Convergence of some deterministic Mean Field Games to aggregation and flocking models

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joint work with Pierre Cardaliaguet

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Plan

1. Connecting MFGs to kinetic models ?
   - Mean Field Games and their system of PDEs
   - Agent-based models
   - Large interest rate limit for stochastic MFG: Bertucci-Lasry-Lions
   - The setting of Degond-Herty-Liu
2. Convergence of MFGs to nonlocal continuity equations
   - a) MFG with controlled velocity $\rightarrow$ aggregation equation
   - b) MFG with controlled acceleration $\rightarrow$ kinetic equations of flocking type
   - Outline of the proof for controlled velocity
   - Outline of the proof for controlled acceleration

(MFE)
$$\begin{cases} -\frac{\partial u}{\partial t} - \nu \Delta u + H(\nabla u) = F(x, m) \quad \text{in } (0, T) \times \mathbf{R}^d \\ \frac{\partial m}{\partial t} - \nu \Delta m - div(m \nabla H(\nabla u)) = 0 \quad \text{in } (0, T) \times \mathbf{R}^d \\ u(T, x) = g(x), \quad m(0, x) = m_o(x), \end{cases}$$

$m(x, t) =$ equilibrium distribution of the agents at time t;

$u(x, t)$ = value function of the representative agent

Data: $\nu \geq 0$, $H = L^*$, e.g., $H(p) = \frac{|p|^2}{2}$,
$F : \mathbf{R}^d \times \mathcal{P}_1(\mathbf{R}^d) \to \mathbf{R}$ = running cost , g = terminal cost,
$m_o \geq 0$ = initial distribution of the agents, $\int_{\mathbf{R}^d} m_o(x) dx = 1$.

1st equation is backward H-J-B, 2nd equation is forward K-F-P eq.

Control interpretation of the MFE

- $$u(x,t) = \inf E[\int_t^T L(\alpha(s)) + F(y(s), m(s))ds + g(y(T))]$$

  over controls α and trajectories of

  $$dy(s) = \alpha(s)ds + \sqrt{2\nu}dW(s), \quad y(t) = x$$

- $$dy(s) = -\nabla H(\nabla u(y(s), s))ds + \sqrt{2\nu}dW(s)$$

  = optimal trajectory of the representative agent

- $m(x,t) =$ distribution of particles moving along optimal trajectories
- In particular, for $\nu = 0$ and $H(p) = |p|^2/2$ the dynamics with optimal feedback is $\dot{y}(s) = -\nabla u(y(s), s)$

  and the KFP equation becomes

  $$\frac{\partial m}{\partial t} - div(m\nabla u) = 0$$

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Agent-based models

They typically are nonlocal continuity equations of the form

$$\partial_t m - div(m\, Q[m]) = 0 \qquad Q : \mathcal{P}_p(\mathbf{R}^d) \to C^1(\mathbf{R}^d, \mathbf{R}^d)$$

- The aggregation equation (Bertozzi, Carrillo, Laurent and many others):

$$Q[m](x,t) = \nabla \int_{\mathbf{R}^d} k(x-y)dm(y)$$

  - $$k(x) = -|x|e^{-a|x|}, \qquad a > 0,$$
  - $$k(x) = e^{-|x|} - Fe^{-|x|/L}, \qquad 0 < F < 1, \quad L > 1$$

- Nonlinear friction equation of granular flows (Toscani et al.): same form with

$$k(x) = |x|^\alpha/\alpha, \qquad \alpha > 0$$

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- Models of crowd dynamics (Cristiani-Piccoli-Tosin)
  - $\partial_t m - div(m(v + Q[m])) = 0, \; v = v(x), \; Q[m] = \nabla \int_{\mathbf{R}^d} k(x-y) dm(y)$

    $k = \phi(|x|)$ with compact support, ϕ decreasing for small $|x|$, then increasing
  - models with "social forces", or mesoscopic, or kinetic: state variables: position and velocity $(x, v) \in \mathbf{R}^{2d}$

    $$\partial_t m + v \cdot D_x m - div_v(mQ[m]) = 0 \qquad \text{in } (0,T) \times \mathbf{R}^{2d}$$

    $$Q[m](x,v) = \nabla_v \int_{\mathbf{R}^{2d}} k(x-y, v-v_*) m(y, v_*, t) dy dv_*$$

- Flocking models: as the last one with different k, e.g.

