Linear-Quadratic N-person and Mean-Field Games with Ergodic Cost

Martino Bardi
University of Padova - Italy

joint work with: Fabio S. Priuli (Univ. Roma Tor Vergata)

Large population dynamics and mean field games, Roma Tor Vergata, April 14th, 2014

Mean Fields Games are macroscopic models of multi-agent decision problems with noise (stochastic differential games) coupled only in the cost and with a large number of indistinguishable agents

- The approach by Huang, Caines & Malhamé (2003, 06, 07,....)
  - solve mean field equations
  - synthesize a ϵ-Nash feedback equilibrium for the game with N agents, N large enough;

Mean Fields Games are macroscopic models of multi-agent decision problems with noise (stochastic differential games) coupled only in the cost and with a large number of indistinguishable agents

- The approach by Huang, Caines & Malhamé (2003, 06, 07,....)
  - solve mean field equations
  - synthesize a ϵ-Nash feedback equilibrium for the game with N agents, N large enough;
- the approach by Lasry & Lions (2006, 07,....)
  - synthesize a Nash equilibrium for the game with N players via a new system of N HJB and N KFP PDEs,
  - prove convergence of solutions of such system to a solution of a single pair of HJB-KFP PDEs,
  - show uniqueness for the MFG PDEs under a monotonicity condition on the costs.

Mean Fields Games are macroscopic models of multi-agent decision problems with noise (stochastic differential games) coupled only in the cost and with a large number of indistinguishable agents

- The approach by Huang, Caines & Malhamé (2003, 06, 07,....)
  - solve mean field equations
  - synthesize a ϵ-Nash feedback equilibrium for the game with N agents, N large enough;
- the approach by Lasry & Lions (2006, 07,....)
  - synthesize a Nash equilibrium for the game with N players via a new system of N HJB and N KFP PDEs,
  - prove convergence of solutions of such system to a solution of a single pair of HJB-KFP PDEs,
  - show uniqueness for the MFG PDEs under a monotonicity condition on the costs.
- Recent developments and applications: 2 special issues on MFG of Dynamic Games and Applications, Dec. 2013 and April 2014 ed. by M.B., I.Capuzzo Dolcetta and P. Caines.

N–person ergodic LQG games in $\mathbb{R}^d$

We consider games with

linear stochastic dynamics w.r.t. state & control	quadratic ergodic cost w.r.t. state & control

N–person ergodic LQG games in $\mathbb{R}^d$

We consider games with

linear stochastic dynamics w.r.t. state & control	quadratic ergodic cost w.r.t. state & control

Main assumption:

- the dynamics of the players are independent

under which we can give necessary and sufficient conditions for the synthesis of Nash feedback equilibria.

In this talk we limit for simplicity to the symmetric case:

- player j and k affect in the same way the cost of player i, they are indistinguishable for player i;

N–person ergodic LQG games in $\mathbb{R}^d$

We consider games with

linear stochastic dynamics w.r.t. state & control	quadratic ergodic cost w.r.t. state & control

Main assumption:

- the dynamics of the players are independent

under which we can give necessary and sufficient conditions for the synthesis of Nash feedback equilibria.

In this talk we limit for simplicity to the symmetric case:

- player j and k affect in the same way the cost of player i, they are indistinguishable for player i;

and focus on nearly identical players, i.e., we assume also

- all players have the same dynamics
- all players have same cost of control and of primary interactions

Consider for $i = 1, \dots, N$

$$dX_t^i = (AX_t^i - \alpha_t^i)dt + \sigma\, dW_t^i \qquad X_0^i = x^i \in \mathbb{R}^d$$

Consider for $i = 1, \dots, N$

$$dX_t^i = (AX_t^i - \alpha_t^i)dt + \sigma\, dW_t^i \qquad X_0^i = x^i \in \mathbb{R}^d$$

where $A \in \mathbb{R}^{d\times d}$, α_t^i controls, σ invertible, W_t^i Brownian,

