Mean-Field Games in Padova

Martino Bardi

Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova

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Plan

- Part I: A brief introduction to MFGs
  - Motivations and heuristic derivation of MFG systems
  - Some basic results and a list of applications.
- Part II: some contributions by the group in Padova
  - 1. Non-uniqueness and Uniqueness [M.B., Fischer] :
  - 2. Robust and risk-sensitive MFGs [M.B., Cirant]
  - 3. Systems driven by jump processes and fractional MFGs [M. Cirant and A. Goffi]
  - 4. Problems with concentration phenomena. [M. Cirant and A. Cesaroni]
  - 5. Time-periodic solutions [M. Cirant et al.]
  - 6. Deterministic MFGs [C. Marchi, P. Mannucci et al.]

Multiagent systems

Consider a population of N agents, whose state is driven by

$$dX_s^i = \alpha_s^i ds + \sigma^i dW_s^i, \ s > t, \quad X_t^i = x^i \in \mathbf{R}^d, \quad i = 1, \dots, N$$

W_s^i independent Brownian motions, $\alpha_s^i =$ control of i-th player, and a finite horizon cost functional of the i-th player:

$$J_T^i(t, x^1, .., x^N, \alpha^1, .., \alpha^N) := \mathbf{E}\left[\int_t^T L^i(\alpha_s^i) + F^i\left(X_s^i, \frac{\sum_{k\neq i}\delta_{X_s^k}}{N-1}\right) ds\right],$$

- L^i is the running cost of using the control α_s^i,
- $F^i : \mathbf{R}^d \times \{$ prob. measures$\} \to \{$ Lip functions $\}, \delta_x$ is the Dirac mass at x

N.B.: J_T^i depends on the players $k \neq i$ only via their empirical measure $\frac{1}{N-1}\sum_{k\neq i}\delta_{X_s^k}$

Nash equilibrium feedbacks for N-person games

N. E. are N-tuple of individual feedbacks $(\overline{\alpha}^1, .., \overline{\alpha}^N)$ such that

$$J_T^i(\overline{\alpha}_1, \dots, \overline{\alpha}_N) \leq J_T^i(\overline{\alpha}_1, .., \overline{\alpha}_{i-1}, \alpha_i, \overline{\alpha}_{i+1}, .., \overline{\alpha}_N) \quad \forall\, \alpha_i \; \forall i.$$

In principle such equilibria can be synthesised by solving a system of N parabolic HJB PDEs in Nd dimensions for the value functions v_i, $i = 1, ..., N$, nonlinear and strongly coupled (e.g., Bensoussan and Frehse books and papers, '80s - now)

This is highly unfeasible in practice, especially if d or N are large (curse of dimensionality already for $N = 1$.....).

Moreover, Nash equilibria are highly non-unique and unstable.

Problem: if the players can be assumed to have the same parameters (e.g., agents in the stock market, families consuming energy...), can we give a macroscopic description of the whole population simpler to handle?
Similar ideas: kinetic models in gas dynamics, mean-field theories in Quantum Physics...

Heuristic derivation of the MFG equations

Assume identical players:

$$\sigma^i = \sigma,\ L^i = L,\ F^i = F, \quad i = 1, ..., N.$$

The dynamics of a generic agent is

$$dX_s = \alpha_s\, ds + \sigma\, dW_s, \quad X_t = x \in \mathbf{R}^d$$

with W_s a Brownian motion, $\alpha_s =$ control, $\sigma > 0$ volatility,
and the his cost functional is

$$J_T(t, x, \alpha.) := \mathbf{E}\left[\int_t^T L(\alpha_s) + F(X_s, m_{env}(s))ds\right] + G(X_T, m_{env}(T)),$$

where $m_{env}(s)$ is the distribution of the other agents in the environment, assumed given, g is the terminal cost.

