An introduction to Mean Field Games and models of segregation

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Plan

1. What are Mean Field Games?
   - a static game with many players
   - a heuristic derivation of the MFG partial differential equations
   - MFG as models of large populations of agents

2. Models of segregation
   [joint work with Yves Achdou (Paris) and Marco Cirant (Milano)]
   - Schelling's model of urban settlements
   - Mean-Field Games with two populations
   - Qualitative properties: segregation?
   - Numerical experiments

Ingredients:

- a bit of Game Theory (Nash equilibria)
- stochastic control
- partial differential equations

1. Introduction to MFG: motivations

Want to model dynamical phenomena with

- many very similar rational agents
- subject to noise
- non-cooperative

Examples of applications:

- Economics
  - financial markets (price formation and dynamic equilibria, formation of volatility)
  - general economic equilibrium with rational expectations
  - environmental policy,
- Engineering
  - wireless power control
  - demand side management in electric power networks,
  - traffic problems

- Social sciences
  - crowd motion (mexican wave "la ola", pedestrian dynamics, congestion phenomena,...)
  - opinion dynamics and consensus problems,
  - models of population distribution (e.g., segregation).

Goals and methods:

- get macroscopic "mean field" continuum models, simpler than the discrete models for N agents,
- in analogy with the Mean Field theories in
  - Statistical Physics (kinetic theory of gases, Boltzmann and Vlasov equations)
  - Quantum Mechanics and Quantum Chemistry (Hartree-Fock models...)
- mostly using Partial Differential Equations and Stochastic methods.

Basic references

Mathematical theory:

- J.-M. Lasry, P.-L. Lions: C.R.A.S. Paris 2006, Jpn. J. Math. 2007
- P.-L. Lions: movies of courses at College de France

Engineering problems with L-Q models:

- M. Huang, P.E. Caines, R.P. Malhamé: Proc. IEEE Conf. 2003, IEEE Trans. Automat. Control 2007, etc....

Applications:

- O. Guéant, J.-M. Lasry, P.-L. Lions: Springer Lecture Notes 2011.
- D.A. Gomes, L. Nurbekian, E.A. Pimentel, Economics models and MFG theory, book to appear

Numerical methods and discrete models

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Games with many players

A (static) N-person game is defined by

- Q = a (compact) metric space
- $F_i : Q^N \to \mathbf{R}$ continuous, $i = 1, ..., N$

Goal of the ith player : minimise F_i.

Definition of Nash equilibrium: $(\overline{x}_1, \ldots, \overline{x}_N) \in Q^N$ such that

$$F_i(\overline{x}_1, \ldots, \overline{x}_N) \leq F_i(\overline{x}_1, .., \overline{x}_{i-1}, x_i, \overline{x}_{i+1}, .., \overline{x}_N) \quad \forall\, x_i \in Q, \forall i.$$

Existence of the equilibria is classical, but there are many and can have a complicate structure.

We're interested in problems with N large.

Question: is there a simpler macroscopic model for large populations?

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Indistinguishable players

Main assumption: "homogeneous population", i.e.,

each cost is a symmetric function of the state of the other players.

For N large, symmetric functions can be approximated by functions only of the empirical measure of their variables.

Then assume, for $\mathcal{P}(Q) := \{\text{probability measures on } Q\}$

$\exists\, F : Q \times \mathcal{P}(Q) \to \mathbf{R}$ such that the cost of the i-th player is

$$F_i = F\left(x_i, \frac{1}{N-1}\sum_{k\neq i} \delta_{x_k}\right),$$

depending on the other players only via their empirical measure,

with F continuous w.r.t. weak* convergence on $\mathcal{P}(Q)$,

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Theorem [Lions, about 2006]

If $(\overline{x}_1^N, \dots, \overline{x}_N^N)$ is a Nash equilibrium for the N-person game, then

(i) $$\frac{1}{N}\sum_{k=1}^{N} \delta_{\overline{x}_k^N} \to^* \overline{m} \quad \text{as } N \to \infty,$$

up to subsequences, $\overline{m}$ solution of

(1) $$\forall\, x \in supp\, \overline{m} \quad F(x, \overline{m}) = \min_{y \in Q} F(y, \overline{m}).$$

(ii) $$\int_Q (F(x, m_1) - F(x, m_2))\, d(m_1 - m_2) > 0 \quad \forall m_1 \neq m_2,$$

i.e., F increasing $\implies$ at most one solution of (1).

N.B.: F increasing means that players don't like the crowd

The large-population limit $N \to \infty$

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A very simple example

Where do I put my towel on the beach?

$x_i \in \mathbf{R}$ is the distance of the towel of the $i-$th person from the sea.

