounded-time controllability by the 1st player

The system is BTC by 1st player if $\exists\, S > 0$ and $\forall\, x, \tilde{x} \in \mathbb{T}^m$ there is a strategy $\alpha \in \Gamma$ such that $\forall b \in \mathcal{B}$

$$y_x(\tilde{t}) = \tilde{x} \quad \text{for some } \tilde{t} \leq S.$$

Theorem

System BTC by 1st player $\Rightarrow$ the lower game is ergodic.

It is easy to give examples of such systems, but they are very unbalanced.

Separate controllability

Let the state be split into $(y(t), z(t)) \in \mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$

$$\text{(SpS)} \qquad \begin{aligned} \dot{y}(t) &= f(y(t), a(t), b(t)), \qquad & y(0) = x \in \mathbb{T}^{m_1}, \\ \dot{z}(t) &= g(z(t), a(t), b(t)) \qquad & z(0) = w \in \mathbb{T}^{m_2}, \end{aligned}$$

$v(t, x, w) := \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} \int_0^t l(y_x(s), z_w(s))\, ds.$

Note: l independent of a, b !

Theorem

$y(\cdot)$ BTC by 1st player, $z(\cdot)$ BTC by 2nd player, and l has a saddle:

$$\bar{l} = \min_{x \in \mathbb{T}^{m_1}} \max_{w \in \mathbb{T}^{m_2}} l(x, w) = \max_{w \in \mathbb{T}^{m_2}} \min_{x \in \mathbb{T}^{m_1}} l(x, w)$$

$\Rightarrow$ lower game is ergodic and $\lambda = \bar{l}$.

Separate controllability

Let the state be split into $(y(t), z(t)) \in \mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$

$$\text{(SpS)} \quad \begin{array}{ll} \dot{y}(t) = f(y(t), a(t), b(t)), & y(0) = x \in \mathbb{T}^{m_1}, \\ \dot{z}(t) = g(z(t), a(t), b(t)) & z(0) = w \in \mathbb{T}^{m_2}, \end{array}$$

$v(t, x, w) := \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} \int_0^t l(y_x(s), z_w(s))\, ds.$

Note: l independent of a, b !

Theorem

$y(\cdot)$ BTC by 1st player, $z(\cdot)$ BTC by 2nd player, and l has a saddle:

$$\bar{l} = \min_{x \in \mathbb{T}^{m_1}} \max_{w \in \mathbb{T}^{m_2}} l(x, w) = \max_{w \in \mathbb{T}^{m_2}} \min_{x \in \mathbb{T}^{m_1}} l(x, w)$$

$\Rightarrow$ lower game is ergodic and $\lambda = \bar{l}$.

Separate controllability

Let the state be split into $(y(t), z(t)) \in \mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$

$$\text{(SpS)} \qquad \begin{array}{ll} \dot{y}(t) = f(y(t), a(t), b(t)), & y(0) = x \in \mathbb{T}^{m_1}, \\ \dot{z}(t) = g(z(t), a(t), b(t)) & z(0) = w \in \mathbb{T}^{m_2}, \end{array}$$

$v(t, x, w) := \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} \int_0^t l(y_x(s), z_w(s))\, ds.$

Note: l independent of a, b !

Theorem

$y(\cdot)$ BTC by 1st player, $z(\cdot)$ BTC by 2nd player, and l has a saddle:

$$\bar{l} = \min_{x \in \mathbb{T}^{m_1}} \max_{w \in \mathbb{T}^{m_2}} l(x, w) = \max_{w \in \mathbb{T}^{m_2}} \min_{x \in \mathbb{T}^{m_1}} l(x, w)$$

$\Rightarrow$ lower game is ergodic and $\lambda = \bar{l}$.