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Define the value function

$$v(t,x) := \inf_{\alpha.} J_T(t,x,\alpha.).$$

Then $v(t,x)$ solves the Hamilton-Jacobi-Bellman equation

$$\begin{cases} -\frac{\partial v}{\partial t} - \nu \Delta v + H(\nabla v) = F(x, m_{env}) & \text{in } (0,T)\times \mathbf{R}^d \\ v(T,x) = G(x, m_{env}) \end{cases}$$

where $\nu := \sigma^2/2$, $\Delta := \Delta_x$, $\nabla := \nabla_x$, and H is the Hamiltonian associated to L:

$$H(p) := \sup_{\alpha\in\mathbf{R}^d} \{p\cdot\alpha - L(\alpha)\}$$

Moreover the feedback control

$$\hat{\alpha}(t,x) = -\nabla H(\nabla v(t,x))$$

is optimal.

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The optimal process

$$d\hat{X}_t = -\nabla H(\nabla v(\hat{X}_t))dt + \sigma dW_t$$

has a distribution whose density m solves the

Kolmogorov-Fokker-Plank equation

$$\begin{cases} \frac{\partial m}{\partial t} - \nu\Delta m - div(m\nabla H(\nabla v)) = 0 & \text{in } (0,T)\times \mathbf{R}^d \\ m(0,x) = m_o(x) \end{cases}$$

where $m_o \geq 0, \ \int_{\mathbf{R}^d} m_o(x)dx = 1,$

is the distribution of the initial position of the system.

The PDEs for value and density of the optimal process are

$$\begin{cases} -\frac{\partial v}{\partial t} - \nu \Delta v + H(\nabla v) = F(x, m_{env}) & \text{in } (0,T) \times \mathbf{R}^d \\ \frac{\partial m}{\partial t} - \nu \Delta m - div(m \nabla H(\nabla v)) = 0 & \text{in } (0,T) \times \mathbf{R}^d \\ v(T,x) = G(x, m_{env}), \quad m(0,x) = m_o(x), \end{cases}$$

and $m_{env} \mapsto v$, $v \mapsto m$ are well-defined maps.

If $m_{env} \mapsto v \mapsto m$ has a fixed point , i.e. $m = m_{env}$,

then m is an equilibrium distribution of the agents,

each behaving optimally as long as the population distribution remains the same.

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Mean Field Games PDEs

We have heuristically derived the basic system of 2 evolutive PDEs of MFGs

$$(MFE) \qquad \begin{cases} -\frac{\partial v}{\partial t} - \nu\Delta v + H(\nabla v) = F(x, m(t)) \quad \text{in } (0,T)\times \mathbf{R}^d \\ \\ \frac{\partial m}{\partial t} - \nu\Delta m - div(m\nabla H(\nabla v)) = 0 \quad \text{in } (0,T)\times \mathbf{R}^d \\ \\ v(T,x) = G(x, m(T)), \quad m(0,x) = m_o(x), \end{cases}$$

Data: ν, H, F, m_o, g ; Unknowns:
$m(t,x) =$ equilibrium distribution of the agents at time t;
$v(t,x)$= value function of the representative agent

1st equation is backward parabolic H-J-B with a possibly non-local cost term $F(x,m)$,
2nd equation is forward parabolic K-F-P equation, linear in m.
3rd line: terminal and initial conditions.

Well-posedness?

Existence: proved under various different sets of assumptions:
J.M. Lasry and P.L. Lions (2006 -...) , mostly for periodic data and boundary conditions, i.e. on the torus $\mathbb{T}^d$ instead of $\mathbf{R}^d$,
see [P. Cardaliaguet, Notes 2010],
M. Huang, P. Caines and R. Malhame (2006 -...), mostly for LQG models,
Carmona and Delarue by probabilistic methods, two huge books 2018.

Uniqueness: is not expected in general, true for

- H convex under the increasing monotonicity condition on the costs

$$\int_{\mathbf{R}^d} [F(x, m_1) - F(x, m_2)] d(m_1 - m_2)(x) > 0, \quad \forall m_1 \neq m_2,$$

  which means crowd aversion (Lasry-Lions), idem for G.
- for small data or short horizon T: less known until recently, see below...

The large population limit

1. Synthesis of ε-Nash equilibria (Huang-Caines-Malhame 2006):

given a solution (v, m) of the PDE system (MFE) the candidate optimal feedback is $\hat{\alpha}(t, x) := -\nabla H(\nabla v(t, x))$.