The cost of the $i-$th player is

$$F_i(x_1, .., x_N) = f(x_i) + g\left(\frac{\#\{k : |x_i - x_k| < \varepsilon\}}{(N-1)|B_\varepsilon|}\right)$$

which becomes a function of the empirical density by choosing

$$F(x, m) = f(x) + g(m * 1_{B_\varepsilon}/|B_\varepsilon|).$$

Note: f is minimal at the preferred position $\overline{x}$,

$g \uparrow$ means aversion to crowd ($\Longrightarrow$ uniqueness in (1)),

$g \downarrow$ means that people like crowd.

An explicit solution

Letting formally $\varepsilon \to 0$ get $F(x,m) = f(x) + g(m(x))$ and the MFG equation (1) becomes

$$supp\, \overline{m} \subseteq \arg\min \left(f(x) + g(\overline{m}(x)) \right).$$

Sometimes can be solved explicitly, e.g.,

$$F(x,m) = \frac{|x-\overline{x}|^2}{2} + \log(m(x)).$$

Must solve

$$\text{if } \overline{m}(x) > 0 \qquad \frac{|x-\overline{x}|^2}{2} + \log(\overline{m}(x)) = \overline{\lambda} := \min F(y, \overline{m}(y))$$

Then $\quad \overline{m}(x) = e^{\overline{\lambda}} e^{-|x-\overline{x}|^2/2} \quad$ and $\overline{\lambda}$ must be such that $\int \overline{m}(x) = 1$

$\Longrightarrow$ the unique solution $\overline{m}$ is Gaussian with mean $\overline{x}$.

Comments

- If the monotonicity of F fails can guess from the example the non-uniqueness and singular solutions....

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Heuristic derivation of the main equations

Basic facts from stochastic control: an agent has dynamics

$$dX_s = \alpha_s \, ds + \sigma \, dW_s, \quad X_t = x \in \mathbf{R}^d$$

with W_s a Brownian motion, $\alpha_s =$ control, $\sigma > 0$ volatility,
and the finite horizon cost functional:

$$J_T(t, x, \alpha.) := E\left[\int_t^T L(\alpha_s) + F(X_s, m_{env}) ds\right] + g(X_T).$$

Here

- L is the running cost of using the control α_s,
- $F : \mathbf{R}^d \times \{ \text{ prob. measures} \} \to \{ \text{ Lip functions } \}$
  is the running cost of being in the state X_s, depending on the distribution of the other agents in the environment m_{env},
- g is the terminal cost.

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- g is the terminal cost.

Define the value function

$$v(t,x) := \inf_{\alpha.} J_T(t,x,\alpha.).$$

Then $v(t,x)$ solves the Hamilton-Jacobi-Bellman equation

$$\begin{cases} -\frac{\partial v}{\partial t} - \nu\Delta v + H(\nabla v) = F(x, m_{env}) & \text{in } (0,T)\times \mathbf{R}^d \\ v(T,x) = g(x) \end{cases}$$

where $\nu := \sigma^2/2$, $\Delta := \Delta_x$, $\nabla := \nabla_x$, and H is the Hamiltonian associated to L:

$$H(p) := \sup_{\alpha\in\mathbf{R}^d} \{p\cdot\alpha - L(\alpha)\}$$

Moreover the feedback control

$$\hat{\alpha}(t,x) = -\nabla H(\nabla v(t,x))$$

is optimal.

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The optimal process

$$d\hat{X}_t = -\nabla H(\nabla v(\hat{X}_t))dt + \sigma dW_t$$

has a distribution whose density m solves the

Kolmogorov-Fokker-Plank equation

$$\begin{cases} \frac{\partial m}{\partial t} - \nu \Delta m + div(m \nabla H(\nabla v)) = 0 \quad \text{in } (0,T) \times \mathbf{R}^d \\ m(0,x) = m_o(x) \end{cases}$$

where $\quad m_o \geq 0, \ \int_{\mathbf{R}^d} m_o(x)dx = 1,$

is the distribution of the initial position of the system.

The PDEs for value and density of the optimal process are

$$\begin{cases} -\frac{\partial v}{\partial t} - \nu \Delta v + H(\nabla v) = F(x, m_{env}) & \text{in } (0,T) \times \mathbf{R}^d \\ \frac{\partial m}{\partial t} - \nu \Delta m + div(m \nabla H(\nabla v)) = 0 & \text{in } (0,T) \times \mathbf{R}^d \\ v(T,x) = g(x), \quad m(0,x) = m_o(x), \end{cases}$$

and $m_{env} \mapsto v$, $v \mapsto m$ are well-defined maps.

If $m_{env} \mapsto v \mapsto m$ has a fixed point , i.e. $m = m_{env}$,

then m is an equilibrium distribution of the agents,

each behaving optimally as long as the population distribution remains the same.