Consider for $i = 1, \dots, N$

$$dX_t^i = (AX_t^i - \alpha_t^i)dt + \sigma\, dW_t^i \qquad X_0^i = x^i \in \mathbb{R}^d$$

$$J^i(X_0, \alpha^1, \dots, \alpha^N) \doteq \liminf_{T\to\infty} \frac{1}{T}\, \mathbb{E}\Bigg[\int_0^T \frac{(\alpha_t^i)^T R\, \alpha_t^i}{2} + \underbrace{(X_t - \overline{X}_i)^T Q^i (X_t - \overline{X}_i)}_{F^i(X^1,\dots,X^N)}\, dt \Bigg]$$

where $A \in \mathbb{R}^{d\times d}$, α_t^i controls, σ invertible, W_t^i Brownian, Consider for $i = 1, \ldots, N$

$$dX_t^i = (AX_t^i - \alpha_t^i)dt + \sigma\, dW_t^i \qquad X_0^i = x^i \in \mathbb{R}^d$$

$$J^i(X_0, \alpha^1, \ldots, \alpha^N) \doteq \liminf_{T\to\infty} \frac{1}{T}\,\mathbb{E}\left[\int_0^T \frac{(\alpha_t^i)^T R\, \alpha_t^i}{2} + \underbrace{(X_t - \overline{X_i})^T Q^i (X_t - \overline{X_i})}_{F^i(X^1,\ldots,X^N)}\, dt\right]$$

where $A \in \mathbb{R}^{d\times d}$, α_t^i controls, σ invertible, W_t^i Brownian,
$R \in \mathbb{R}^{d\times d}$ symm. pos. def., $X_t = (X_t^1, \ldots, X_t^N) \in \mathbb{R}^{Nd}$ state var.,
$\overline{X_i} = (\overline{X_i^1}, \ldots, \overline{X_i^N}) \in \mathbb{R}^{Nd}$ vector of reference positions,
$Q^i \in \mathbb{R}^{Nd\times Nd}$ symmetric matrix

$\overline{X_i} = (\overline{X_i^1}, \dots, \overline{X_i^N}) \in \mathbb{R}^{Nd}$ s.t.

- $\overline{X_i^i} = h \;\; \forall i$ (preferred own position)
- $\overline{X_i^j} = r \;\; \forall j \neq i$ (reference position of the others)

$\overline{X_i} = (\overline{X_i^1}, \dots, \overline{X_i^N}) \in \mathbb{R}^{Nd}$ s.t.

- $\overline{X_i^i} = h \ \ \forall i$ (preferred own position)
- $\overline{X_i^j} = r \ \ \forall j \neq i$ (reference position of the others)

$Q^i \in \mathbb{R}^{Nd \times Nd}$ block matrix $(Q^i_{jk})_{j,k} \in \mathbb{R}^{d \times d}$ s.t.

$$F^i(X^1, \dots, X^N) = \sum_{j,k=1}^{N} (X_t^j - \overline{X_i^j})^T Q^i_{jk} (X_t^k - \overline{X_i^k})$$

satisfies

$\overline{X_i} = (\overline{X_i^1}, \dots, \overline{X_i^N}) \in \mathbb{R}^{Nd}$ s.t.

- $\overline{X_i^i} = h \quad \forall i$ (preferred own position)
- $\overline{X_i^j} = r \quad \forall j \neq i$ (reference position of the others)

$Q^i \in \mathbb{R}^{Nd \times Nd}$ block matrix $(Q^i_{jk})_{j,k} \in \mathbb{R}^{d \times d}$ s.t.

$$F^i(X^1, \dots, X^N) = \sum_{j,k=1}^{N} (X_t^j - \overline{X_i^j})^T Q^i_{jk} (X_t^k - \overline{X_i^k})$$

satisfies

- $Q^i_{ii} = Q$ symm. pos. def. $\forall i$ (primary cost of self-displacement)
- $Q^i_{ij} = Q^i_{ji} = B \quad \forall j \neq i$ (primary cost of cross-displacement)

$\overline{X_i} = (\overline{X_i^1}, \dots, \overline{X_i^N}) \in \mathbb{R}^{Nd}$ s.t.

- $\overline{X_i^i} = h \quad \forall i$ (preferred own position)
- $\overline{X_i^j} = r \quad \forall j \neq i$ (reference position of the others)

$Q^i \in \mathbb{R}^{Nd \times Nd}$ block matrix $(Q^i_{jk})_{j,k} \in \mathbb{R}^{d \times d}$ s.t.

$$F^i(X^1, \dots, X^N) = \sum_{j,k=1}^{N} (X_t^j - \overline{X_i^j})^T Q^i_{jk} (X_t^k - \overline{X_i^k})$$

satisfies

- $Q^i_{ii} = Q$ symm. pos. def. $\forall i$ (primary cost of self-displacement)
- $Q^i_{ij} = Q^i_{ji} = B \quad \forall j \neq i$ (primary cost of cross-displacement)
- $Q^i_{jj} = C_i \quad \forall j \neq i$ (secondary cost of self-displacement)
- $Q^i_{jk} = Q^i_{kj} = D_i \quad \forall j \neq k \neq i \neq j$ (secondary cost of cross-displacement)

Admissible strategies

A control α_t^i adapted to W_t^i is an *admissible strategy* if

- $\mathbb{E}[X_t^i], \mathbb{E}[X_t^i (X_t^i)^T] \leq C$ for all $t > 0$
- $\exists$ probability measure m_{α^i} s.t. the process X_t^i is ergodic

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E}\left[\int_0^T g(X_t^i)\, dt\right] = \int_{\mathbb{R}^d} g(\xi)\, dm_{\alpha^i}(\xi)$$

for any polynomial g, with $\deg(g) \leq 2$, loc. unif. in X_0^i.

