Liouville properties of fully nonlinear elliptic operators and some applications

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Liouville properties

Fully nonlinear (degenerate) elliptic equations in $\mathbb{R}^N$

(E) $$F(x, u, Du, D^2u) = 0 \quad \text{in } \mathbb{R}^N.$$

Questions:

- are subsolutions bounded from above constant?
- are supersolutions bounded from below constant?

N.B.: different from the same question for solutions, which follows from Harnack inequality.

Outline

- Liouville properties for
  1. Hamilton-Jacobi-Bellman operators
  2. quasilinear hypoelliptic operators
  3. fully nonlinear uniformly elliptic operators (via comparison with Pucci)
- Application 1 : large-time stabilization in parabolic equations

$$u_t + F(x, Du, D^2u) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N.$$

- Application 2 - Ergodic HJB equation, i.e., critical value of the operator F:

$$F(x, D\chi, D^2\chi) = c \quad + \text{growth at } \infty \text{ of } \chi.$$

in the unknowns (c, χ).

Liouville for subsolutions: some known results

- $F = -\Delta u$ Liouville for subsols. $\iff$ $N \leq 2$;
- Cutrì - Leoni (2000): $F = \tilde{F}(x, D^2u) + h(x)u^p$, $\tilde{F}$ uniformly elliptic, via Hadamard-type theorems
- Capuzzo Dolcetta - Cutrì (2003): $F = \tilde{F}(x, D^2u) + g(|x|)|Du| + h(x)u^p$ with g "small at ∞"
- Chen - Felmer (2013): similar smallness conditions on the terms in Du
- for subelliptic operators: Capuzzo Dolcetta - Cutrì (1997), Bonfiglioli-Lanconelli-Uguzzoni (book 2007), Kogoj - Lanconelli (2009, 2015), ...

Our goal: use the terms with Du as "good terms".

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Liouville for HJB operators: abstract assumptions

$$L^{\alpha} u := tr(a(x,\alpha)D^2 u) + b(x,\alpha)\cdot Du$$

Concave H-J-B operator

$$G[u] := \inf_{\alpha\in A}\{-L^{\alpha} u + c(x,\alpha)u\}$$

with coefficients a, b, c continuous in x uniformly in $\alpha \in A$, A a metric space. Assume

1. Comparison Principle in bounded open sets Ω: u, v sub- and supersolutions of $G[u] = 0$ in Ω, $u \le v$ on $\partial\Omega$, $\implies u \le v$ in Ω;
2. Strong Maximum Principle: $G[u] \le 0$ in Ω and $\exists\, x_o$ such that $u(x_o) = \max_{\Omega} u \implies u$ is constant.
3. there exist Lyapunov function: $\exists\, R_o \ge 0$, $w \in LSC(\mathbb{R}^N)$ s. t.

$$G[w] \ge 0 \text{ for } |x| > R_o, \quad \lim_{|x|\to\infty} w(x) = +\infty.$$

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Theorem 1

Assume (1), (2) and (3), $u \in USC(\mathbb{R}^N)$ satisfies $G[u] \leq 0$ in $\mathbb{R}^N$,

(G)
$$\limsup_{|x|\to\infty} \frac{u(x)}{w(x)} \leq 0,$$

and either $u \geq 0$ or $c(x,\alpha) \equiv 0 \implies u$ is constant.

Rmk.: (G) is satisfied if u is bounded above.

Sketch of proof: assume $c(x,\alpha) \equiv 0$.

- Step 1: $\eta > 0$ $\quad u_\eta(x) := u(x) - \eta w(x)$, $\quad C_\eta := \max_{|x|\leq R_o} u_\eta(x)$.