Assume all the players use this feedback: $\tilde{\alpha}^i_s := \hat{\alpha}(s, X^i_s)$.
Then $\forall\, \varepsilon > 0\ \exists\, N_\varepsilon$ such that $\forall\, N \geq N_\varepsilon$, $\forall\, i = 1, .., N$, $\forall$ admissible α^i,

$$J^i_T(t, x^1, .., x^N, \tilde{\alpha}^1, .., \tilde{\alpha}^N) \leq J^i_T(t, x^1, .., x^N, \tilde{\alpha}^1, .., \alpha^i, .., \tilde{\alpha}^N) + \varepsilon$$

2. Convergence of Nash equilibria for N person games to a MF equilibrium and rigorous derivation of the MFG system: much more complicated.

2a. For problem with ergodic cost functional: Lasry-Lions 2006, M.B. - F. Priuli 2014 (LQG models)

2b. By the Probabilistic approach to MFG : M. Fischer 14, D. Lacker 14 -18,

2c. Convergence of solutions of the system of N HJB PDEs for the finite horizon problem to a solution of the evolutive MFG system of PDEs (MFE):
Cardaliaguet - Delarue - Lasry - Lions preprint 9/2015, book to appear.

Problem is related to propagation of chaos in statistical phisics.

Covers also the case of common noise, i.e., the noises W_s^i are NOT independent.

Main tool: the master equation, a fully nonlinear PDE in infinite dimensions.

Examples of applications:

- Economics
  - financial markets (price formation and dynamic equilibria, formation of volatility)
  - general economic equilibrium with rational expectations
  - environmental policy,
- Engineering
  - wireless power control
  - demand side management in electric power networks,
  - traffic problems
- Social sciences
  - crowd motion (mexican wave "la ola", pedestrian dynamics, congestion phenomena,...)
  - opinion dynamics and consensus problems,
  - models of population distribution (e.g., segregation).

Part II. 1. Non-uniqueness of solutions

Notation: Mean of $\mu \in \mathcal{P}_1(\mathbb{R}^n)$, $\quad M(\mu) := \int_{\mathbb{R}} y\, \mu(dy)$.

Theorem (An explicit example, M.B. - M. Fischer 2018)

Assume $d = 1$, $H(p) = |p|$ *(i.e., control* $\alpha \in [-1, 1]$*),*

$F, G \in C^1$, $\nu > 0$, $M(m_0) = 0$ *, and*

$$F(x, \mu) = \beta x M(\mu), \quad G(x, \mu) = \gamma x M(\mu), \quad \beta, \gamma \in \mathbf{R}.$$

Then, if $\beta \leq 0$, $\gamma < 0$ *,* $\forall\, T > 0\ \exists$ *solutions* (v, m) *,* $(\bar{v}, \bar{m})$ *with*

$$v_x(t, x) < 0, \quad \bar{v}_x(t, x) > 0 \quad \textit{for all } 0 < t < T.$$

If, instead, $\beta > 0$, $\gamma \geq 0$ *, the solution is unique.*

If $\beta < 0, \gamma < 0$ an agent has a negative cost, i.e., a reward, for having a position x with the same sign as the average position $M(m)$ of the whole population.
Conversely, the conditions for uniqueness express aversion to crowd.

Thm. [B.-Fischer]: uniqueness for short horizon in $\mathbf{R}^d$

Assume $H \in C^2$ with respect to p, $m_0 \in \mathcal{P} \cap L^\infty(\mathbf{R}^d)$,

$$\|F(\cdot,\mu) - F(\cdot,\bar{\mu})\|_2 \leq L_F \|\mu - \bar{\mu}\|_2,$$

$$\|DG(\cdot,\mu) - DG(\cdot,\bar{\mu})\|_2 \leq L_G \|\mu - \bar{\mu}\|_2$$

Then any $(v_1, m_1), (v_2, m_2)$ two classical solutions of the MFG PDEs $(v_1, m_1), (v_2, m_2)$ coincide if

- either the time horizon T is small enough
- or L_F, L_G are small
- or $\sup |D_p^2 H(x, Dv_i)|$ $i = 1, 2$, is small.

Remark: no convexity assumption on H, nor monotonicity of F and G, but the minimal regularity of H is $C^{1,1}$: see the preceding counterexample!