Admissible strategies

A control α_t^i adapted to W_t^i is an *admissible strategy* if

- $\mathbb{E}[X_t^i], \mathbb{E}[X_t^i (X_t^i)^T] \leq C$ for all $t > 0$
- $\exists$ probability measure m_{α^i} s.t. the process X_t^i is ergodic

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \left[\int_0^T g(X_t^i) \, dt \right] = \int_{\mathbb{R}^d} g(\xi) \, dm_{\alpha^i}(\xi)$$

for any polynomial g, with $\deg(g) \leq 2$, loc. unif. in X_0^i.

Example. Any affine $\alpha^i(x) = Kx + c$ with "$K - A > 0$" is admissible and the corresponding diffusion process

$$dX_t^i = \big((A - K)X_t^i - c\big)dt + \sigma dW_t^i$$

is ergodic with $m_{\alpha^i} =$ multivariate Gaussian

Admissible strategies

A control α_t^i adapted to W_t^i is an *admissible strategy* if

- $\mathbb{E}[X_t^i], \mathbb{E}[X_t^i (X_t^i)^T] \leq C$ for all $t > 0$
- $\exists$ probability measure m_{α^i} s.t. the process X_t^i is ergodic

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E}\left[\int_0^T g(X_t^i)\, dt\right] = \int_{\mathbb{R}^d} g(\xi)\, dm_{\alpha^i}(\xi)$$

for any polynomial g, with $\deg(g) \leq 2$, loc. unif. in X_0^i.

Nash equilibria

Any set of admissible strategies $\overline{\alpha}^1, \dots, \overline{\alpha}^N$ such that

$$J^i(X, \overline{\alpha}^1, \dots, \overline{\alpha}^N) = \min_{\omega} J^i(X, \overline{\alpha}^1, \dots, \overline{\alpha}^{i-1}, \omega, \overline{\alpha}^{i+1}, \dots, \overline{\alpha}^N)$$

for any $i = 1, \dots, N$

The HJB+KFP PDEs of Lasry-Lions for the N–person game are

$$\begin{cases} -\mathrm{tr}(\nu D^2 v^i) + H(x, \nabla v^i) + \lambda^i = f^i(x; m^1, \dots, m^N) \\ -\mathrm{tr}(\nu D^2 m^i) + \mathrm{div}\left(m^i \frac{\partial H}{\partial p}(x, \nabla v^i) \right) = 0, \qquad x \in \mathbb{R}^d \\ m^i > 0, \qquad \int_{\mathbb{R}^d} m^i(x)\, dx = 1, \qquad i = 1, ..., N \end{cases} \tag{1}$$

where

$$\nu = \frac{\sigma^T \sigma}{2} \qquad\qquad H(x,p) = p^T \frac{R^{-1}}{2} p - p^T A x$$

$$f^i(x; m^1, \dots, m^N) \doteq \int_{\mathbb{R}^{(N-1)d}} F^i(\xi^1, \dots, \xi^{i-1}, x, \xi^{i+1}, \dots \xi^N) \prod_{j \neq i} dm^j(\xi^j)$$

The HJB+KFP PDEs of Lasry-Lions for the N–person game are

$$\begin{cases} -\mathrm{tr}(\nu D^2 v^i) + H(x, \nabla v^i) + \lambda^i = f^i(x; m^1, \dots, m^N) \\ -\mathrm{tr}(\nu D^2 m^i) + \mathrm{div}\left(m^i \frac{\partial H}{\partial p}(x, \nabla v^i) \right) = 0, \qquad x \in \mathbb{R}^d \\ m^i > 0, \qquad \int_{\mathbb{R}^d} m^i(x)\, dx = 1, \qquad i = 1, ..., N \end{cases} \tag{1}$$

where

$$\nu = \frac{\sigma^T \sigma}{2} \qquad\qquad H(x,p) = p^T \frac{R^{-1}}{2} p - p^T A x$$

$$f^i(x; m^1, \dots, m^N) \doteq \int_{\mathbb{R}^{(N-1)d}} F^i(\xi^1, \dots, \xi^{i-1}, x, \xi^{i+1}, \dots \xi^N) \prod_{j \neq i} dm^j(\xi^j)$$