$\implies \quad \lim_{|x|\to\infty} u_\eta(x) = -\infty \quad$ and $\exists\, M$ such that

$$u_\eta(x) \leq C_\eta \quad \forall\, |x| \geq M.$$

- Step 2: $G[w] \geq 0 \implies -L^\alpha w \geq 0 \quad \forall \alpha, |x| > R_o$, thus

$$G[u_\eta] = \inf_{\alpha \in A}\{-L^\alpha u + \eta L^\alpha w\} \leq \inf_{\alpha \in A}\{-L^\alpha u\} = G[u] \leq 0, |x| > R_o.$$

- Step 3: Comparison Principle in $\Omega = \{R_o < |x| < M\}$ with C_η:

$G[C_\eta] = 0 \Rightarrow \quad u_\eta(x) \leq C_\eta$ in $\Omega \quad \Rightarrow u_\eta(x) \leq C_\eta \; \forall |x| \geq R_o$.

As $\eta \to 0+ \implies \quad u(x) \leq \max_{|x| \leq R_o} u(x) \quad \forall x$

$\implies u$ attains its maximum over $\mathbb{R}^N$.

- Step 4: Strong Maximum Principle $\implies$ u is constant.

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HJB: explicit sufficient conditions

(a) $a = \sigma\sigma^T(x,\alpha)$ and

$\forall R > 0 \ \exists K_R \ : \quad \sup_{|x|\le R}(|\sigma| + |b| + |c|) \le K_R,$

$\sup_{|x|,|y|\le R, \alpha\in A}(|\sigma(x,\alpha) - \sigma(y,\alpha)| + |b(x,\alpha) - b(y,\alpha)|) \le K_R|x-y|;$

(b) $c(x,\alpha) \ge 0, \quad c$ continuous in x uniformly in $|x| \le R, \alpha \in A$;

(c) $\xi^T a(x,\alpha)\xi \ge |\xi|^2/K_R \quad \forall \xi \in \mathbb{R}^N, \ |x| \le R, \alpha \in A.$

(d) $\sup_{\alpha\in A}(tr\, a(x,\alpha) + b(x,\alpha)\cdot x - c(x,\alpha)|x|^2/2) \le 0 \quad$ for $|x| \ge R_o$.

(a), (b), (c) $\implies$ Comparison Principle on bounded sets and Strong Maximum Principle.

HJB: Liouville for subquadratic subsolutions

Corollary

Assume L^α satisfy (a), (b), (c), (d), $u \in USC(\mathbb{R}^N)$ satisfies $G[u] \leq 0$ and

$$\limsup_{|x|\to+\infty} \frac{u(x)}{|x|^2} \leq 0.$$

Assume either $u \geq 0$ or $c(x,\alpha) \equiv 0$, then u is a constant.

Proof.

Take the Lyapunov function $w(x) = |x|^2/2$. Since

$$L^\alpha w = tr\, a(x,\alpha) + b(x,\alpha)\cdot x$$

(d) implies $G[w] = \inf_{\alpha\in A}\{-L^\alpha w + c(x,\alpha)|x|^2/2\} \geq 0$ for $|x| \geq R_o$. □

Example

(d) is satisfied $\forall\, c \geq 0$ if

$$a = o(|x|^2) \quad \text{as } |x| \to \infty$$

and the drift b is a controlled perturbation of a mean reverting drift of Ornstein-Uhlenbeck type:

$$b(x,\alpha) = \gamma(m - x) + \tilde{b}(x,\alpha), \qquad \lim_{x\to\infty} \sup_{\alpha\in A} \frac{\tilde{b}(x,\alpha)\cdot x}{|x|^2} = 0$$

for some $m \in \mathbb{R}^N$, $\gamma > 0$.

Quasilinear hypoelliptic equations

Consider

$$\text{(Q)} \qquad -tr(\sigma\sigma^T(x)D^2u) + \inf_{\alpha\in A}\{-b(x,\alpha)\cdot Du + c(x,\alpha)u\} = 0, \quad \text{in } \mathbb{R}^N$$

with σ, b, c loc. Lipschitz in x. Assume

- $\forall\, R > 0$, either $\inf_{|x|\le R, \alpha\in A} c(x,\alpha) > 0$ or

  $\exists i : \inf_{|x|\le R} \inf_{j=1,..,m} \sigma_{ij}^2(x) > 0$, (non-degeneracy in one direction)

