On the contribution of Partial Differential Equations to Differential Games. A personal view

Martino Bardi

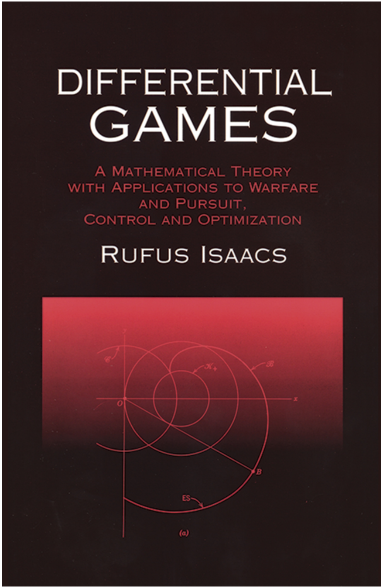
Department of Mathematics
Università di Padova

Isaacs Award presentation
17^{th} International Symposium on Dynamic Games and Applications
Urbino, July 12-15, 2016

Plan

- Isaacs' equation
- Viscosity solution
- Applications: numerical methods, stochastic problems, nonlinear H^∞, multiscale systems...
- Alternative - equivalent theories
- N-player games and systems of Hamilton-Jacobi equations
- Mean-Field Games

Two-person 0-sum differential games



Isaacs' equation

P. 67 of Isaacs' book (1965):
the Main Equation for the value function $V(x)$

(IE) $$\min_a \max_b \{f(x,a,b)\cdot\nabla V(x) + \ell(x,a,b)\} = 0$$

where f is the vector field driving the dynamics of the game

$$\dot{X}(t) = f(X(t),a(t),b(t)), \quad X(0) = x$$

and ℓ is the running cost-payoff of the game:

$$\int_0^{t_x(a,b)} \ell(X(t),a(t),b(t))\,dt$$

and $t_x(a,b)$ is the arrival time of the system on some terminal set.
(IE) is a 1st order, fully nonlinear PDE of Hamilton-Jacobi type.

Isaacs' methods and the 70s

For low–dimensional model problems

- construct piecewise C^1 solutions
- try to synthesize from them saddle points in feedback form
- main difficulty: handling the singularities of the solution

Theory of Singular Surfaces was developed in the 70s-90s by John Breakwell, Pierre Bernhard, J. Lewin, Arik Melikyan, and others. See Lewin's book, Springer 1994.

Notions of value and connections with the Isaacs' equation :

- Wendell Fleming 1961-69
- Avner Friedman, book 1971
- N. Krassovskii , A. Subbotin, books 1974-1988
- Varaiya , Roxin, R. Elliott - N. Kalton 1967 - 1972

Viscosity solution

Michael Crandall, Pierre-Louis Lions, Viscosity solutions of Hamilton-Jacobi equations. Trans. Amer. Math. Soc. 277 (1983)

Craig Evans

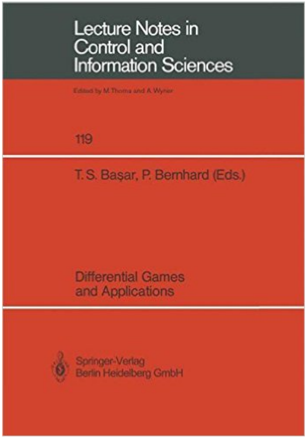
Crandall, M. G.; Evans, L. C.; Lions, P.-L. Some properties of viscosity solutions of Hamilton-Jacobi equations. Trans. Amer. Math. Soc. 282 (1984), 487–502.

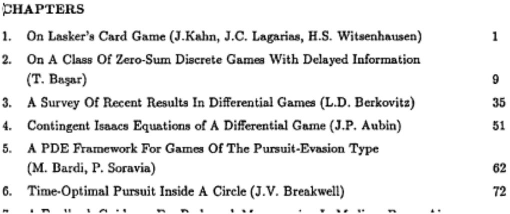
Value functions of diff. games are viscosity solution

Finite horizon games: $-V_t + H(x, \nabla_x V) = 0, t < T; \quad V(x, T) = g(x)$
using Elliott-Kalton notion of value:

Evans, L. C.; Souganidis, P. E.: Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. Indiana Univ. Math. J. 33 (1984)

Sophia-Antipolis 1988





CHAPTERS

M.B. - Pierpaolo Soravia: A PDE framework for games of pursuit-evasion type.

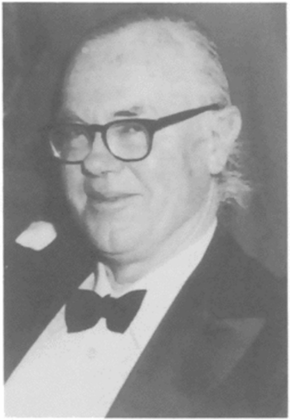


Figure: John Breakwell (1917–1991) and Pierre Bernhard, Isaacs Award 2008

Why are visosity solutions useful for games?