$2N$ equations with unknowns λ^i, v^i, m^i, but always $x \in \mathbb{R}^d$,

different form the classical strongly coupled system of N HJB PDEs in $\mathbb{R}^{Nd}$! (e.g. Bensoussan-Frehse 1995)

The HJB+KFP PDEs of Lasry-Lions for the N–person game are

$$\begin{cases} -\mathrm{tr}(\nu D^2 v^i) + H(x, \nabla v^i) + \lambda^i = f^i(x; m^1, \dots, m^N) \\ -\mathrm{tr}(\nu D^2 m^i) + \mathrm{div}\left(m^i \frac{\partial H}{\partial p}(x, \nabla v^i)\right) = 0, \qquad x \in \mathbb{R}^d \\ m^i > 0, \qquad \int_{\mathbb{R}^d} m^i(x)\, dx = 1, \qquad i = 1, ..., N \end{cases} \tag{1}$$

Search for solutions

- identically distributed, i.e., $v^i = v^j = v$, $m^i = m^j = m$, $\forall\, i, j = 1, ..., N$,

and

- Quadratic–Gaussian (QG): v polynomial of degree 2, $m \sim \mathcal{N}(\mu, \Sigma^{-1})$,

$$\lambda^i \in \mathbb{R} \qquad v^i(x) = x^T \frac{\Lambda}{2} x + \rho x \qquad m^i(x) = \gamma \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma (x-\mu)\right\}$$

Theorem 1. For N–players LQG game

- Existence & uniqueness λ^i, v^i, m^i sol. to (1) with v^i, m^i QG $\Leftrightarrow$ ALGEBRAIC CONDITIONS

Theorem 1. For N–players LQG game

- Existence & uniqueness λ^i, v^i, m^i sol. to (1) with v^i, m^i QG $\Leftrightarrow$ ALGEBRAIC CONDITIONS
- $\overline{\alpha}^i(x) = R^{-1}\nabla v^i(x) = R^{-1}(\Lambda x + \rho)$ are Nash feedback equilibrium strategies and
  $\lambda^i = J^i(X_0, \overline{\alpha}^1, \ldots, \overline{\alpha}^N)$ for $i = 1, \ldots, N$

Theorem 1. For N–players LQG game

- Existence & uniqueness λ^i, v^i, m^i sol. to (1) with v^i, m^i QG $\Leftrightarrow$ ALGEBRAIC CONDITIONS
- $\overline{\alpha}^i(x) = R^{-1}\nabla v^i(x) = R^{-1}(\Lambda x + \rho)$ are Nash feedback equilibrium strategies and
  $\lambda^i = J^i(X_0, \overline{\alpha}^1, \dots, \overline{\alpha}^N)$ for $i = 1, \dots, N$

Proof.
(i) By plugging into (1)

$$v^i(x) = x^T \frac{\Lambda}{2} x + \rho x \qquad m^i(x) = \gamma \exp\left\{ -\frac{1}{2}(x-\mu)^T \Sigma (x-\mu) \right\}$$

$\rightsquigarrow$ algebraic conditions on $\rho, \mu \in \mathbb{R}^d$, $\Lambda, \Sigma \in \mathbb{R}^{d\times d}$

(ii) Verification theorem, using Dynkin's formula and ergodicity.

Algebraic conditions:

$$\nabla v^i(x) = \Lambda x + \rho \qquad \nabla m^i(x) = -m^i(x)\Sigma(x-\mu)$$

Algebraic conditions:

$$\nabla v^i(x) = \Lambda x + \rho \qquad \nabla m^i(x) = -m^i(x)\Sigma(x - \mu)$$

KFP

$$\Lambda = R\big(\nu\Sigma + A\big) \qquad \rho = -R\nu\Sigma\mu$$

Algebraic conditions:

$$\nabla v^i(x) = \Lambda x + \rho \qquad \nabla m^i(x) = -m^i(x)\Sigma(x-\mu)$$

KFP

$$\Lambda = R\big(\nu\Sigma + A\big) \qquad \rho = -R\nu\Sigma\mu$$

HJB

$$\Sigma\,\frac{\nu R\nu}{2}\,\Sigma - \frac{A^T RA}{2} = Q$$

$$-\left(\frac{A^T RA}{2} + Q + (N-1)B\right)\mu = -Qh - (N-1)Br$$

$$(\mu)^T\,\frac{\Sigma\nu R\nu\Sigma}{2}\,\mu - \mathrm{tr}(\nu R\nu\Sigma + \nu RA) + \lambda^i = \mathfrak{f}^i(\Sigma,\mu)$$