$\Longrightarrow$ Comparison Principle on bounded sets [M.B. - P. Mannucci 2006]

- the columns σ_j of σ are smooth and $rank\ Lie[\sigma_1, .., \sigma_m](x) = N, \forall x$

$\Longrightarrow$ Strong Maximum principle [M.B. - F. Da Lio 2003]

- $|\sigma(x)|^2 + \sup_{\alpha\in A}(b(x,\alpha)\cdot x - c(x,\alpha)|x|^2/2) \le 0, \quad |x| \ge R_o$

which implies $w(x) = |x|^2/2$ is a Lyapunov function.

Liouville for subquadratic subsolutions of (Q)

Corollary

Under the assumptions on (Q), let $u \in USC(\mathbb{R}^N)$ be a subsolution to (Q) ,

$$\limsup_{|x| \to +\infty} \frac{u(x)}{|x|^2} \leq 0,$$

and either $u \geq 0$ or $c(x, \alpha) \equiv 0$,

Then u is a constant.

Uniformly elliptic operators

Pucci extremal operators, for $0 < \lambda \leq \Lambda$:

$$\mathcal{M}^-(X) := \inf\{-\mathrm{tr}(MX) : M \in S^N, \lambda I \leq M \leq \Lambda I\}$$
$$= -\Lambda \sum_{e_i>0} e_i - \lambda \sum_{e_i<0} e_i$$

$$\mathcal{M}^+(X) \quad := \quad \sup\{...\text{same}...\} \quad = \quad -\lambda \sum_{e_i>0} e_i \; - \; \Lambda \sum_{e_i<0} e_i.$$

F is uniformly elliptic if there exist constants $0 < \lambda \leq \Lambda$ such that

$$\lambda tr(Q) \leq F(x,t,p,X) - F(x,t,p,X+Q) \leq \Lambda tr(Q),$$

for all $x, p \in \mathbb{R}^N$, $t \in \mathbb{R}$, $X, Q \in S^N$, $Q \geq 0$, or equivalently

$$\mathcal{M}^-(X) \leq F(x,t,p,X) - F(x,t,p,0) \leq \mathcal{M}^+(X).$$

Liouville for Pucci + H

(P+H) $\mathcal{M}^-(D^2u) + \inf_{\alpha\in A}\{c(x,\alpha)u - b(x,\alpha)\cdot Du\} = 0 \quad \text{in } \mathbb{R}^N$

Previous sufficient condition ($w = |x|^2/2$):

$$\sup_{\alpha\in A}(b(x,\alpha)\cdot x - c(x,\alpha)|x|^2/2) \le -N\Lambda \quad \text{for } |x| \ge R_o.$$

We improve it to

(P) $\sup_{\alpha\in A}(b(x,\alpha)\cdot x - c(x,\alpha)|x|^2\log(|x|)) \le \lambda - (N-1)\Lambda$ for $|x| \ge R_o$.

(P) , u subsolution to (P+H) , $\limsup_{|x|\to+\infty} \frac{u(x)}{\log|x|} \le 0$, either $u \ge 0$ or $c(x,\alpha) \equiv 0 \implies u$ is a constant

Proof: $w(x) = \log|x| \implies \mathcal{M}^-(D^2w) = (\lambda - (N-1)\Lambda)/|x|^2$.

Liouville for uniformly elliptic operators

Corollary

F uniformly elliptic, $\quad F(x,t,p,0) \geq \inf_{\alpha \in A}\{c(x,\alpha)t - b(x,\alpha)\cdot p\}$,

b, c satisfy (P) , u subsolution to (E) , $\quad \limsup_{|x|\to+\infty} \frac{u(x)}{\log|x|} \leq 0,$

either $u \geq 0$ or $c(x,\alpha) \equiv 0 \quad \Longrightarrow \quad u$ is a constant.