  Cucker-Smale : $k(x,v) = \frac{|v|^2}{(\alpha + |x|^2)^\beta}, \quad \alpha > 0, \beta \geq 0$

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Question: connection among MFGs and ABMs?

For MFG with dynamics $\dot{y} = \alpha$ the equation for the density *m* is

$$\frac{\partial m}{\partial t} - div(m \nabla H(\nabla u)) = 0$$

which is a continuity equation with $Q[m] = \nabla H(\nabla u)$ and *u* depends on *m* in a non-local way via the HJB equation, so the dependence is not explicit.

For MFG with dynamics $\ddot{y} = \alpha$ the density *m* solves

$$\partial_t m + v \cdot D_x m - div_v(m \nabla_v H(\nabla u)) = 0$$

which is a kinetic equation with $Q[m] = \nabla_v H(\nabla u)$ and *u* depends on *m* via the HJB equation.

Q.: can one connect in a rigorous way the classical ABMs to some MFGs ?

Stochastic case: Bertucci-Lasry-Lions 2018

$$
(MF0) \quad \begin{cases} -\partial_t u_\lambda + \lambda u_\lambda - \Delta u_\lambda + Q[m_\lambda] \cdot Du_\lambda + \dfrac{|Du_\lambda|^2}{2} = F(x), \\ \partial_t m_\lambda - \Delta m_\lambda - div(m_\lambda(Du_\lambda + Q[m_\lambda])) = 0 \quad \text{in } \mathbf{R}^d \times \mathbf{R}_+ \\ m_\lambda(0) = m_0, \qquad \text{in } \mathbf{R}^d \end{cases}
$$

$\lambda =$ the discount factor in the cost functional associated to the HJB equation =
"inter temporal preference parameter that measures the weight of anticipation for a given agent",

the dynamics of an agent in the MFG is

$$
dy(s) = (\alpha(s) - Q[m_\lambda])ds + \sqrt{2\nu}dW(s), \quad y(t) = x
$$

The limit $\lambda \to \infty$

Theorem (Bertucci-Lasry-Lions)

$Q : \mathcal{P}_1(\mathbf{R}^d) \to Lip(\mathbf{R}^d), \quad \|Q[m]\|_\infty \le C, \ \forall m \quad \Longrightarrow$

any solution (u_λ, m_λ) *of (MF0) is bounded uniformly in* λ *and*

for any $\lambda_n \to \infty$ *such that* $m_{\lambda_n} \to m$ *the limit m is a solution of the continuity equation*

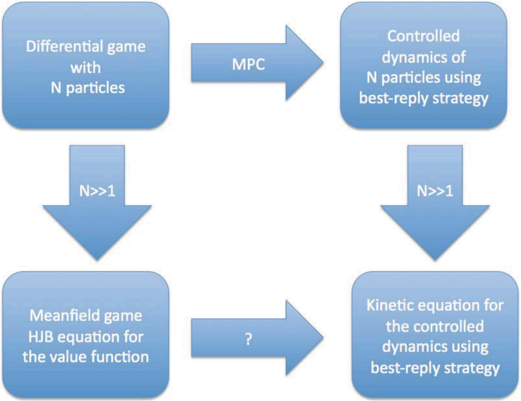
$$\partial_t m - \Delta m - div(m\, Q[m]) = 0.$$

So "any" ABM model (with diffusion), defined by Q, has at least one solution that is the limit of the solution of a MFG.

The setting of Degond-Herty-Liu 2017

Here MPC = Model Predictive Control

We address the horizontal ? $\Longrightarrow$? with a different approach.



The control problem for a single agent is

$$\dot{y} = v(y) + \alpha, \quad y(t) = x, \quad \inf_{\alpha(\cdot)} \int_t^T [\frac{|\alpha|^2}{2} + F(y(s), m(s))]ds$$

MPC approximation:

$$y(t+\Delta t) = x + \Delta t(v(x) + \alpha), \quad \min_{\alpha} \left[\Delta t \frac{|\alpha|^2}{2} + F(y(t+\Delta t), m(t)) \right]$$

Note that the scaling with Δt means that the control is cheap.
Taking the derivative w.r.t. α we get the optimal control $\bar{\alpha}$ if

$$\Delta t\, [\bar{\alpha} + DF(x, m(t))] = 0.$$

This suggests that, for short horizon T and cheap control, the optimal feedback should be approximated by the

steepest decent of the running cost $\bar{\alpha} \approx -DF(x, m(t))$.