Counterexamples

Take $l(x,w)$ without a saddle, e.g., $m_1 = m_2 = m/2$ and $l(x,w) = n(x-w)$

$$\min_x \max_w l(x,w) = \max n > \min n = \max_w \min_x l(x,w).$$

Take the system

$$\dot{y}(t) = h(y(t), z(t))a(t), \qquad y(0) = x \in \mathbb{T}^{m/2}, \qquad |a(t)| \le 1,$$
$$\dot{z}(t) = h(y(t), z(t))b(t) \qquad z(0) = w \in \mathbb{T}^{m/2}, \qquad |b(t)| \le 1,$$

with $h : \mathbb{T}^m \to R$. If $h > 0$ $y(\cdot)$ is BTC by 1st player and $z(\cdot)$ BTC by 2nd player. But the game is NOT ergodic:

$$\lim_{t\to+\infty} \frac{v(t,x,w)}{t} = n(x-w) \neq \text{ constant }.$$

Proof

Set $u(t,x,w) := t\,n(x-w)$ and suppose $n \in C^1$. Then
$D_x u = Dn(x-w) = -D_w u$, so

$$\frac{\partial u}{\partial t} + h(x,w)|D_x u| - h(x,w)|D_w u| = l, \qquad u(0,x.w) = 0.$$

This is the Cauchy problem and the H-J-Isaacs equation satified by the value function v. By uniqueness of the (viscosity) solution $v \equiv u$, so

$$\frac{v(t,x,w)}{t} \equiv n(x-w).$$

Remark. This proof can be extended to a slightly more general class of systems.
By a comparison argument we can also prove non-ergodicity for

$$l(x,w) = n(x-w) + q(x,w), \quad \text{if} \quad \max q - \min q < \max n - \min n.$$

Proof

Set $u(t,x,w) := t\,n(x-w)$ and suppose $n \in C^1$. Then
$D_x u = Dn(x-w) = -D_w u$, so

$$\frac{\partial u}{\partial t} + h(x,w)|D_x u| - h(x,w)|D_w u| = l, \qquad u(0,x.w) = 0.$$

This is the Cauchy problem and the H-J-Isaacs equation satified by the value function v. By uniqueness of the (viscosity) solution $v \equiv u$, so

$$\frac{v(t,x,w)}{t} \equiv n(x-w).$$

Remark. This proof can be extended to a slightly more general class of systems.
By a comparison argument we can also prove non-ergodicity for

$$l(x,w) = n(x-w) + q(x,w), \quad \text{if} \quad \max\, q - \min\, q < \max\, n - \min\, n.$$

A model problem: convex-concave eikonal equation

Take the simple example ($h \equiv 1$)

$$\dot{y}(t) = a(t), \qquad y(0) = x \in \mathbb{T}^{m/2}, \qquad |a(t)| \leq 1,$$
$$\dot{z}(t) = b(t) \qquad z(0) = w \in \mathbb{T}^{m/2}, \qquad |b(t)| \leq \gamma,$$

whose H-J-Isaacs equation is

$$v_t + |D_x v| - \gamma |D_w v| = l(x, w).$$

So far we can say

- game is ergodic $\forall\, \lambda > 0$ if l has a saddle,
- game is NOT ergodic if $\gamma = 1$ and $l(x, w) = n(x - w)$.

What about some other cases, e.g., $l(x, w) = n(x - w)$ but $\gamma \neq 1$?

We'll settle this case and prove the game is ergodic.

A model problem: convex-concave eikonal equation

Take the simple example ($h \equiv 1$)

$$\dot{y}(t) = a(t), \qquad y(0) = x \in \mathbb{T}^{m/2}, \qquad |a(t)| \leq 1,$$
$$\dot{z}(t) = b(t) \qquad z(0) = w \in \mathbb{T}^{m/2}, \qquad |b(t)| \leq \gamma,$$

whose H-J-Isaacs equation is

$$v_t + |D_x v| - \gamma |D_w v| = l(x, w).$$

So far we can say

- game is ergodic $\forall \lambda > 0$ if l has a saddle,
- game is NOT ergodic if $\gamma = 1$ and $l(x, w) = n(x - w)$.

What about some other cases, e.g., $l(x, w) = n(x - w)$ but $\gamma \neq 1$?

We'll settle this case and prove the game is ergodic.

A model problem: convex-concave eikonal equation

Take the simple example ($h \equiv 1$)

$$\dot{y}(t) = a(t), \qquad y(0) = x \in \mathbb{T}^{m/2}, \qquad |a(t)| \le 1,$$
$$\dot{z}(t) = b(t) \qquad z(0) = w \in \mathbb{T}^{m/2}, \qquad |b(t)| \le \gamma,$$

whose H-J-Isaacs equation is

$$v_t + |D_x v| - \gamma |D_w v| = l(x, w).$$

So far we can say

- game is ergodic $\forall \lambda > 0$ if l has a saddle,
- game is NOT ergodic if $\gamma = 1$ and $l(x, w) = n(x - w)$.