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II.2: Robust control

Consider a stochastic control system with an additional disturbance β

$$dX_s = [f(X_s) + g(X_s)\alpha_s + \tau(X_s)\beta_s]\, ds + \sigma\, dW_s, \quad X_t = x$$

with τ a given matrix, and β an UNKNOWN disturbance modeled as an adversary choosing a control function whereas α plays causal strategies: a 0-sum differential game. The value function is, for $\delta > 0$

$$V(x) := \inf_{\alpha.} \sup_{\beta.} \mathbf{E}\left[\int_t^T L(\alpha[\beta]_s) + F(X_s) - \frac{\delta}{2}|\beta_s|^2 ds + G(X_T)\right]$$

with β open loop control and $\alpha[\cdot]$ non-anticipating strategy.
The H-J-Isaacs equation associated to V by Dynamic Programming is

$$-v_t + H(x, Dv) = \nu \Delta v + F(x)$$

$$H(x,p) := \inf_{b \in \mathbf{R}^m} \sup_{a \in A} [-(f(x) + g(x)a + \tau(x)b) \cdot p - L(a) + \frac{\delta}{2}|b|^2]$$

$$= \sup_{a \in A} [(-f(x) - g(x)a) \cdot p - L(a)] - |\tau(x)^T p|^2$$

The PDE system for robust MFG

For the previous nonconvex H we derive heuristically the MFG system

$$\begin{cases} -v_t + H(x, Dv) = \nu \Delta v + F(x, m(t, \cdot)) \\ v(T, x) = G(x, m(T, \cdot)) \\ m_t - div(D_p H(x, Dv) m) = \nu \Delta m \\ m(0, x) = m_0(x), \end{cases}$$

Known partial results

- Bauso, Tembine, Basar 2016: model and numerical simulations, no rigorous results
- Moon, Basar 2017: LQ risk-sensitive and robust MFG (also Bauso, Mylvaganam, Astolfi 2016; Huang, Huang 2017)

Robust MFG with bounded state domain

For a bounded smooth domain Ω consider the trajectories of the system that are reflected at the boundary of Ω : this leads to Neumann boundary conditions for the H-J-I and KFP equations (n is the exterior normal):

$$\partial_n v = 0, \quad \partial_n m + mD_pH(x, Du) \cdot n = 0 \quad on\ \partial\Omega \times (0, T)$$

Theorem (M.B. - Cirant 2018)

Assume F, G regularizing and Lip in L^2, $m_0 \in C^{2,\beta}(\overline{\Omega})$ satisfying the compatibility condition

$$\partial_n m_0(x) - m_0(x) f(x) \cdot n(x) = 0 \quad \forall x \in \partial\Omega. \quad \textit{Then}$$

- *for all $T > 0$ there is a classical solution of the robust MFG system with Neumann conditions,*
- *there exists $\bar{T} > 0$ such that for all $T \in (0, \bar{T}]$ such solution is unique.*

II.3 The fractional MFG system, [Cirant - A. Goffi '18]

The MFG system involving the fractional Laplacian operator $(-\Delta)^s$ instead of the usual Laplacian is

$$(1) \quad \begin{cases} -\partial_t u + (-\Delta)^s u + H(x, Du) = F[m](x) & \text{in } \mathbb{T}^d \times (0,T) \\ \partial_t m + (-\Delta)^s m - \operatorname{div}(m D_p H(x, Du)) = 0 & \text{in } \mathbb{T}^d \times (0,T) \\ m(x,0) = m_0(x),\ u(x,T) = u_T(x) & \text{in } \mathbb{T}^d\ , \end{cases}$$

where $H = H(x,p) \sim C_H|Du|^\gamma,\ \gamma > 1$ for $|p|$ large, F is a regularizing coupling, $u_T \in C^{4+\alpha}(\mathbb{T}^d)$, $m_0 \in C^{4+\alpha}(\mathbb{T}^d)$, $\mathbb{T}^d = d-$dimensional torus.

Remark. Such systems arise when the underlying dynamics is driven by a $2s$-stable Lévy process instead of the Brownian motion.

Motivations. Systems with nonlocal diffusion have been introduced by [Chan-Sircar '17] in MFG economic models; diffusion processes with jumps have an increasing importance in financial models.

Remark . Stationary case in [A. Cesaroni, M. Cirant et al '17].