2. Convergence: a) the basic model

(MF1) $$\begin{cases} -\partial_t u_\lambda + \lambda u_\lambda - v(x)\cdot Du_\lambda + \frac{\lambda}{2}|Du_\lambda|^2 = F(x, m_\lambda(t)) \\ \partial_t m_\lambda - div(m_\lambda(\lambda Du_\lambda - v(x))) = 0 \qquad \text{in } \mathbf{R}^d \times \mathbf{R}_+ \\ m_\lambda(0) = m_0, \qquad \text{in } \mathbf{R}^d \qquad u_\lambda \text{ bounded.} \end{cases}$$

$\lambda > 0, \quad v \in W^{2,\infty}$,

$m_0 \in \mathcal{P}_1(\mathbf{R}^d)$ has bounded density and compact support

$F : \mathbf{R}^d \times \mathcal{P}_1(\mathbf{R}^d) \to \mathbf{R}$ continuous and

$\|F(\cdot, m)\|_{\mathcal{C}^2} \le C \ \forall m \in \mathcal{P}_1(\mathbf{R}^d), \quad \|DF(\cdot, m) - DF(\cdot, \bar{m})\|_\infty \le C\mathbf{d}_1(m, \bar{m})$

Note: 1. no terminal condition for the HJB equation.

2. $H(p) = \lambda|p|^2/2 \implies DH(p) = \lambda p$

Existence and representation of (u_λ, m_λ)

Theorem (see Cardaliaguet's Lect. Notes on Lions' lectures)

(MF1) has a solution (viscosity sense for HJB, distribution sense for KFP). Any solution satisfies

$$u_\lambda(x,t) = \inf \int_t^{+\infty} e^{-\lambda(s-t)} \left[\frac{1}{2\lambda} |\alpha(s)|^2 + F(y(s), m_\lambda(s)) \right] ds,$$

for $\quad \dot{y}(s) = v(y(s)) + \alpha(s), \; s > t, \quad y(t) = x.$

Rmks.:

1. Meaning of λ large: high discount factor (the near future counts much more than the far future) and cheap control.

2. In general the solution of (MF1) is NOT UNIQUE, the monotonicity condition on F of Lasry and Lions in NOT satisfied for F modelling aggregation.

Convergence Theorem for (MF1)

Under the previous assumptions, as $\lambda \to \infty$, any solution of (MF1) satisfies

$$m_\lambda \to m \quad \text{in } C([0,T],\mathcal{P}_1) \text{ and weak}^* \text{ in } L^\infty([0,T]\times \mathbf{R}^d)\ \forall\, T>0,$$

m the unique solution (distribution sense) of the continuity equation

$$\begin{cases} \partial_t m - div(m(DF(x,m) - v(x))) = 0 & \text{in } \mathbf{R}^d \times \mathbf{R}_+ \\ m(0) = m_0, \qquad \text{in } \mathbf{R}^d. \end{cases}$$

$\lambda u_\lambda(x,t) \to F(x,m(t))$ loc. uniformly, $\lambda D u_\lambda(x,t) \to DF(x,m(t))$ a.e..

Remarks.

1. Thm. says that the optimal feedback $-Du_\lambda$ is close to the scaled gradient descent of the running cost $-\frac{1}{\lambda}DF$, which is perhaps new even for pure control problems with frozen m.

Remarks continued

2. $F(x, m) = k * m(x)$ with $k \in C^2 \cap W^{2,\infty}$

fits the assumptions of the Theorem.

3. The MFG system (MF1) has many solutions in general, and ALL of them converge to the limit continuity equation that has a unique solution (by, e.g, Piccoli-Rossi 2013).

Applications and extensions

- The aggregation equation and model 1 of crowd dynamics are

$$\partial_t m - div(m(D\,k * m - v(x))) = 0$$

but $k = \phi(|x|)$ NOT C^1 in $x = 0$, but semiconcave if there is repulsion at short distance.

We can replace the condition $\|D^2 F(x,m)\|_\infty \le C$ for all m with the semiconcavity of F uniformly in m

$$F(x+h,m) - 2F(x,m) + F(x-h,m) \le C|h|^2 \quad \forall m \in \mathcal{P}_1.$$

- In the nonlinear friction equation $k \notin L^\infty$, so $\|F(x,m)\|_\infty \le C$ for all m is false.