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Mean Field Games PDEs: evolutive

We have heuristically derived the basic system of 2 evolutive PDEs of MFGs

(MFE)
$$\begin{cases} -\frac{\partial v}{\partial t} - \nu \Delta v + H(\nabla v) = F(x, m) \quad \text{in } (0, T) \times \mathbf{R}^d \\ \\ \frac{\partial m}{\partial t} - \nu \Delta m + div(m \nabla H(\nabla v)) = 0 \quad \text{in } (0, T) \times \mathbf{R}^d \\ \\ v(T, x) = g(x), \quad m(0, x) = m_o(x), \end{cases}$$

Data: ν, H, F, m_o, g ; Unknowns:
$m(t, x) =$ equilibrium distribution of the agents at time t;
$v(t, x)$= value function of the representative agent

1st equation is backward parabolic H-J-B with a possibly non-local cost term $F(x, m)$,
2nd equation is forward parabolic K-F-P equation, linear in m.
3rd line: terminal and initial conditions.

Well-posedness?

Existence was proved by Lasry and Lions under various sets of assumptions (mostly for periodic data).
A simple example [P. Cardaliaguet, Notes 2010] is

- $H(p) = |p|^2$
- g, F bounded and Lipschitz (w.r.t. Kantorovitch-Rubinstein distance of prob measures)
- m_o Hölder, $\int_{\mathbf{R}^d} |x|^2 m_o(x) dx < +\infty$.

Uniqueness is not expected in general, true for H convex under the monotonicity condition (as in the static game)

$$\int_{\mathbf{R}^d} [F(x, m_1) - F(x, m_2)] d(m_1 - m_2)(x) > 0, \quad \forall m_1 \neq m_2,$$

which means crowd aversion.

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which means crowd aversion.

MFG with long-time-average cost

For the same control system

$$dX_s = \alpha_s ds + \sigma dW_s, \quad X_0 = x \in \mathbf{R}^d$$

take the long-time-average (or "ergodic") cost functional:

$$J(x, \alpha.) := \liminf_{T\to+\infty} \frac{1}{T} E\left[\int_0^T L(\alpha_t) + F(X_t, m) dt\right],$$

Assume the dynamics is on the torus $\mathbb{T}^d$, i.e., $F(\cdot, m)$ is $\mathbb{Z}^d$-periodic, so all admissible controls produce an ergodic diffusion process X_s.
Now the Hamilton-Jacobi-Bellman equation is

$$-\nu\Delta v + H(\nabla v) + \lambda = F(x, m) \quad \text{in } \mathbf{R}^d$$

If it has a solution pair $\lambda, v(\cdot)$, then the value and the optimal control are

$$\lambda = \inf_{\alpha.} J(x, \alpha.) = J(x, \hat{\alpha}), \quad \hat{\alpha}(x) = -\nabla H(\nabla v(x))$$

Mean Field Games PDEs: stationary

The MFG PDEs now are elliptic, with an additive eigenvalue

$$(MFS) \qquad \begin{cases} -\nu \Delta v + H(\nabla v) + \lambda = F(x, m) \quad \text{in } \mathbb{T}^d, \\ \nu \Delta m + div(\nabla H(\nabla v) m) = 0 \quad \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} m(x) dx = 1, \quad m > 0, \end{cases}$$

Data: ν, H, F ;
Unknowns:
$m(x) =$ equilibrium distribution of the agents = invariant measure of the optimal process;
$\lambda =$ value
$v(x)$, such that $\nabla H(\nabla v) =$ optimal feedback.

Microscopic justification?

Main mathematical question: how are these systems of PDEs related to Nash equilbria of N-person differential games, with large N?
The state of the i-th player is

$$dX_s^i = \alpha_s^i ds + \sigma dW_s^i, \quad X_s^i = x^i \in \mathbf{R}^d, \quad i = 1, \ldots, N$$

W_s^i independent Brownian motions, $\alpha_s^i =$ control of i-th player,
long-time-average cost functional of the i-th player:

$$J_T^i(t, x^1, .., x^N, \alpha^1, .., \alpha^N) := E\left[\int_t^T L(\alpha_s^i) + F\left(X_s^i, \frac{\sum_{k\neq i} \delta_{X_s^k}}{N-1}\right) ds\right],$$

depending on the players $k \neq i$ only via their empirical measure $\frac{1}{N-1} \sum_{k\neq i} \delta_{X_s^k}$, where δ_x is the Dirac mass at x.

Nash equilibrium feedbacks for N players

Such equilibria can be synthesised by solving a system of N parabolic HJB PDEs in Nd dimensions for the value functions v_i, $i = 1, ..., N$, nonlinear and strongly coupled.

There is large theory on existence of solutions, mostly by Bensoussan and Frehse ('80s - now).

Question: in what sense are they "close to" solutions of the MFG system of PDEs as $N \to \infty$?

This is very hard in general and was largely open until this year, with some (interesting) partial answers.

On the large population limit 1: ε-equilibria

1. Synthesis of ε-Nash equilibria (Huang-Caines-Malhame 2006).

Given a solution (v, m) of the evolutive MFG system of PDEs (MFE)

the candidate optimal feedback is $\hat{\alpha}(t,x) := -\nabla H(\nabla v(t,x))$.

Assume all the players use this feedback: $\tilde{\alpha}^i_s := \hat{\alpha}(s, X^i_s)$.