The Fractional Laplacian

Classical definitions: for $s \in (0, 1)$

- As integro-differential operator

$$(-\Delta)^s u(x) = c_{d,s} \text{P.V.} \int_{\mathbf{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy$$

- Via Fourier transform

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u))$$

PDEs with fractional diffusion: Local vs Nonlocal

The Laplacian may be defined in **R** as follows

(2) $$(-\Delta u)(x) = \lim_{y\to 0} 2\frac{u(x) - \frac{u(x+y)+u(x-y)}{2}}{|y|^2}\ .$$

The fractional Laplacian can be written as

$$(-\Delta)^s u(x) = c_s \int_{\mathbf{R}} \frac{u(x) - u(x-y)}{|y|^{1+2s}} dy = c_s \int_{\mathbf{R}} \frac{u(x) - u(x+\tilde{y})}{|\tilde{y}|^{1+2s}} d\tilde{y}\ .$$

After computing the average we get

(3) $$(-\Delta)^s u(x) = c_s \int_{\mathbf{R}} \frac{u(x) - \frac{u(x+y)+u(x-y)}{2}}{|y|^{1+2s}} dy$$

The main difference is that to compute (2) it is sufficient to know only the value of u in a neighborhood of x, while for (3) we need to know the value of u in the whole real line.

Remark. This is why such PIDE are usually called nonlocal.

Existence is tackled via vanishing viscosity, namely one considers the viscous system

$$
(4) \qquad \begin{cases} -\partial_t u - \sigma\Delta u + (-\Delta)^s u + H(x, Du) = F[m](x) & \text{in } Q_T \\ \partial_t m - \sigma\Delta m + (-\Delta)^s m - div(m D_p H(x, Du)) = 0 & \text{in } Q_T \\ m(x,0) = m_0(x), \; u(x,T) = u_T(x) & \text{in } \mathbb{T}^d . \end{cases}
$$

and then pass to the limit as $\sigma \to 0$.

Theorem (Existence for the viscous system (4))

Under the previous assumptions on H and F, for all $\sigma > 0$ and $s \in (0,1)$, there exists a classical solution (u_σ, m_σ) to the viscous (fractional) MFG system (4).

- Semiconcavity estimates independent of σ for the solution u of the HJB equation $\rightsquigarrow$ uniform Lipschitz estimates via duality methods.
- Some estimates independent of σ for the solution m of the FP equation and its time derivative in some energy spaces of the form $L^2(0,T; W^{\mu,2}(\mathbb{T}^d))$ for some $\mu \in \mathbf{R}$.

Existence and regularity of solutions

Theorem

Let H be convex. Under the same assumptions of the above result, let (u_σ, m_σ) *be a solution to* (4). *Then, as* $\sigma \to 0$ *and up to subsequences,* $u_\sigma \to u$ *uniformly,* $Du_\sigma \to Du$ *strongly,* $m_\sigma \to m$ *weakly.*

- *If* $s \in (0, 1/2]$, *then* (u, m) *is a weak energy solution to* (1).
- *If* $s \in (1/2, 1)$, *then* $\partial_t u, \partial_t m, (-\Delta)^s u, (-\Delta)^s m$ *belong to some* $\mathcal{C}^{\bar{\alpha}, \frac{\bar{\alpha}}{2s}}(Q_T)$, $\bar{\alpha} \in (0, 1)$, *and* (u, m) *is classical solution to* (1) .

Regularity: Generalize the classical spaces $C^{2+\alpha,1+\alpha/2}$ and $W_p^{2,1}$, associated to the heat operator $\partial_t - \Delta$, to the operator $\partial_t + (-\Delta)^s$:

- **Fractional parabolic Hölder spaces**
- **Parabolic Bessel potential spaces**

Uniqueness of solutions, perspectives

Theorem

System (1) *admits a unique solution in the following cases:*

(a) *F monotone as before and H convex ;*

(b) *for short horizon: for $s \in (\frac{1}{2}, 1)$, there exists $T^* > 0$, depending on d, s, H, F, m_0, u_T such that for all $T \in (0, T^*]$,* (1) *has at most a solution (u, m).*

- The approach extends to $\mathbf{R}^d$ and to more general integro-differential operators

$$\mathcal{I}[u](x,t) = \int_{\mathbb{R}^d} u(x+z,t) - u(x,t) - Du(x,t) \cdot z\nu(dz)$$

where ν is a Lévy measure, as appearing in finance models.