Algebraic conditions:

$$\nabla v^i(x) = \Lambda x + \rho \qquad \nabla m^i(x) = -m^i(x)\Sigma(x-\mu)$$

KFP

$$\Lambda = R\big(\nu\Sigma + A\big) \qquad \rho = -R\nu\Sigma\mu$$

HJB

Σ solves ARE $$X\,\frac{\nu R\nu}{2}\,X - \left(\frac{A^TRA}{2} + Q\right) = 0$$

$$-\left(\frac{A^TRA}{2} + Q + (N-1)B\right)\mu = -Qh - (N-1)Br$$

$$(\mu)^T\,\frac{\Sigma\nu R\nu\Sigma}{2}\,\mu - \operatorname{tr}(\nu R\nu\Sigma + \nu RA) + \lambda^i = \mathfrak{f}^i(\Sigma,\mu)$$

Algebraic conditions:

$$\nabla v^i(x) = \Lambda x + \rho \qquad \nabla m^i(x) = -m^i(x)\Sigma(x-\mu)$$

KFP

$$\Lambda = R\big(\nu\Sigma + A\big) \qquad \rho = -R\nu\Sigma\mu$$

HJB

Σ solves ARE $\qquad X\,\frac{\nu R\nu}{2}\,X - \left(\frac{A^T RA}{2} + Q\right) = 0$

μ solves linear system $\qquad \mathcal{B}y = \mathcal{C} \quad$ for $\quad \mathcal{B} \doteq \frac{A^T RA}{2} + Q + (N-1)B$

$$(\mu)^T\,\frac{\Sigma\nu R\nu\Sigma}{2}\,\mu - \operatorname{tr}(\nu R\nu\Sigma + \nu RA) + \lambda^i = \mathfrak{f}^i(\Sigma,\mu)$$

Algebraic conditions:

$$\nabla v^i(x) = \Lambda x + \rho \qquad \nabla m^i(x) = -m^i(x)\Sigma(x-\mu)$$

KFP

$$\Lambda = R\big(\nu\Sigma + A\big) \qquad \rho = -R\nu\Sigma\mu$$

HJB

Σ solves ARE $\quad X\frac{\nu R\nu}{2}X - \left(\frac{A^T RA}{2} + Q\right) = 0$

μ solves linear system $\quad \mathcal{B}y = \mathcal{C}$ for $\mathcal{B} \doteq \frac{A^T RA}{2} + Q + (N-1)B$

$\lambda^i =$ explicit function of Σ and μ

FACT:
under our conditions on the matrices ν, R, A, Q the Algebraic matrix Riccati Equation

$$X\,\frac{\nu R \nu}{2}\,X - \left(\frac{A^T R A}{2} + Q\right) = 0$$

has a unique solution positive definite solution $X = \Sigma > 0$

Proof: follows with some work from the theory of ARE.

Refs: books by Engwerda (2005) and Lancaster - Rodman (1995)

Algebraic conditions in Thm. 1

Existence

Uniqueness

ALGEBRAIC CONDITIONS IN THM. 1

EXISTENCE

- $$\text{rank } \mathcal{B} = \text{rank } [\mathcal{B}, \mathcal{C}]$$

  [$\iff$ the system $\mathcal{B}y = \mathcal{C}$ has solutions]

UNIQUENESS

Algebraic conditions in Thm. 1

Existence

- $$\text{rank } \mathcal{B} = \text{rank } [\mathcal{B}, \mathcal{C}]$$

  [$\iff$ the system $\mathcal{B}y = \mathcal{C}$ has solutions]

- the unique $\Sigma > 0$ that solves ARE also solves Sylvester's eq.

$$X\nu R - R\nu X = RA - A^T R$$

[$\iff \Lambda = R(\nu\Sigma + A)$ symmetric matrix]

Uniqueness

Algebraic conditions in Thm. 1

Existence

- $$\text{rank } \mathcal{B} = \text{rank } [\mathcal{B}, \mathcal{C}]$$

  [$\Longleftrightarrow$ the system $\mathcal{B}y = \mathcal{C}$ has solutions]

- the unique $\Sigma > 0$ that solves ARE also solves Sylvester's eq.

  $$X\nu R - R\nu X = RA - A^T R$$

  [$\Longleftrightarrow \Lambda = R\big(\nu\Sigma + A\big)$ symmetric matrix]

Uniqueness

- $\mathcal{B}$ is invertible

  [$\Longleftrightarrow$ the system $\mathcal{B}y = \mathcal{C}$ has unique solution]