Then $\forall\, \varepsilon > 0\ \exists\, N_\varepsilon$ such that $\forall\, N \geq N_\varepsilon$, $\forall\, i = 1, .., N$, $\forall$ admissible α^i,

$$J^i_T(t, x^1, .., x^N, \tilde{\alpha}^1, .., \tilde{\alpha}^N) \leq J^i_T(t, x^1, .., x^N, \tilde{\alpha}^1, .., \alpha^i, .., \tilde{\alpha}^N) + \varepsilon$$

On the large population limit 2: ergodic costs

2. For the long-time-average cost functional J the system of PDEs producing the Nash equilibrium feedback can be simplified to

$$\begin{cases} -\nu\Delta v_i + H(\nabla v_i) + \lambda_i = \int_{\mathbb{T}^{d(N-1)}} F\left(x, \frac{\sum_{k\neq i}\delta_{x^k}}{N-1}\right) \prod_{k\neq i} dm_k(x^k), \\ \nu\Delta m_i + div\,(\nabla H(\nabla v_i) m_i) = 0, \qquad \text{in } \mathbb{T}^d, \quad i = 1, \dots, N, \\ \int_{\mathbb{T}^d} m_i(x)dx = 1, \quad m_i > 0, \end{cases}$$

weakly coupled and in dimension d (instead of Nd!).

Theorem [Lasry-Lions '06]

(i) the system has a solution $\lambda_i^N, v_i^N, m_i^N$, $i = 1, .., N$ and for any solution $(\lambda_i^N, v_i^N, m_i^N)_{i,N}$ is relatively compact in $\mathbf{R} \times C^2(\mathbb{T}^d) \times W^{1,p}(\mathbb{T}^d)$,

(ii) fixed i, the lim of any converging subsequence as $N \to \infty$ solves (MFS)

3. Linear-Quadratic MFG with ergodic cost [M.B. - F. Priuli 2014]:

$$dX_t^i = (AX_t^i - \alpha_t^i)dt + \sigma dW_t^i, \quad X_0^i = x^i \in \mathbf{R}^d, \quad i = 1, \ldots, N$$

running cost = quadratic form in α_t^i and X_t^i :

the system of $2N$ PDEs for N-person Nash equilibria can be solved by matrix Riccati equations,

the solution v_i^N are quadratic and m_i^N are Gaussian,

they converge as $N \to \infty$ to a solution of (MFE).

4. Probabilistic approach to MFG [M. Fischer '14, D. Lacker '14]:

convergence of Nash equilbria for N-person game with finite horizon to an equilibrium of MFG by weak convergence methods, without PDEs.

On the large population limit: the main result

Convergence of solutions of the system of N HJB PDEs for the finite horizon problem to a solution of the evolutive MFG system of PDEs (MFE):
Cardaliaguet - Delarue - Lasry - Lions preprint 9/2015.

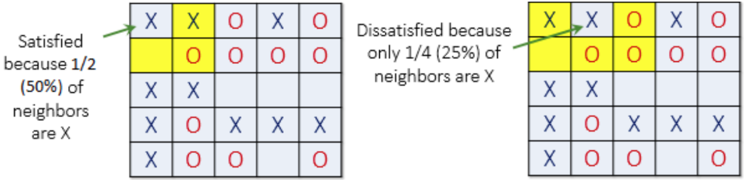
Problem si related to propagation of chaos in statistical phisics.

Covers also the case of common noise, i.e., the noises W_s^i are NOT independent.

Main tool: the master equation, a fully nonlinear PDE in infinite dimensions.

2. Models of segregation: Schelling's neighborhoods

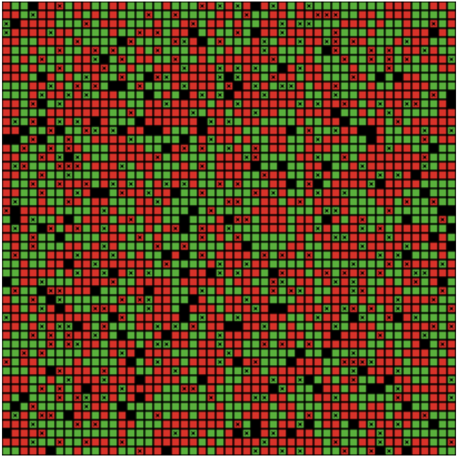
In the 70s the economist Thomas Schelling made some simple simulations to understand the formation of segregated neighbourhoods in US cities.
Blue people and red people live in a chessboard.
Each individual is happy if the percentage of same-color individuals among his neighbors is above a given threshold **a**.
If he's not happy, he moves to another free house.