Proof.

u satisfies

$$\mathcal{M}^-(D^2u) + \inf_{\alpha\in A}\{c(x,\alpha)u - b(x,\alpha)\cdot Du\} \leq F(x,u,Du,D^2u) \leq 0 \quad \text{in } \mathbb{R}^N,$$

so we apply the result for (P+H). □

Remarks

- Symmetric results hold for supersolutions $v \in LSC(\mathbb{R}^N)$ of (E) such that

$$\liminf_{|x|\to+\infty} \frac{v(x)}{\log|x|} \geq 0$$

if F is uniformly elliptic and

$$F(x,t,p,0) \leq \sup_{\alpha\in A}\{c(x,\alpha)t - b(x,\alpha)\cdot p\}.$$

- Example: Liouville holds for sub- and supersolutions of (E) if $F = F(x,p,X)$ is "almost 1-homogeneous" in p

(Fh) $\quad -b_1(x)\cdot p - g_1(x)|p| \leq F(x,p,0) \leq -b_2(x)\cdot p + g_2(x)|p|$

with $g_i \geq 0$ bounded and locally Lipschitz, $i = 1, 2$, if

(P') $\quad b_i(x)\cdot x + g_i(x)|x| \leq \lambda - (N-1)\Lambda \quad$ for $|x| \geq R_o, i = 1, 2.$

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$$\text{(P')} \quad b_i(x)\cdot x + g_i(x)|x| \leq \lambda - (N-1)\Lambda \quad \text{for } |x| \geq R_o,\ i = 1,2.$$

Application 1: parabolic equations

(CP) $$\begin{cases} u_t + F(x, Du, D^2u) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \\ u(0, x) = h(x) & \text{in } \mathbb{R}^n, \end{cases}$$

with F uniformly elliptic, satisfying (Fh) , (P') , and the standard assumptions for the Comparison Principle ("Lipschitz in x") ,

$h \in BUC(\mathbb{R}^N)$.

Proposition

There exist a unique solution u bounded and Hölder continuous in $[0, +\infty) \times \mathbb{R}^N$.

Proof based on viscosity methods, e.g. Perron, and Krylov-Safonov estimates.

Large-time stabilization (in space)

Theorem

There exist constants $\overline{u}, \underline{u} \in \mathbb{R}$ such that

$$\limsup_{t\to+\infty} u(t,x) = \overline{u}, \quad \liminf_{t\to+\infty} u(t,x) = \underline{u}, \quad \forall\, x \in \mathbb{R}^N.$$

Idea of proof: the relaxed semi-limits

$$\limsup_{t\to+\infty, y\to x} u(t,y), \qquad \liminf_{t\to+\infty, y\to x} u(t,y)$$

are sub- and supersolutions of (E).

- In some cases (e.g., F, h periodic in x) $\overline{u} = \underline{u}$ and $\lim_{t\to+\infty} u(t,x) =$constant is locally uniform [Alvarez - M.B. 2010]
- In general, for some (bounded) initial data h, $\overline{u} > \underline{u}$, even for $u_t = u_{xx}$ in dim. $N = 1$. [Eidelman-Kamin-Tedeev 2009, Collet-Eckmann 1992]

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Application 2: ergodic HJB equations

(EE) $$\inf_{\alpha \in A} \{-tr\, a(x,\alpha) D^2 \chi - b(x,\alpha) \cdot D\chi - l(x,\alpha)\} = c \quad x \in \mathbb{R}^N$$

Theorem

Assume $\forall\, M > 0\ \exists\, R > 0$ such that

(L) $$\sup_{a \in A} \{tr\, a(x,\alpha) + b(x,\alpha) \cdot x\} \leq -M \quad \text{for } |x| \geq R.$$

Then exists a unique constant $c \in \mathbb{R}$ for which (EE) has a solution χ such that

(G) $$\lim_{|x| \to +\infty} \frac{\chi(x)}{|x|^2} = 0.$$

Moreover $\chi \in C^2(\mathbb{R}^N)$ and is unique up to additive constants.