Main result in viscosity theory is a Comparison Principle between sub- and supersolutions of Hamilton-Jacobi-Isaacs type equations + boundary or initial conditions. This allows to prove easily

- lower value $\leq$ upper value
- value exists if max min $=$ min max in the definition of Hamiltonian: the "Isaacs condition"
- all different notions of value coincide, provided they satisfy the Isaacs equation in viscosity sense + boundary conditions.

For generalized pursuit-evasion games this program in carried out in M.B. - Soravia, P.: Hamilton-Jacobi equations with singular boundary conditions on a free boundary and applications to differential games. Trans. Amer. Math. Soc. 325 (1991)

Applications of viscosity theory: Numerical methods

Finite difference schemes based on discrete Dynamic Programming, also called semi-Lagrangian schemes, can be shown to converge by methods of viscosity theory.
Can also find

- approximate saddle in feedback form
- error estimates.

Some contributions :

- single player, discounted infinite horizon: I. Capuzzo Dolcetta 1983, M. Falcone 1987
- Finite horizon games:
  P.E. Souganidis PhD thesis and Nonlinear Anal. 9 (1985)
- Generalized pursuit-evasion games:
  M.B. - Pierpaolo Soravia - Maurizio Falcone 1991-94

Pierpaolo Soravia and Maurizio Falcone

Applications of viscosity theory: stochastic control and differential games

1 player :
- P.L. Lions 1983 ,
- W.H. Fleming, H.M. Soner: book, Springer 1993

2 players : W.H. Fleming, P.E. Souganidis 1989.

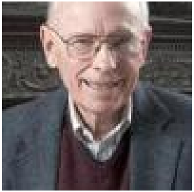


Figure: Wendell Fleming, Isaacs Award 2006

Applications of viscosity theory: nonlinear H^∞

Connection between H^∞ control and games:
Tamer Basar, Pierre Bernhard, H^∞-optimal control and related minimax design problems. A dynamic game approach. Birkhäuser Boston, 1991

Use of viscosity theory for nonlinear systems:
W. Fleming, W. McEneaney: Risk sensitive control on an infinite time horizon, SIAM J. Control Optim. 33 (1995)

Pierpaolo Soravia:
H^∞ control of nonlinear systems: differential games and viscosity solutions. SIAM J. Control Optim. 34 (1996)
Equivalence between nonlinear H^∞ control problems and existence of viscosity solutions of Hamilton-Jacobi-Isaacs equations. Appl. Math. Optim. 39 (1999)

Applications of viscosity theory: multiscale systems

$x =$ "slow" state variables, $y =$ "fast" state variables.

$$\dot{x} = f(x, y, a, b) \qquad \dot{y} = \frac{1}{\varepsilon} g(x, y, a, b)$$

$a =$ control of 1st player, $b =$ control of 2nd player, $\varepsilon > 0$ is small,

For an optimal control problem or 0-sum differential game with running cost ℓ the H-J-Bellman-Isaacs equation is

$$-\frac{\partial V^\varepsilon}{\partial t} + H\left(x, y, D_x V^\varepsilon, \frac{1}{\varepsilon} D_y V^\varepsilon\right) = 0,$$

$$H(x, y, p, q) := \min_{b \in B} \max_{a \in A} \{-f(x, a, b) \cdot p - g(x, a, b) \cdot q - \ell(x, a, b)\}$$

the limit $\varepsilon \to 0$ is a Singular Perturbation problem for the H-J-Isaacs equation.

Applications of viscosity theory: multiscale systems

$x =$ "slow" state variables, $y =$ "fast" state variables.

$$\dot{x} = f(x, y, a, b) \qquad \dot{y} = \frac{1}{\varepsilon} g(x, y, a, b)$$

$a =$ control of 1st player, $b =$ control of 2nd player, $\varepsilon > 0$ is small,

For an optimal control problem or 0-sum differential game with running cost ℓ the H-J-Bellman-Isaacs equation is

$$-\frac{\partial V^\varepsilon}{\partial t} + H\left(x, y, D_x V^\varepsilon, \frac{1}{\varepsilon} D_y V^\varepsilon\right) = 0,$$

$$H(x, y, p, q) := \min_{b \in B} \max_{a \in A} \{-f(x, a, b) \cdot p - g(x, a, b) \cdot q - \ell(x, a, b)\}$$

the limit $\varepsilon \to 0$ is a Singular Perturbation problem for the H-J-Isaacs equation.

S.P. can be applied to model simplification via dimension reduction.

First paper on differential games: Vladimir Gaitsgory 1996.