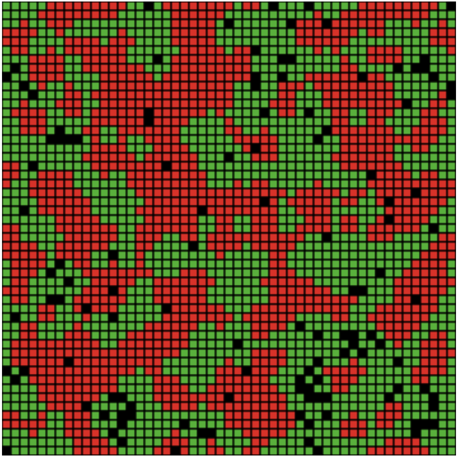
T. Schelling: Micromotives and Macrobehavior, 1978.

Schelling's experiments

30% similar wanted, 15% unhappy initially (x)

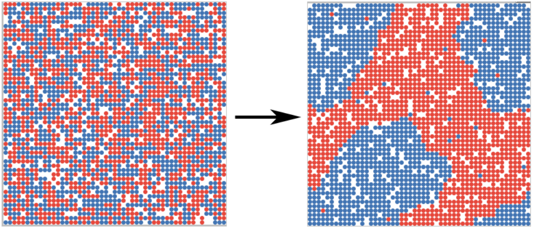


converges quickly to 0 unhappy and 75% similar in average !
Islands form: segregation.

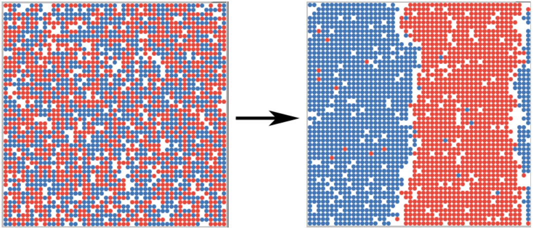


... and with some noise

$a = 35\%$, 85% similar in the end:



$a = 70\%$, 96% similar in the end, but it keeps oscillating:



Schelling's conclusions

"The interplay of individual choices, where unorganized segregation is concerned, is a complex system with collective results that bear no close relation to the individual intent"
I.o.w., even in this oversimplified model, knowing individuals' intent does not allow you to foresee the social outcome, and knowing the social outcome does not give you an accurate picture of individuals' intent.
There are several videos on YouTube showing experiments of Schelling's neighbourhoods, and various free software is available online to make such experiments.
This model is considered as a prototype of the modern field of artificial societies.
Schelling also got the Nobel Prize in Economics in 2005 with R. Aumann,
"for having enhanced our understanding of conflict and cooperation through game-theory analysis".

Cost functionals for $N + N$-person games

Want to build differential games and MFG with cost functionals reproducing Schelling's ideas and see the qualitative properties of solutions, e.g., if segregation occurs.

Cost for the i-th player of the 1st population:
for $0 < a_j < 1$, $a_j = \%$ similar wanted by population j

$$F_i^{1,N}(x_1, \ldots, x_N, y_1, \ldots, y_N) = \left(\frac{\sharp\{x_k \in \mathcal{U}(x_i) : k \neq i\}}{\sharp\{x_k \in \mathcal{U}(x_i) : k \neq i\} + \sharp\{y_k \in \mathcal{U}(x_i)\}} - a_1 \right)^{-},$$

Cost for the i-th player of the 2nd population:

$$F_i^{2,N}(x_1, \ldots, x_N, y_1, \ldots, y_N) = \left(\frac{\sharp\{y_k \in \mathcal{U}(y_i) : k \neq i\}}{\sharp\{y_k \in \mathcal{U}(y_i) : k \neq i\} + \sharp\{x_k \in \mathcal{U}(y_i)\}} - a_2 \right)^{-}$$

Can also be written as

$$F_i^{1,N}(x_1,\dots,x_N,y_1,\dots,y_N) = F^{1,N}\left(x_i, \frac{1}{N-1}\sum_{i\neq k}\delta_{x_k}, \frac{1}{N}\sum \delta_{y_k}\right)$$

$$F^{1,N}(x_i,m_1,m_2) := \left(\frac{\int_{\mathcal{U}(x_i)} m_1}{\int_{\mathcal{U}(x_i)} m_1 + \frac{N}{N-1}\int_{\mathcal{U}(x_i)} m_2} - a_1\right)^{-},$$

$$F_i^{2,N}(x_1,\dots,x_N,y_1,\dots,y_N) = F^{2,N}\left(y_i, \frac{1}{N}\sum_{i\neq k}\delta_{x_k}, \frac{1}{N-1}\sum \delta_{y_k}\right)$$

$$F^{2,N}(y_i,m_1,m_2) := \left(\frac{\int_{\mathcal{U}(y_i)} m_2}{\int_{\mathcal{U}(y_i)} m_2 + \frac{N}{N-1}\int_{\mathcal{U}(y_i)} m_1} - a_2\right)^{-}.$$