- Similar procedures work for MFG models with Caputo time-fractional derivative [F. Camilli - A. Goffi, ongoing], being mostly based on results for abstract evolution equations.

II.4 Beyond the monotonicity of the costs

MFGs with costs F increasing with respect to m has been treated in generality with several techniques and carry many good properties:

- uniqueness of solutions,
- regularity of solutions as a byproduct of “competition” effects,
- stability in the long time regime, i.e. convergence of the time-dependent problem to the stationary one corresponding to ergodic control.

However, very seldom this assumption is satisfied in applications, e.g. [Gonzalez, Gualdani, Sola-Morales, 16], [Carmona, Graves, 18], [Hongler et al., 16], multi-population [Achdou-Bardi-Cirant, 16] , ...

M. Cirant explored MFG systems when $f(\cdot, x)$ is not increasing, in particular: non-uniqueness, concentration phenomena and pattern formation in the long time regime.

MFGs with decreasing costs [Cirant, 16]

The MFG PDEs for ergodic control problems, i.e.,

cost funtional $= \lim_{T\to+\infty} \frac{1}{T} \mathbf{E}[\int_0^T (L(\alpha_s) + F(X_s, m(s)))ds]$

are stationary with an additive eigenvalue $\lambda =$ constant value function:

$$(5) \qquad \begin{cases} -\epsilon\Delta u + H(\nabla u) + \lambda = f(x, m(x)) \approx -c\, m^\alpha + V(x) \\ -\epsilon\Delta m - \operatorname{div}(m\nabla H(\nabla u)) = 0, \quad \int m = 1, \quad m > 0. \end{cases}$$

model problem describing "aggregation", i.e., cost decreasing w.r.t m.

Theorem

Suppose that $H \approx |p|^\gamma$, $\gamma > 1$. Two main regimes are identified:

- **Subcritical**: if $\alpha < \gamma'/d$, there exists a smooth solution (u, λ, m); if $\gamma'/d \leq \alpha < \gamma'/(d-\gamma')$ a solution exists under additional smallness conditions on c .
- **Supercritical**: if $\alpha > \gamma'/(d-\gamma')$, (5) may not have solutions.

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Concentration of mass in the small noise limit [Cirant-Cesaroni, 19]

Concentration phenomena for ergodic problems in the vanishing viscosity regime $\epsilon \to 0$

$$(6) \qquad \begin{cases} -\epsilon\Delta u + H(\nabla u) + \lambda = f(x, m(x)) \approx -m^\alpha + V(x) \\ -\epsilon\Delta m - \mathrm{div}(m\nabla H(\nabla u)) = 0, \quad \int_{\mathbf{R}^d} m = 1, \quad m > 0. \end{cases}$$

Theorem

Under the assumptions that $V(\infty) = +\infty$ and α be "subcritical",

- For all $\epsilon \to 0$ there exists a triple $(u_\epsilon, m_\epsilon, \lambda_\epsilon)$ solving (6).
- There exist sequences $\epsilon \to 0$ and $x_\epsilon \in \mathbf{R}^d$, such that for all $\eta > 0$ there exists $R_\epsilon \to 0$ s.t.,

$$\int_{B(x_\epsilon, R_\epsilon)} m_\epsilon \, dx \geq 1 - \eta,$$

and x_ϵ approaches one of the (flattest) minima of V.

II. 5 Time-periodic solutions [Cirant et al.]

The existence of periodic in time solutions to the parabolic system

$$\begin{cases} -\partial_t u - \Delta u + H(\nabla u) = f(m(x,t)) & \text{on } \mathbb{T}^d \times (-\infty, +\infty) \\ \partial_t m - \Delta m - \operatorname{div}(m \nabla H(\nabla u)) = 0 \end{cases}$$

has been obtained in [Cirant, Cirant-Nurbekyan 18] . The issue arises naturally in economic models, and was numerically observed in multi-population models of segregation [Achdou-M.B.-Cirant, 17] .