What about some other cases, e.g., $l(x, w) = n(x - w)$ but $\gamma \neq 1$?

We'll settle this case and prove the game is ergodic.

A model problem: convex-concave eikonal equation

Take the simple example ($h \equiv 1$)

$$\dot{y}(t) = a(t), \qquad y(0) = x \in \mathbb{T}^{m/2}, \qquad |a(t)| \leq 1,$$
$$\dot{z}(t) = b(t) \qquad z(0) = w \in \mathbb{T}^{m/2}, \qquad |b(t)| \leq \gamma,$$

whose H-J-Isaacs equation is

$$v_t + |D_x v| - \gamma |D_w v| = l(x, w).$$

So far we can say

- game is ergodic $\forall \lambda > 0$ if l has a saddle,
- game is NOT ergodic if $\gamma = 1$ and $l(x, w) = n(x - w)$.

What about some other cases, e.g., $l(x, w) = n(x - w)$ but $\gamma \neq 1$?

We'll settle this case and prove the game is ergodic.

A model problem: convex-concave eikonal equation

Take the simple example ($h \equiv 1$)

$$\dot{y}(t) = a(t), \qquad y(0) = x \in \mathbb{T}^{m/2}, \qquad |a(t)| \leq 1,$$
$$\dot{z}(t) = b(t) \qquad z(0) = w \in \mathbb{T}^{m/2}, \qquad |b(t)| \leq \gamma,$$

whose H-J-Isaacs equation is

$$v_t + |D_x v| - \gamma |D_w v| = l(x, w).$$

So far we can say

- game is ergodic $\forall \lambda > 0$ if l has a saddle,
- game is NOT ergodic if $\gamma = 1$ and $l(x, w) = n(x - w)$.

What about some other cases, e.g., $l(x, w) = n(x - w)$ but $\gamma \neq 1$?

We'll settle this case and prove the game is ergodic.

Combined conditions on the system and the cost l

A sample result for the system in general form (S).
Define the target $\mathcal{T} := \operatorname{argmin} l$, with $l = l(y)$ independent of the controls.
The system is asymptotically controllable to $\mathcal{T}$ in the mean by the first player if $\forall x \in \mathbb{T}^m$, there is a strategy $\alpha \in \Gamma$ such that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \operatorname{dist}(y_x(t), \mathcal{T})\, dt = 0 \quad \text{uniformly in } x \text{ and } b \in \mathcal{B}.$$

Theorem

System asymptotically controllable to $\mathcal{T}$
$\implies$ the lower game is ergodic with $\lambda = \min l$.

Example: system bounded-time controllable to $\mathcal{T}$ and stoppable on $\mathcal{T}$ by the first player (turnpike strategy).
[Still weaker than the 1st controllability assumption].

Example:

$$l(x,w) = n(x-w)$$

$$\dot{y}(t) = h(y(t),z(t))a(t), \qquad y(0) = x \in \mathbb{T}^{m/2}, \qquad |a(t)| \le 1,$$

$$\dot{z}(t) = h(y(t),z(t))b(t), \qquad z(0) = w \in \mathbb{T}^{m/2}, \qquad |b(t)| \le \gamma,$$

with $h > 0$ and $\gamma < 1$.

Since the dynamics of the $y(\cdot)$ and $z(\cdot)$ are the same, but the first player can drive $y(\cdot)$ at higher speed, for any fixed $\overline{w} \in \mathbf{R}^{m/2}$ the first player can drive the system from any initial position to the set $\{y - z = \overline{w}\}$ in finite time for all controls of the second player, and then stop it there. By choosing $\overline{w} \in \mathcal{T} := \operatorname{argmin} n$, we verify the assumptions of the last Theorem.

The same property holds for the 2nd player if $\gamma > 1$ and we take $\mathcal{T} := \operatorname{argmax} l$.

This proves the claim that the game is ergodic for $\gamma \neq 1$ in the convex-concave eikonal equation with $l = n(x-w)$.