Regularized cost functionals

$$F^{1,N}(x, m_1, m_2) :=$$

$$\left(\frac{\int_{\overline{\Omega}} K(x-y)dm_1(y)}{\int_{\overline{\Omega}} K(x-y)dm_1(y) + \frac{N}{N-1}\int_{\overline{\Omega}} K(x-y)dm_2(y) + \eta_1} - a_1 \right)^{-},$$

where K is a regularizing kernel with support in $B(0, \rho)$, $\eta_1 > 0$;

$$F^{2,N}(x, m_1, m_2) :=$$

$$\left(\frac{\int_{\overline{\Omega}} K(x-y)dm_2(y)}{\int_{\overline{\Omega}} K(x-y)dm_2(y) + \frac{N}{N-1}\int_{\overline{\Omega}} K(x-y)dm_1(y) + \eta_2} - a_2 \right)^{-},$$

$\eta_2 > 0$. They are continuous on $\mathcal{P}(\overline{\Omega}) \times \mathcal{P}(\overline{\Omega})$ and tend to an obvious limit F^i as $N \to \infty$, since $\frac{N}{N-1} \to 1$.

MFG PDEs for two populations: stationary

For two populations with ergodic costs the stationary equations are

$$\text{(MFGs)}\qquad \begin{cases} -\nu\Delta v_i + H^i(x,\nabla v_i) + \lambda_i = F^i(x, m_1, m_2), \\ -\nu\Delta m_i - \operatorname{div}(m_i \nabla_p H^i(x,\nabla v_i)) = 0, \qquad\qquad i = 1,2. \end{cases}$$

1. Periodic boundary conditions:
   - existence of solutions and estimates [M.B. - E. Feleqi 2014]
   - convergence of $N+N$ system of HJB-KFP equations to a solution of (MFGs) [Feleqi 2013]
2. Neumann boundary conditions:

$$\begin{cases} \partial_n v_i = 0, & \textit{on } \partial\Omega \\ \nu\partial_n m_i + m_i \nabla_p H^i(x,\nabla v_i)\cdot n = 0, & i = 1,2. \end{cases}$$

   - existence of solutions and estimates [M. Cirant 2015]

Evolutive MFG with Neumann boundary conditions

$$
\text{(MFGe)}\quad \begin{cases}
-\partial_t v_i - \nu\Delta v_i + H^i(x, \nabla v_i) = F^i(x, m_1, m_2) & \text{in } \Omega \times [0, T] \\
\partial_t m_i - \nu\Delta m_i = \mathrm{div}(\nabla_p H^i(x, m_i \nabla v_i)), & i = 1, 2, \\
\\
\partial_n v_i(x) = 0, & \text{on } \partial\Omega, \\
\nu\partial_n m_i(x,t) + m_i D_p H^i(x, Dv_i(x,t)) \cdot n(x) = 0, & \\
\\
v_i(x, T) = g(x), \qquad m_i(x, 0) = m_{i,0}(x), & i = 1, 2.
\end{cases}
$$

Theorem [Y. Achdou - M.B. - M. Cirant]

Assume H^i satisfy $\quad D_p H^i(x,p) \cdot p \geq -C(1 + |p|^2)$,
F^i takes value in a bounded set of $W^{1,\infty}(\overline{\Omega})$, $g \in W^{1,\infty}(\overline{\Omega})$,
$m_{i,0} \in C^{2,\beta}(\overline{\Omega})$ + compatibility conditions at $\partial\Omega$.
Then (MFGe) has a classical solution.

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Qualitative properties: segregation?

The simplest example: no noise $\nu = 0$, $d = 1$ and $H^i(x,p) = |p|^2$.
Then (MFGs) becomes

$$\begin{cases} \frac{(v_k')^2}{2} + \lambda_k = F^k(x, m_1, m_2) \quad \text{in } (c,d), \\ (v_k' m_k)' = 0, \qquad k = 1,2, \\ \text{Neumann B.C. in viscosity sense at } c, d. \end{cases}$$

Explicit multiple solutions, if
– the threshold is below xenophobia: $a_k < 0.5$, $k = 1,2$,
– the size ρ of the neighbourhood $\mathcal{U}(x)$ is not large,
– F^k is constant if both m_k are constant:

1. uniform distribution: $m_k = \frac{1}{d-c}$, $v_k = 0$, $\lambda_k = F^k(x, m_1, m_2)$, $k = 1,2$

2. segregated solution (m_1 and m_2 have disjoint support):

$$m_1(x) = \frac{1}{x_2 - x_1} \chi_{[x_1,x_2]}(x), \quad m_2(x) = \frac{1}{x_4 - x_3} \chi_{[x_3,x_4]}(x)$$

for any choice $c = x_0 < x_1 < ... < x_4 < x_5 = d$ with $x_{i+1} - x_i \geq \rho$.

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for any choice $c = x_0 < x_1 < ... < x_4 < x_5 = d$ with $x_{j+1} - x_j > \rho$.