Comments

- The "critical value problem" (EE) arises in
  - ergodic stocastic control [P.L.Lions,..., Ichihara, Borkar, Cirant, ...]
  - homogenization [P.L.Lions-Papanicolaou-Varadhan, Evans,...]
  - singular perturbations [O. Alvarez - M.B., C. Marchi, ...],
  - weak KAM theory [Fathi,....],
  - Mean-Field Games [Lasry-Lions, M.B., ...]
- Condition (L) is stronger than the previous ones and implies that the Lyapunov function $w(x) = |x|^2/2$ satifies

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Steps of the proof

- Discounted HJB equation: for $\delta > 0$

$$\delta u_\delta + F(x, Du_\delta, D^2 u_\delta) = 0 \qquad \text{in } \mathbb{R}^N,$$

has a unique solution s.t. $\|u_\delta\|_\infty \le \|l\|_\infty/\delta$ and $\forall\, h \in (0,1]\ \exists\, R_h$:

$$(\mathrm{G}\delta) \qquad -h\frac{|x|^2}{2} + \min_{|x|\le R_h} u_\delta \le u_\delta(x) \le \max_{|x|\le R_h} u_\delta + h\frac{|x|^2}{2},$$

- $v_\delta = u_\delta - u_\delta(0)$ are uniformly bounded and Hölder continuous on compact sets (by Krylov-Safonov)

- $\delta u_\delta(0) \to -c$ and $v_\delta \to \chi$ loc. uniformly as $\delta \to 0$ and $(\mathrm{G}\delta)$ implies the sub-quadratic growth (G) of χ.

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- Uniqueness of c and of χ up to additive constants.
  Assume there exist two solutions of (EE) (c_1, χ_1), (c_2, χ_2) and suppose $c_1 \leq c_2$. Use Theorem 1:

$$
\begin{aligned}
& G[\chi_1 - \chi_2] \\
= & \inf_{\alpha \in A} \{ -tr\, a(x,\alpha)(D^2\chi_1 - D^2\chi_2) - b(x,\alpha) \cdot (D\chi_1 - D\chi_2) \} \\
\leq & \inf_{\alpha \in A} \{ -tr\, a(x,\alpha) D^2\chi_1 - b(x,\alpha) \cdot D\chi_1 - l(x,\alpha) \} \\
+ & \sup_{\alpha \in A} \{ tr\, a(x,\alpha) D^2\chi_2 + b(x,\alpha) \cdot D\chi_2 + l(x,\alpha) \} \\
= & c_1 - c_2 \leq 0
\end{aligned}
$$

The first Corollary of Thm. 1 $\implies$ $\chi_1 - \chi_2 = \text{constant}$

$\implies$ $c_1 = c_2$.

- Uniqueness of c and of χ up to additive constants.
  Assume there exist two solutions of (EE) (c_1, χ_1), (c_2, χ_2) and suppose $c_1 \leq c_2$. Use Theorem 1:

$$
\begin{aligned}
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= \;& \inf_{\alpha \in A} \{ -tr\, a(x,\alpha)(D^2\chi_1 - D^2\chi_2) - b(x,\alpha) \cdot (D\chi_1 - D\chi_2) \} \\
\leq \;& \inf_{\alpha \in A} \{ -tr\, a(x,\alpha) D^2\chi_1 - b(x,\alpha) \cdot D\chi_1 - l(x,\alpha) \} \\
+ \;& \sup_{\alpha \in A} \{ tr\, a(x,\alpha) D^2\chi_2 + b(x,\alpha) \cdot D\chi_2 + l(x,\alpha) \} \\
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Some references

- M.B., A. Cesaroni: Liouville property and critical value... J. D. E. to appear

Related papers:

- M.B., A. Cesaroni, L. Manca: Convergence by viscosity methods in multiscale financial models..., Siam J. Financial Math. (2010)
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- D. Castorina, A. Cesaroni and L. Rossi: A parabolic HJB equation degenerating at the boundary, Comm. Pure Appl. Anal. 2016
- A. Arapostathis, V.S. Borkar, M.K. Ghosh: Ergodic control of diffusion processes. Cambridge University Press 2012.

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