Approach via H-J-Isaacs equations uses as a tool games with long-time average cost, or ergodic cost, and the associated H-J-I PDEs

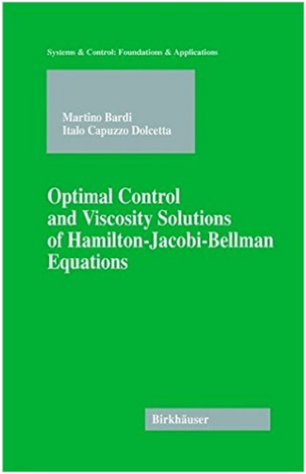
$$\liminf_{T\to+\infty} \frac{1}{T}\int_0^T L(X(t), a(t), b(t))\, dt$$

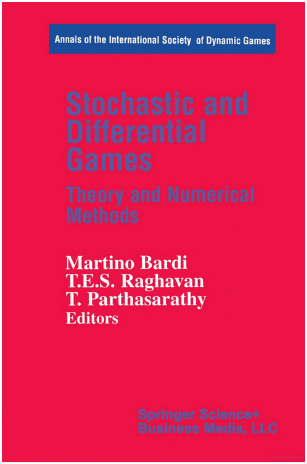
which are of independent interest.

My coworkers on this: Fabio Bagagiolo 1998

Olivier Alvarez 2001 – 2010, also with Claudio Marchi 2007.

Books on viscosity solutions and games: 1997, 1999





Italo Capuzzo Dolcetta and me

Alternative theories: minimax solutions

N. N.Krasovskii, A. I.Subbotin: Positional differential games (Russian), Nauka, Moscow, 1974.
(also Game-theoretical control problems, Springer-Verlag, 1988)

Subbotin, A. I.: Generalization of the fundamental equation of the theory of differential games. (Russian) Dokl. Akad. Nauk SSSR 254 (1980), 293–297.

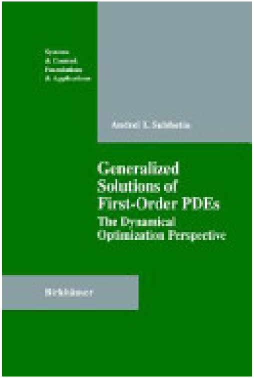
Subbotin, A. I. Generalization of the main equation of differential game theory. J. Optim. Theory Appl. 43 (1984), 103–133

Figure: Nikolai N. Krasovskii, Isaacs Award 2006, and Andrei I. Subbotin

Equivalence of viscosity and minimax solutions

Lions, P.-L.; Souganidis, P. E. Differential games, optimal control and directional derivatives of viscosity solutions of Bellman's and Isaacs' equations. SIAM J. Control Optim. 23 (1985)

Subbotin, Andrei I. Generalized solutions of first-order PDEs. The dynamical optimization perspective. Birkhäuser Boston, 1995



Alternative theories: set-valued analysis and viability theory

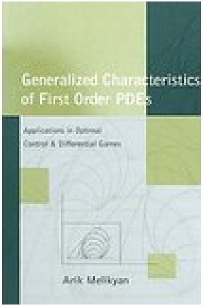
Jean-Pierre Aubin 1988 -....

Pierre Cardaliaguet, Marc Quincampoix 1994 -.....

Numerics: also Patrick Saint-Pierre 1994 - ...

Alternative theories: generalized characteristics

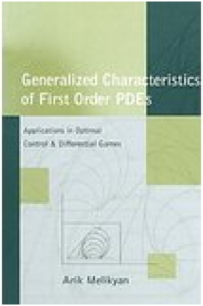
Arik A. Melikyan: Generalized characteristics of first order PDEs. Applications in optimal control and differential games. Birkhäuser Boston, 1998



Evans, L. C. Envelopes and nonconvex Hamilton-Jacobi equations. Calc. Var. PDE 50 (2014), dedicated to A. Melikyan.

Alternative theories: generalized characteristics

Arik A. Melikyan: Generalized characteristics of first order PDEs. Applications in optimal control and differential games. Birkhäuser Boston, 1998



Evans, L. C. Envelopes and nonconvex Hamilton-Jacobi equations. Calc. Var. PDE 50 (2014), dedicated to A. Melikyan.

N-person games and systems of Hamilton-Jacobi equations

Avner Friedman, Differential Games, Wiley 1971: verification theorem for Nash equilibria in feedback form, if $\exists$ smooth solution of

$$\begin{cases} \frac{\partial v_i}{\partial t} + H^i(x^i, \nabla_{x^i} v_i) = \sum_{k \neq i} f^k(x^k, \hat{\alpha}^k) \cdot \nabla_{x^k} v_i \quad \text{in } \mathbf{R}^{dN} \\ \hat{\alpha}^k = \text{argmin}_{\alpha^k} \{ \nabla v_k(x) \cdot f(x^k, \alpha^k) + L(x^k, \alpha^k) \}, \quad i, k = 1, \dots, N \end{cases}$$

system of PDEs strongly coupled via $\hat{\alpha}^k$ in $\sum_{k \neq i} f^k(x^k, \hat{\alpha}^k) \cdot D_{x^k} v_i$ and strongly nonlinear.