Segregation in a simplified problem

Each V^k is local and linear:

$$
(SS) \qquad \begin{cases}
-\nu v_1'' + \frac{(v_1')^2}{2} + \lambda_1 = m_2 & \text{in } (c,d), \\
-\nu v_2'' + \frac{(v_2')^2}{2} + \lambda_2 = m_1 & \\
-\nu m_k'' + (v_k' m_k)' = 0, & k = 1,2, \\
v_k'(c) = v_k'(d) = m_k'(c) = m_k'(d) = 0. &
\end{cases}
$$

Theorem (Cirant JMPA 2015)

If $0 < \nu < \nu_o$ then (SS) has at least two different solutions, and the non-constant solution satisfies $\int_\Omega m_1 m_2 \leq C\nu^2$.

This says that there is segregation in the vanishing viscosity limit. Similar results hold also in higher space dimension for "variational" systems.

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Numerical methods: stationary case [A. - B. - C.]

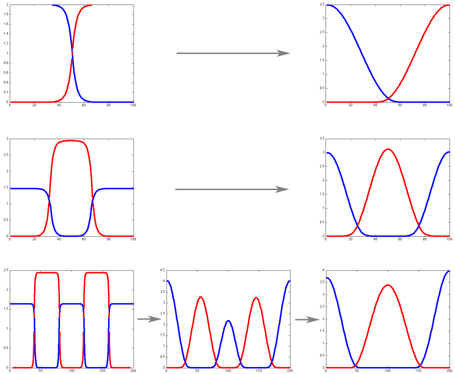
The system (MFGs) with the eigenvalues λ_i has no standard approximation.
A natural approximation would be via the (MFGe) with large T (by Cardaliaguet, Lasry, Lions, Porretta), but this is very heavy form the computationally point of view.
We consider a *finite difference* version of the forward-forward system

$$\begin{cases} \partial_t v_i - \nu_i \Delta v_i + H^i(x, Dv_i) = V^i[m_1, m_2], & \Omega \times (0, T) \\ \partial_t m_i - \nu_i \Delta m_i - \mathrm{div}(D_p H^i(x, Dv_i) m_i) = 0, & \\ \partial_n v_i = 0,\ \partial_n m_i + m_i D_p H^i(x, Dv_i) \cdot n = 0 & \partial\Omega \times (0, T) \\ v_i = 0,\ m_i = m_i^0 & \Omega \times \{0\}, \end{cases}$$

for a large number of iterations (large T).
Motivated by the ergodic theory for HJB equations, we expect that for the numerical time-derivatives $\partial_t u_i \to \mathit{constant} =: \lambda_i$ we are approximating a solution of the stationary system (MFGs).

1D simulations

There is always convergence for large number of time-steps.
If ν is large, convergence to constant m_1, m_2.
Here $\nu = .05$, $a = 0.4$ (NOT xenophobic), and we see the segregation.



Solutions with many peaks are not detected by this method.
Same qualitative behavior for $a = 0.7$.

Numerical methods: evolutive case

How to deal with the backward-forward time structure?
Define the operator $m_i \mapsto \mu_i$,
by solving discrete versions of HJB, KFP

$$\begin{cases} -\partial_t v_i - \nu \Delta v_i + H^i(x, v_i) = F^i(x, m_1, m_2), \quad in\, \Omega \times [0, T], \\ \partial_t \mu_i - \nu \Delta \mu_i - \mathrm{div}(D_p H^i(x, Dv_i)\mu_i) = 0, \\ \text{Neumann B.C.}, \\ v(x, T) = v_T(x), \quad \mu_i(x, 0) = m_{i,0}(x), \quad i = 1, 2. \end{cases}$$

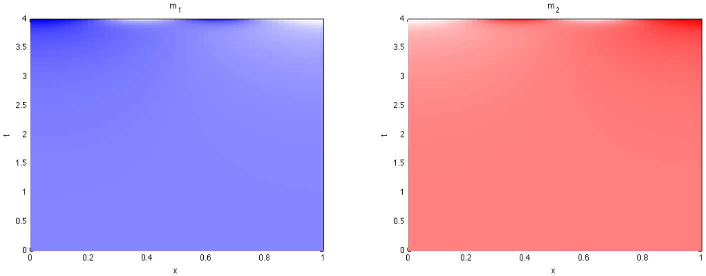
Find an approximate FIXED POINT m_i via a Newton's method.

- Positivity of m_i is preserved; any $\nu \geq 0.01$ is ok.
- Initial guess $m^0(x, t)$ for fixed point of $m_i \mapsto \mu_i$ is extremely important.

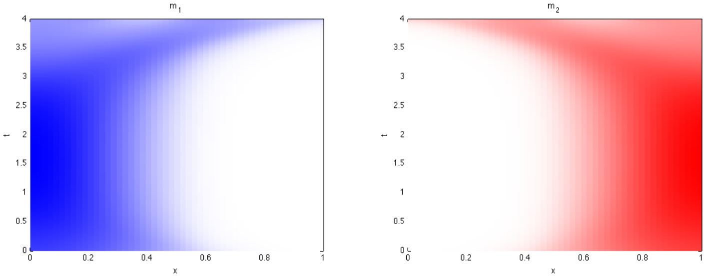
The experiments are done (for simplicity) with localized cost functionals F^i (depend only on $m_k(x)$) and with a term that penalises overcrowding.