Mean field equations

The symmetry assumption on the costs $F^i(X^1, \dots, X^N)$ imply they can be written as function of the empirical density of other players

$$F^i(X^1, \dots, X^N) = V^i_N \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{X^j} \right] (X^i)$$

Mean field equations

The symmetry assumption on the costs $F^i(X^1, \dots, X^N)$ imply they can be written as function of the empirical density of other players

$$F^i(X^1, \dots, X^N) = V_N^i \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{X^j} \right] (X^i)$$

where

$$V_N^i : \{\text{prob. meas. on } \mathbb{R}^d\} \to \{\text{quadratic polynomials on } \mathbb{R}^d\}$$

Mean field equations

The symmetry assumption on the costs $F^i(X^1, \dots, X^N)$ imply they can be written as function of the empirical density of other players

$$F^i(X^1, \dots, X^N) = V_N^i \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{X^j} \right] (X^i)$$

where

$$V_N^i \colon \{\text{prob. meas. on } \mathbb{R}^d\} \to \{\text{quadratic polynomials on } \mathbb{R}^d\}$$

$$\begin{aligned} V_N^i[m](X) \doteq{}& (X-h)^T Q^N (X-h) \\ &+ (N-1) \int_{\mathbb{R}^d} \left((X-h)^T \frac{B^N}{2} (\xi - r) + (\xi - r)^T \frac{B^N}{2} (X-h) \right) dm(\xi) \\ &+ (N-1) \int_{\mathbb{R}^d} (\xi - r)^T (C_i^N - D_i^N)(\xi - r)\, dm(\xi) \\ &+ \left((N-1) \int_{\mathbb{R}^d} (\xi - r)\, dm(\xi) \right)^T D_i^N \left((N-1) \int_{\mathbb{R}^d} (\xi - r)\, dm(\xi) \right) \end{aligned}$$

Assuming that the coefficients scale as follows as $N \to \infty$

$$Q^N \to \hat{Q} > 0 \qquad B^N(N-1) \to \hat{B}$$
$$C_i^N(N-1) \to \hat{C} \qquad D_i^N(N-1)^2 \to \hat{D}$$

then for any prob. measure m on $\mathbb{R}^d$ and all $i = 1, \dots, N$

$$V_N^i[m](X) \to \hat{V}[m](X) \qquad \text{loc. unif. in } X$$

Assuming that the coefficients scale as follows as $N \to \infty$

$$Q^N \to \hat{Q} > 0 \qquad B^N(N-1) \to \hat{B}$$
$$C_i^N(N-1) \to \hat{C} \qquad D_i^N(N-1)^2 \to \hat{D}$$

then for any prob. measure m on $\mathbb{R}^d$ and all $i = 1, \ldots, N$

$$V_N^i[m](X) \to \hat{V}[m](X) \qquad \text{loc. unif. in } X$$

where

$$\begin{aligned}\hat{V}[m](X) \doteq\ & (X-h)^T \hat{Q}(X-h) \\ & + \int_{\mathbb{R}^d} \left((X-h)^T \frac{\hat{B}}{2}(\xi - r) + (\xi - r)^T \frac{\hat{B}}{2}(X-h) \right) dm(\xi) \\ & + \int_{\mathbb{R}^d} (\xi - r)^T \hat{C}(\xi - r)\, dm(\xi) \\ & + \left(\int_{\mathbb{R}^d} (\xi - r)\, dm(\xi) \right)^T \hat{D} \left(\int_{\mathbb{R}^d} (\xi - r)\, dm(\xi) \right)\end{aligned}$$

Then we expect as $N \to \infty$ that the N HJB + N KFP reduce to just ONE HJB+KFP

$$\begin{cases} -\mathrm{tr}(\nu D^2 u) + H(x, Du) + \lambda = \hat{V}[m](x) \\ -\mathrm{tr}(\nu D^2 m) - \mathrm{div}\left(m \frac{\partial H}{\partial p}(x, Du)\right) = 0, \quad x \in \mathbb{R}^d \\ m > 0 \qquad \int_{\mathbb{R}^d} m(x)\, dx = 1 \end{cases} \qquad \text{(MFE)}$$

We look for solutions λ, u, m such that u, m is QG

$$u(x) = x^T \frac{\Lambda}{2} x + \rho x \qquad\qquad m(x) = \gamma \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma (x-\mu)\right\}$$

Existence and uniqueness for the single HJB+KFP

$$\begin{cases} -\mathrm{tr}(\nu D^2 u) + H(x, Du) + \lambda = \hat{V}[m](x) \\ -\mathrm{tr}(\nu D^2 m) - \mathrm{div}\left(m \frac{\partial H}{\partial p}(x, Du)\right) = 0, \quad x \in \mathbb{R}^d \\ m > 0 \qquad \int_{\mathbb{R}^d} m(x)\, dx = 1 \end{cases} \tag{MFE}$$

Theorem 2.