Example:

$$l(x, w) = n(x - w)$$
$$\dot{y}(t) = h(y(t), z(t))a(t), \qquad y(0) = x \in \mathbb{T}^{m/2}, \qquad |a(t)| \leq 1,$$
$$\dot{z}(t) = h(y(t), z(t))b(t), \qquad z(0) = w \in \mathbb{T}^{m/2}, \qquad |b(t)| \leq \gamma,$$

with $h > 0$ and $\gamma < 1$.
Since the dynamics of the $y(\cdot)$ and $z(\cdot)$ are the same, but the first player can drive $y(\cdot)$ at higher speed, for any fixed $\overline{w} \in \mathbf{R}^{m/2}$ the first player can drive the system from any initial position to the set $\{y - z = \overline{w}\}$ in finite time for all controls of the second player, and then stop it there. By choosing $\overline{w} \in \mathcal{T} := \text{argmin}\, n$, we verify the assumptions of the last Theorem.

The same property holds for the 2nd player if $\gamma > 1$ and we take $\mathcal{T} := \text{argmax}\, l$.
This proves the claim that the game is ergodic for $\gamma \neq 1$ in the convex-concave eikonal equation with $l = n(x - w)$.

Example:

$$
\begin{aligned}
l(x,w) &= n(x-w) \\
\dot{y}(t) &= h(y(t),z(t))a(t), \qquad y(0)=x\in\mathbb{T}^{m/2}, \qquad |a(t)|\le 1, \\
\dot{z}(t) &= h(y(t),z(t))b(t), \qquad z(0)=w\in\mathbb{T}^{m/2}, \qquad |b(t)|\le \gamma,
\end{aligned}
$$

with $h>0$ and $\gamma<1$.
Since the dynamics of the $y(\cdot)$ and $z(\cdot)$ are the same, but the first player can drive $y(\cdot)$ at higher speed, for any fixed $\overline{w}\in\mathbf{R}^{m/2}$ the first player can drive the system from any initial position to the set $\{y-z=\overline{w}\}$ in finite time for all controls of the second player, and then stop it there. By choosing $\overline{w}\in\mathcal{T}:=\text{argmin } n$, we verify the assumptions of the last Theorem.

The same property holds for the 2nd player if $\gamma>1$ and we take $\mathcal{T}:=\text{argmax } l$.
This proves the claim that the game is ergodic for $\gamma\neq 1$ in the convex-concave eikonal equation with $l=n(x-w)$.

Variants and open problems

- For systems in split form (SpS) can give conditions of asymptotic controllability to suitable targets so that the game is ergodic with

$$\lambda = \min_{x \in \mathbb{T}^{m_1}} \max_{w \in \mathbb{T}^{m_2}} l(x, w)$$

  or with $\lambda = \max_{w \in \mathbb{T}^{m_2}} \min_{x \in \mathbb{T}^{m_1}} l(x, w)$.
- We do not know sharp conditions for games in split form with l without a saddle but not of the form $n(x - w)$.
- The ergodicity for cost l depending on the controls a, b is largely open (and interesting for applications).
- The existence of the LTAC value without the ergodicity seems completely open.

Variants and open problems

- For systems in split form (SpS) can give conditions of asymptotic controllability to suitable targets so that the game is ergodic with

$$\lambda = \min_{x\in\mathbb{T}^{m_1}} \max_{w\in\mathbb{T}^{m_2}} l(x,w)$$

  or with $\lambda = \max_{w\in\mathbb{T}^{m_2}} \min_{x\in\mathbb{T}^{m_1}} l(x,w)$.

- We do not know sharp conditions for games in split form with l without a saddle but not of the form $n(x-w)$.
- The ergodicity for cost l depending on the controls a, b is largely open (and interesting for applications).
- The existence of the LTAC value without the ergodicity seems completely open.

Variants and open problems

- For systems in split form (SpS) can give conditions of asymptotic controllability to suitable targets so that the game is ergodic with

$$\lambda = \min_{x \in \mathbb{T}^{m_1}} \max_{w \in \mathbb{T}^{m_2}} l(x, w)$$

  or with $\lambda = \max_{w \in \mathbb{T}^{m_2}} \min_{x \in \mathbb{T}^{m_1}} l(x, w)$.

- We do not know sharp conditions for games in split form with l without a saddle but not of the form $n(x - w)$.
- The ergodicity for cost l depending on the controls a, b is largely open (and interesting for applications).
- The existence of the LTAC value without the ergodicity seems completely open.

Variants and open problems

- For systems in split form (SpS) can give conditions of asymptotic controllability to suitable targets so that the game is ergodic with

$$\lambda = \min_{x \in \mathbb{T}^{m_1}} \max_{w \in \mathbb{T}^{m_2}} l(x, w)$$

  or with $\lambda = \max_{w \in \mathbb{T}^{m_2}} \min_{x \in \mathbb{T}^{m_1}} l(x, w)$.