We have extensions to the case

$$-C_o \le F(x,m) \le C_o(1+|x|^2) \qquad \text{for all } m.$$

N.B. Even existence of solutions to (MF1) is new in this case.

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N.B. Even existence of solutions to (MF1) is new in this case.

2. b) Convergence for controlled acceleration

(MF2)
$$\begin{cases} -\partial_t u_\lambda + \lambda u_\lambda - v \cdot D_x u_\lambda + \frac{\lambda}{2}|D_v u_\lambda|^2 = F(x, v, m_\lambda(t)) \\ \partial_t m_\lambda + v \cdot D_x m_\lambda - div_v(m_\lambda \lambda D_v u_\lambda) = 0 \quad \text{in } \mathbf{R}^{2d} \times \mathbf{R}_+ \\ m_\lambda(0) = m_0, \qquad \text{in } \mathbf{R}^{2d}. \end{cases}$$

$\lambda > 0$, $m_0 \in \mathcal{M}$, i.e., $m_0 \in \mathcal{P}_1(\mathbf{R}^{2d})$ with bounded density and compact support

$F : \mathbf{R}^d \times \mathcal{M} \to \mathbf{R}$ continuous and $\forall x, v \in \mathbf{R}^d$, $m \in \mathcal{M}$,

$$-C_o \le F(x, v, m) \le C_o(1 + |v|^2),$$

$$|D_x F(x, v, m(x, v)| \le C, \quad |D_v F(x, v, m(x, v))| \le C(1 + |v|)$$

$$|D^2 F(x, v, m)| \le C$$

Representation of u_λ

Any solution (u_λ, m_λ) satisfies

$$u_\lambda(x,v,t) = \inf \int_t^{+\infty} e^{-\lambda(s-t)} \left[\frac{1}{2\lambda} |\alpha(s)|^2 + F(y(s), v(s), m_\lambda(s)) \right] ds,$$

$$\dot{y}(s) = v(s), \quad \dot{v}(s) = \alpha(s), \ s > t, \quad y(t) = x, \ v(t) = v.$$

There is no existence theory for (MF2) available in the literature:

we prove directly the existence of a solution satisfying the estimates we need, and show convergence for it.

See also ongoing work by Achdou-Mannucci-Marchi-Tchou.

Convergence Theorem for (MF2)

Under the previous assumptions there is a solution to (MF2) such that

$$m_\lambda \to m \quad \text{in } C([0,T],\mathcal{P}_1) \text{ and weak}^* \text{ in } L^\infty([0,T]\times\mathbf{R}^{2d})\ \forall\, T>0,$$

as $\lambda \to \infty$, $m =$ unique solution of the continuity equation

$$\begin{cases} \partial_t m + v\cdot D_x m - div(m\, D_v F(x,v,m)) = 0 & \text{in } \mathbf{R}^{2d}\times\mathbf{R}_+ \\ m(0) = m_0, & \text{in } \mathbf{R}^{2d}, \end{cases}$$

$$\lambda u_\lambda(x,v,t) \to F(x,v,m(t)) \text{ loc. uniformly,}$$

$$\lambda D_v u_\lambda(x,v,t) \to D_v F(x,v,m(t)) \text{ a.e..}$$

1. Thm. says that the optimal feedback $-D_v u_\lambda$ is close to the scaled gradient descent of the running cost $-\frac{1}{\lambda} D_v F$, which is perhaps new even for pure control problems with frozen m.

Examples

1. $F(x, v, m) = k * m(x, v)$ with $k \in C^2 \cap L^\infty$.

2. Same with Cucker-Smale kernel :

$$k(x, v) = \frac{|v|^2}{(\alpha + |x|^2)^\beta}, \quad \alpha > 0,\ \beta \geq 0.$$

3. in crowd dynamics with social forces $k = \phi(|x|)$ may be not C^1 in $x = 0$, but it is semiconcave if there is repulsion at short distance: we can extend the result to this case.