T. Basar, G.J. Olsder Dynamic Noncooperative Game Theory. Academic Press, 1982.

Alberto Bressan 2010: if space dimension $d \geq 2$ the system is generically NOT well-posed $\Rightarrow$ no hope to find a good general theory as for Issacs equation in 0-sum games.

N-person with noise: systems of parabolic or elliptic H-J equations

If the dynamical system for the players has a non-degenerate noise, the system of H-J PDEs has 2nd order terms that regularize solutions.

Alain Bensoussan and Jens Frehse: many papers for elliptic problems with unbounded controls - coercive Hamiltonians, 1984 - now.
Book: Regularity results for nonlinear elliptic systems and applications. Springer-Verlag, 2002.

Paola Mannucci: systems of parabolic H-J equations for problems with bounded controls 2004 - 2014

Probabilistic approach: Vivek Borkar, M. Ghosh,...

Theory very technical and hard to use, especially if the number of players N is not small.

What if the players are many, but very similar and indistinguishable, and weakly coupled?
Use ideas for mean-field theories in Physics.
J.-M. Lasry and P.-L. Lions (2006) first simplified the previous system of PDEs in the weakly coupled case, then took the limit as $N \to \infty$ and got a system of just two PDEs:

$$
\text{(MFE)} \qquad \begin{cases} -\frac{\partial v}{\partial t} - \nu \Delta v + H(\nabla v) = F(x, m) \quad \text{in } (0,T) \times \mathbf{R}^d \\ \frac{\partial m}{\partial t} - \nu \Delta m + div(m \nabla H(\nabla v)) = 0 \quad \text{in } (0,T) \times \mathbf{R}^d \\ v(T,x) = g(x), \quad m(0,x) = m_o(x), \end{cases}
$$

the 1st is a backward H-J-Bellman equation for the value function of a representative player, the 2nd is a forward Kologorov-Fokker-Planck equation for the density of the players.

Mean-Field Games

What if the players are many, but very similar and indistinguishable, and weakly coupled?
Use ideas for mean-field theories in Physics.
J.-M. Lasry and P.-L. Lions (2006) first simplified the previous system of PDEs in the weakly coupled case, then took the limit as $N \to \infty$ and got a system of just two PDEs:

$$
\text{(MFE)} \qquad \begin{cases} -\frac{\partial v}{\partial t} - \nu \Delta v + H(\nabla v) = F(x, m) \quad \text{in } (0, T) \times \mathbf{R}^d \\ \\ \frac{\partial m}{\partial t} - \nu \Delta m + div(m \nabla H(\nabla v)) = 0 \quad \text{in } (0, T) \times \mathbf{R}^d \\ \\ v(T, x) = g(x), \quad m(0, x) = m_o(x), \end{cases}
$$

the 1st is a backward H-J-Bellman equation for the value function of a representative player, the 2nd is a forward Kologorov-Fokker-Planck equation for the density of the players.

Jean-Michel Lasry and P.-L. Lions

Nash certainty equivalence principle

Similar ideas were developed independently by different methods by M. Huang, P. Caines and R. Malhamé:

- Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions. Proc.42nd IEEE Conference Decision December 2003.
- Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. Commun. Inf. Syst. 6 (2006),

Eventually the two theories merged and go under the name MFG.

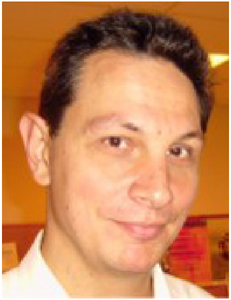
Minyi Huang, Peter Caines and Roland Malhame

My collaborators on MFG

- Linear-Quadratic MFG with long-time average cost, aka ergodic cost: 2012 and 2015 with Fabio Priuli
- MFG with multiple populations: Ermal Feleqi 2013 - 2016,
- MFG models of segregation, after Schelling: Marco Cirant and Yves Achdou.

MFG: recent result and perspectives

- P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, Alessio Porretta. Long time average of mean field games. Netw. Heterog. Media 2012
- Diogo Gomes, L. Nurbekyan, E. Pimentel. Economic models and mean-field games theory. IMPA Mathematical Publications, 2015
- Pierre Cardaliaguet, F. Delarue, J.-M. Lasry, P.-L. Lions, The master equation and the convergence problem in mean field games, preprint arXiv



Thanks for your attention!