$\mathbf{a} = \mathbf{0.4}$

$\nu = 0.15$: large noise $\Rightarrow$ uniform distribution

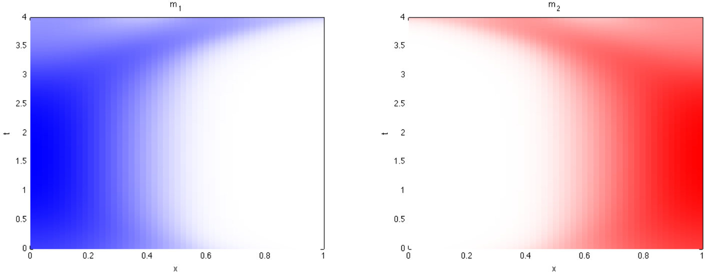


$\nu = 0.05$: small noise $\Rightarrow$ segregation

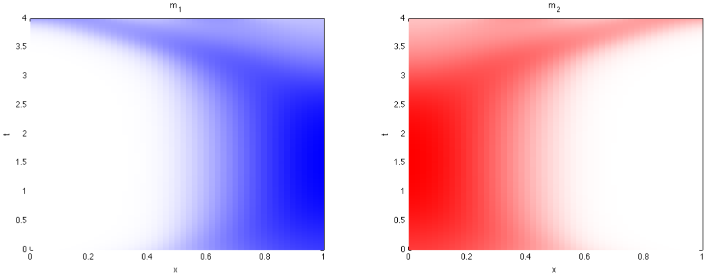


a = 0.4

$\nu = 0.05$

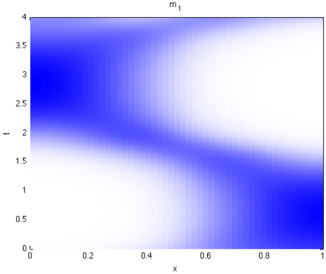


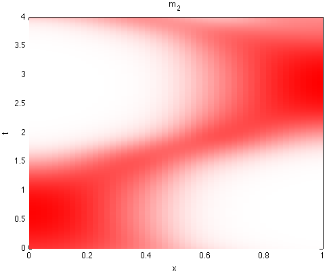
$\nu = 0.05$, different initial guess in Newton's method $\Rightarrow$ different numerical solution!



Large threshold: oscillations

$\mathbf{a = 0.7}$: xenophobic populations,
same behavior as $a = 0.4$ for some time....





but then the populations move in the opposite direction.....
and later they keep oscillating: see the movie!

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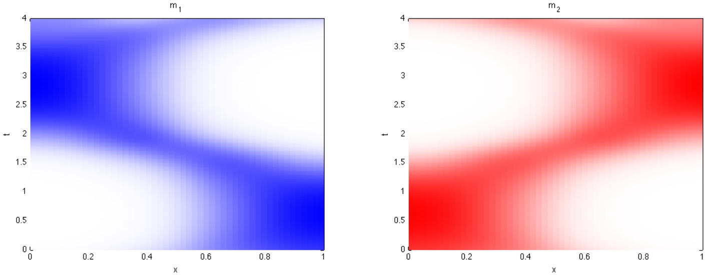

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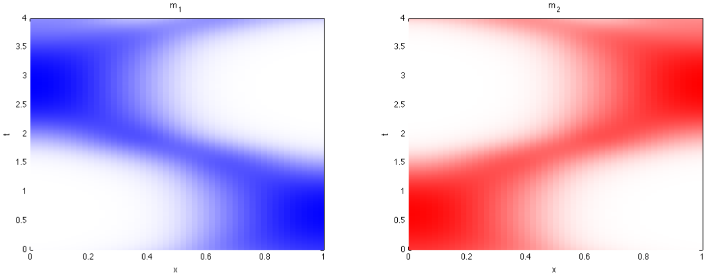

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Low threshold of happiness, i.e., not-xenophobic populations: segregation

Higher threshold of happiness, i.e., xenophobic populations: oscillations

Some conclusions

- MFG is a young theory with many challenging open problems;
- there are many potential applications, most yet to be found, especially to economics and social sciences;
- MFG with several interacting populations is at a very early stage and much can be done, e.g.,
  proving rigorously qualitative properties in segregation or aggregation models.

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Further references

- P. Cardaliaguet, J.M. Lasry, P.L. Lions, A. Porretta : long time behaviour of MFG and convergence to the stationary PDEs
- A. Bensoussan, J. Frehse, P. Yam: short book on MFG and connections with Mean Field control
- R. Carmona, F. Delarue: Probabilistic approach to MFG (also book in progress)
- F. Camilli, E. Carlini, C. Marchi 2015: Mean Field Games on networks.

Thanks for your attention!