Outline fo the proof for (MF1): estimates for HJB

Step 1: Convergence in HJB

$$-\partial_t u_\lambda + \lambda u_\lambda - v(x) \cdot Du_\lambda + \frac{\lambda}{2}|Du_\lambda|^2 = F(x, m_\lambda(t))$$

For a suitable $C > 0$, $\frac{F(x,m_\lambda(t))}{\lambda} - \frac{C}{\lambda^2}$ is a subsolution and $\frac{F(x,m_\lambda(t))}{\lambda} + \frac{C}{\lambda^2}$ is a supersolution, so

$$\sup_{\mathbf{R}^d \times \mathbf{R}_+} |\lambda u_\lambda - F(\cdot, m_\lambda)| \leq \frac{C}{\lambda}, \quad \forall \lambda \geq 1.$$

Step 2: Semiconcavity estimate for u_λ:

$$u_\lambda(x+h,t) - 2u_\lambda(x,t) + u_\lambda(x-h,t) \leq \frac{C}{\lambda}|h|^2,$$

$\Longrightarrow$ $\lambda Du_\lambda(x,t) \to DF(x,m)$ a.e. if $F(x,m_\lambda) \to F(x,m)$ loc. unif.

Step 3: Lipschitz estimate for u_λ:

$$|u_\lambda(x,t) - u_\lambda(x+h,t)| \le \frac{C|h|}{\lambda}$$

Step 4: can define a flow $\Phi(x,t,s)$ such that $m_\lambda(s) = \Phi(\cdot, 0, s)\# m_0$ and

$$|\Phi(x,0,s) - \Phi(x,0,s')| \le \|\lambda D u_\lambda\|_\infty |s-s'| \le C|s-s'|$$

Step 5: Estimates on the KFP equation:

- $\mathbf{d}_1(m_\lambda(s), m_\lambda(s')) \le \int_{\mathbf{R}^d} |\Phi(x,0,s) - \Phi(x,0,s')| dm_0(x) \le C|s-s'|$

  where $\mathbf{d}_1$ is the Kantorovich-Rubinstein or 1-Wasserstein distance on $\mathcal{P}_1(\mathbf{R}^d)$.

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- $$\|m_\lambda(t)\|_\infty \le C_T \|m_0\|_\infty$$
- $$\int_{\mathbf{R}^d} |x|^2 m_\lambda(t) dx \le C(M_2(m_0) + T).$$

$\implies$ compactness of $\{m_\lambda\}$ in $C([0,T], \mathcal{P}_1)$

and weak* in $L^\infty(\mathbf{R}^d \times [0,T])$, $\forall T > 0$.

For any sequence λ_n such that $m_{\lambda_n} \to m$ as above

$F(x, m_{\lambda_n}(t)) \to F(x, m(t))$ loc. uniformly by the continuity of F in $\mathcal{P}_1$.

$\implies$ $\lambda_n Du_{\lambda_n}(x,t) \to DF(x, m(t))$ a.e.

$\implies$ m solves $\partial_t m - div(m(DF(x,m) - v(x))) = 0.$

DF Lip in $m \implies$ uniqueness for this equation, so $m_\lambda \to m$,

and then also $\lambda u_\lambda \to F(\cdot, m)$ and $\lambda Du_\lambda \to DF(\cdot, m)$. QED

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Outline fo the proof for (MF2)

We use the vanishing viscosity approximation to (MF2) to build a solution satisfying all the estimates we need.

Estimates for HJB:

- L^∞ estimates $\quad -\frac{C}{\lambda} \leq u_\lambda(x, v, t) \leq \frac{C}{\lambda}(1 + |v|^2)$
- Lipschitz estimates $\quad |u_\lambda(x, v, t) - u_\lambda(x + h, v, t)| \leq C\frac{|h|}{\lambda},$

$$|u_\lambda(x, v, t) - u_\lambda(x, v + h, t)| \leq \frac{|h|}{\lambda}C(1 + |v|)$$

- Semiconcavity estimate for (MF2)

$$u_\lambda(x+h, v+h_1, t) - 2u_\lambda(x, t) + u_\lambda(x-h, v-h_1, t) \leq \frac{C}{\lambda}(|h|^2 + |h_1|^2).$$

Estimates for KFP:

- $\mathbf{d}_1(m_\lambda(s), m_\lambda(s')) \leq C_T(1 + M_1(m_0))|s - s'|$

  where $\mathbf{d}_1$ is the Kantorovich-Rubinstein on $\mathcal{P}_1(\mathbf{R}^{2d})$.

- $\|m_\lambda(t)\|_\infty \leq C_T \|m_0\|_\infty$

- $\int_{\mathbf{R}^d} (|x|^2 + |v|^2) m_\lambda(t) dx \leq C(M_2(m_0) + T).$

After these estimates the proof is the same as for (MF1).

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Thanks for your attention !