- We do not know sharp conditions for games in split form with l without a saddle but not of the form $n(x - w)$.

- The ergodicity for cost l depending on the controls a, b is largely open (and interesting for applications).

- The existence of the LTAC value without the ergodicity seems completely open.

An application: games with multiple time-scales

Consider a system with state variables (x_s, y_τ), $\tau = s/\varepsilon$ and $0 < \varepsilon << 1$,

$$(1) \qquad \begin{aligned} \dot{x}_s &= \phi(x_s, y_s, \alpha_s, \beta_s) & x_s \in \mathbf{R}^n, \\ \dot{y}_s &= \frac{1}{\varepsilon} f(x_s, y_s, \alpha_s, \beta_s) & y_s \in \mathbf{R}^m. \end{aligned}$$

Want to understand the limit as $\varepsilon \to 0$:

a Singular Perturbation problem.

Expect y_s to disappear and the limit system to depend only on the long time regime of y_s .

For a cost functional $J(t, x_0, y_0, \alpha, \beta) := \int_0^t L(x_s, y_s, \alpha_s, \beta_s)\, ds + h(x_t, y_t)$ with value function $v^\varepsilon(t, x_0, y_0)$ we expect to find a limit v independent of y_0

$$v^\varepsilon(t, x_0, y_0) \to v(t, x_0) \quad \text{as } \varepsilon \to 0.$$

An application: games with multiple time-scales

Consider a system with state variables (x_s, y_τ), $\tau = s/\varepsilon$ and $0 < \varepsilon << 1$,

$$(1) \qquad \begin{aligned} \dot{x}_s &= \phi(x_s, y_s, \alpha_s, \beta_s) \qquad x_s \in \mathbf{R}^n, \\ \dot{y}_s &= \frac{1}{\varepsilon} f(x_s, y_s, \alpha_s, \beta_s) \qquad y_s \in \mathbf{R}^m. \end{aligned}$$

Want to understand the limit as $\varepsilon \to 0$:

a Singular Perturbation problem.

Expect y_s to disappear and the limit system to depend only on the long time regime of y_s .

For a cost functional $J(t, x_0, y_0, \alpha, \beta) := \int_0^t L(x_s, y_s, \alpha_s, \beta_s)\, ds + h(x_t, y_t)$ with value function $v^\varepsilon(t, x_0, y_0)$ we expect to find a limit v independent of y_0

$$v^\varepsilon(t, x_0, y_0) \to v(t, x_0) \quad \text{as } \varepsilon \to 0.$$

Connection with LTAC games

Consider the fast subsystem with frozen $\overline{x}$ and $\varepsilon = 1$

(FS) $$\dot{y}_\tau = f(\overline{x}, y_\tau, \alpha_\tau, \beta_\tau),$$

and the running cost with parameters $\overline{x}, \overline{p} \in \mathbf{R}^n$

$$l(y, a, b; \overline{x}, \overline{p}) := \overline{p} \cdot \phi(\overline{x}, y, a, b) + L(\overline{x}, y, a, b).$$

Theorem

Game (FS) with cost l ergodic, with LTAC value $\lambda =: \overline{H}(\overline{x}, \overline{p}) \implies$ $v^\varepsilon \to v$ in the SP problem, and v solves

$$\frac{\partial v}{\partial t} + \overline{H}(x, D_x v) = 0.$$

Connection with LTAC games

Consider the fast subsystem with frozen $\overline{x}$ and $\varepsilon = 1$

(FS) $$\dot{y}_\tau = f(\overline{x}, y_\tau, \alpha_\tau, \beta_\tau),$$

and the running cost with parameters $\overline{x}, \overline{p} \in \mathbf{R}^n$

$$l(y, a, b; \overline{x}, \overline{p}) := \overline{p} \cdot \phi(\overline{x}, y, a, b) + L(\overline{x}, y, a, b).$$

Theorem

Game (FS) with cost l ergodic, with LTAC value $\lambda =: \overline{H}(\overline{x}, \overline{p}) \implies$ $v^\varepsilon \to v$ in the SP problem, and v solves

$$\frac{\partial v}{\partial t} + \overline{H}(x, D_x v) = 0.$$

Thanks for your attention!