- Existence & uniqueness λ, u, m sol. to MFE with u, m QG $\iff$ ALGEBRAIC CONDITIONS similar to the preceding

Existence and uniqueness for the single HJB+KFP

$$\begin{cases} -\mathrm{tr}(\nu D^2 u) + H(x, Du) + \lambda = \hat{V}[m](x) \\ -\mathrm{tr}(\nu D^2 m) - \mathrm{div}\left(m \, \frac{\partial H}{\partial p}(x, Du)\right) = 0, \quad x \in \mathbb{R}^d \\ m > 0 \qquad \int_{\mathbb{R}^d} m(x)\, dx = 1 \end{cases} \qquad \text{(MFE)}$$

Theorem 2.

- Existence & uniqueness λ, u, m sol. to MFE with u, m QG $\iff$ ALGEBRAIC CONDITIONS similar to the preceding
- $\hat{V}$ is a monotone operator in L^2 $\iff$ $\hat{B} \geq 0$, then if $\hat{B} \geq 0$ the QG sol. is the unique C^2 solution to MFE

Rmk: $\hat{B} \geq 0$ means that imitation is not rewarding, [M.B. 2012 for $d = 1$]

Limit as $N \to \infty$

Theorem 3. Assume

(i) $Q^N \to \hat{Q}$ $\quad$ $B^N(N-1) \to \hat{B}$

$C_i^N(N-1) \to \hat{C}$ $\quad$ $D_i^N(N-1)^2 \to \hat{D}$

(ii) HJB+KFP for N–players admit QG sol. $(v_N, m_N, \lambda_N^1, \dots \lambda_N^N)$

(iii) MFE admits unique QG solution (u, m, λ)

Limit as $N \to \infty$

Theorem 3. Assume

(i) $Q^N \to \hat{Q}$ $\quad$ $B^N(N-1) \to \hat{B}$

$C_i^N(N-1) \to \hat{C}$ $\quad$ $D_i^N(N-1)^2 \to \hat{D}$

(ii) HJB+KFP for N–players admit QG sol. $(v_N, m_N, \lambda_N^1, \dots \lambda_N^N)$

(iii) MFE admits unique QG solution (u, m, λ)

Then

- $v_N \to u$ in $\mathbf{C}^2_{loc}(\mathbb{R}^d)$
- $m_N \to m$ in $\mathbf{C}^k(\mathbb{R}^d)$ for all k
- $\lambda_N^i \to \lambda$ for all i

Proof is based on estimates of the maximal eigenvalue of Σ_N solving the ARE for N players.

General conclusions

General conclusions

- Characterization of existence & uniqueness for QG sols to N–person LQG games, and synthesis of feedback Nash equilibrium strategies
  (also without symmetry and identical players conditions)

General conclusions

- Characterization of existence & uniqueness for QG sols to N–person LQG games, and synthesis of feedback Nash equilibrium strategies
(also without symmetry and identical players conditions)

- Characterization of existence & uniqueness for QG sols to MFE
$+$ Characterization of monotonicity of $\hat{V}$ & uniqueness among C^2 non-QG sols to MFE

General conclusions

- Characterization of existence & uniqueness for QG sols to N–person LQG games, and synthesis of feedback Nash equilibrium strategies
(also without symmetry and identical players conditions)

- Characterization of existence & uniqueness for QG sols to MFE
$+$ Characterization of monotonicity of $\hat{V}$ & uniqueness among C^2 non-QG sols to MFE

- Convergence of QG sols of HJB$+$KFP for N-person to sols of MFE as $N \to +\infty$

General conclusions

- Characterization of existence & uniqueness for QG sols to N–person LQG games, and synthesis of feedback Nash equilibrium strategies
  (also without symmetry and identical players conditions)

- Characterization of existence & uniqueness for QG sols to MFE + Characterization of monotonicity of $\hat{V}$ & uniqueness among C^2 non-QG sols to MFE

- Convergence of QG sols of HJB+KFP for N-person to sols of MFE as $N \to +\infty$

- In several examples the algebraic conditions can be easily checked solutions are explicit

Example 1:

N–players game with $R = r\mathrm{I}_d$, $\nu = \overline{\nu}\mathrm{I}_d$, $r, \overline{\nu} \in \mathbb{R}$,
A symmetric, $B \geq 0$

ALGEBRAIC CONDITIONS

- $X\,\dfrac{\nu R \nu}{2}\,X = \dfrac{A^T R A}{2} + Q \implies X\nu R - R\nu X = RA - A^T R$
  becomes $\overline{\nu} r(X - X) = r(A - A^T)$, true for all matrices X
- $\mathcal{B} = Q + r\,\dfrac{A^2}{2} + \dfrac{B}{2} > 0$

Example 1:

N–players game with $R = r\mathrm{I}_d$, $\nu = \overline{\nu}\mathrm{I}_d$, $r, \overline{\nu} \in \mathbb{R}$,
A symmetric, $B \geq 0$

ALGEBRAIC CONDITIONS

- $X \frac{\nu R \nu}{2} X = \frac{A^T R A}{2} + Q \implies X\nu R - R\nu X = RA - A^T R$
  becomes $\overline{\nu} r(X - X) = r(A - A^T)$, true for all matrices X
- $\mathcal{B} = Q + r\frac{A^2}{2} + \frac{B}{2} > 0$

$\implies$ $\exists!$ QG solution, with $\Sigma, \Lambda, \mu, \rho$ satisfying

$$\Sigma^2 = \frac{2}{r\overline{\nu}^2}\left(r\frac{A^2}{2} + Q\right) \qquad \mathcal{B}\mu = \mathcal{C}$$

$$\Lambda = r\left(\overline{\nu}\Sigma + A\right) \qquad \rho = -r\overline{\nu}\Sigma\mu$$

Examples with A NOT symmetric?

Example 2:
the previous example can be adapted to the case of A non-defective, i.e.

$\forall\,\lambda$ eigenvalue the dimension of the eigenspace $=$ multiplicity of λ,

i.e., $\mathbb{R}^d$ has a basis of right eigenvectors of A.

Examples with A NOT symmetric?

Example 2:
the previous example can be adapted to the case of A non-defective, i.e.

$\forall\, \lambda$ eigenvalue the dimension of the eigenspace $=$ multiplicity of λ,
i.e., $\mathbb{R}^d$ has a basis of right eigenvectors of A.

A non-defective $\implies$ A has a positive definite symmetrizer.
Then by changing coordinates can transform the game into one that fits in Example 1.

A Consensus model:

N–players game with cost involving

$$F^i(X^1, \dots, X^N) = \frac{1}{N-1} \sum_{j \neq i} (X^i - X^j)^T P (X^i - X^j),$$

$i = 1, \dots, N$, with $P = P^N > 0$.

Then aggregation is rewarding: $B = -\frac{2}{N-1} P < 0$.

A Consensus model:

N–players game with cost involving

$$F^i(X^1, \dots, X^N) = \frac{1}{N-1} \sum_{j \neq i} (X^i - X^j)^T P (X^i - X^j),$$

$i = 1, \dots, N$, with $P = P^N > 0$.

Then aggregation is rewarding: $B = -\frac{2}{N-1} P < 0$.

Assume (for simplicity) $R = r\mathrm{I}_d,\ \nu = \overline{\nu}\mathrm{I}_d,\ r, \overline{\nu} \in \mathbb{R},\ A$ symmetric.

There is QG solution $m = m_N \sim \mathcal{N}(\mu, \Sigma^{-1}) \iff \frac{r}{2} A^2 \mu = 0$.

The N-person game has an identically distributed QG solution with mean $\mu \iff A\mu = 0$

i.e., for μ stationary point of the system $X_t' = AX_t$.

Large population limit in the Consensus model:

Assume $P^N \to \hat{P} > 0$ as $N \to \infty$, and set

$$\hat{\Sigma} = \frac{1}{\bar{\nu}} \sqrt{\frac{2}{r}\,\hat{P} - A^2}$$

The MFG PDEs have a solution (v, m, λ) with v quadratic and m Gaussian $\iff \quad m \sim \mathcal{N}(\mu, \hat{\Sigma}^{-1})$ with $A\mu = 0$.

- If $\det A = 0$ there are infinitely many QG solutions,
- if $\det A \neq 0$ uniqueness of QG solution, other non-QG solutions?

Ref. for MFG consensus models: Nourian, Caines, Malhame, Huang 2013.

PERSPECTIVES:

- In a very recent paper Priuli studies infinite horizon discounted LQG games, the small discount limit, the vanishing viscosity and other singular limits,
- multi-population MFG models can be studied by these methods,
- ideas for applications are welcome!

PERSPECTIVES:

- In a very recent paper Priuli studies infinite horizon discounted LQG games, the small discount limit, the vanishing viscosity and other singular limits,
- multi-population MFG models can be studied by these methods,
- ideas for applications are welcome!

Thanks for Your Attention!