Theorem

$f'(1) < -4\pi^2 \quad \Longrightarrow \quad \exists$ non-trivial solutions (u_n, m_n), defined for $t \in (-\infty, +\infty)$, approaching a trivial solution $(f(1)t, 1)$ as $n \to \infty$, and $T_n > 0$ such that $\quad m_n(\cdot, t) = m_n(\cdot, t + T_n) \quad$ for all t.

Proof by bifurcation methods. Bifurcation suggests (see also [Cirant, Verzini 17]) that when the cost is not increasing, it is natural to expect the existence of several families of solutions.

II. 6 Deterministic Mean Field Games

If the system is noiseless $\dot{X}_s = \alpha_s$, , i.e., $\sigma = 0 = \nu$,
the MFG PDEs formally become

$$\begin{cases} -\frac{\partial v}{\partial t} + H(\nabla v) = F(x, m) \quad \text{in } (0,T) \times \mathbf{R}^d \\ \frac{\partial m}{\partial t} - div(m \nabla H(\nabla v)) = 0 \quad \text{in } (0,T) \times \mathbf{R}^d \\ v(T, x) = G(x, m(T)), \quad m(0, x) = m_o(x). \end{cases}$$

Troubles: v solving the HJB equation is not smooth, the vector field $\nabla H(\nabla v)$ can be undefined and discontinuos.
However, $H(p) \to +\infty$ as $|p| \to \infty$ and F, G smooth imply v is semiconcave in x, i.e., for some $C > 0$ $x \mapsto v(x,t) - C|x|^2$ is concave, which implies $\nabla H(\nabla v) \in L^\infty$.
Then the system has a solution with the HJB equation satisfied in "viscosity sense" and KFP in the sense of distributions [Cardaliaguet, Notes 2010].

Non-coercive first order Mean Field Games
P. Mannucci, C. Marchi, C. Mariconda, N. Tchou '19

$$\begin{cases} -\partial_t u + H(x, Du) = F(x, m) & R^2 \times (0, T) \\ \partial_t m - div(m\,\partial_p H(x, Du)) = 0 & R^2 \times (0, T) \\ m(x, 0) = m_0(x),\ u(x, T) = G(x, m(T)) & R^2 \end{cases}$$

$$H(x, p) = \frac{1}{2}(p_1^2 + h(x_1)^2 p_2^2), \quad p = (p_1, p_2),\ x = (x_1, x_2).$$

$h \in C^2(R)$ is bounded and possibly vanishing, e.g.,

$$h(x_1) = \frac{x_1}{(1 + x_1^2)^{1/2}} \qquad \text{(or also} \quad h(x_1) = \sin x_1).$$

These systems arise when the generic player wants to choose the control $\alpha = (\alpha_1, \alpha_2) \in L^2([t, T]; \mathbf{R}^2)$ so to minimize the cost

$$\int_t^T \left[\frac{1}{2} |\alpha(\tau)|^2 + F(x(\tau), m) \right] d\tau + G(x(T), m(T))$$

with the dynamics $x(\cdot)$ is

$$\begin{cases} x_1'(s) = \alpha_1(s), & x_1(t) = x_1 \\ x_2'(s) = h(x_1(s))\, \alpha_2(s), & x_2(t) = x_2. \end{cases}$$

When $h = 0$: $x_1 = 0$ is a forbidden direction.

Theorem

$\|F(\cdot, m)\|_{\mathcal{C}^2} \leq C$, $\|G(\cdot, m)\|_{\mathcal{C}^2} \leq C$ *for all* $m \in \mathcal{P}_1$

$\Longrightarrow$ *exists a solution to the MFG system.*

MFG with control on acceleration (double integrator) Y. Achdou, P. Mannucci, C. Marchi, N. Tchou

The generic player wants to choose the control α so to minimize

$$\int_t^T \left[\frac{1}{2}|\alpha(\tau)|^2 + F(x(\tau), v(\tau), m)\right] d\tau + G(x(T), m(T))$$

where F and G are C^2-bounded and $x(\cdot)$ is the double integrator

$$\begin{cases} x'(s) = v(s) \\ v'(s) = \alpha(s). \end{cases}$$

The MFG system is

$$\begin{cases} -\partial_t u + H(x, v, D_x u, D_v u) = F(x, v, m) & R^{2N} \times (0, T) \\ \partial_t m - div(m\, \partial_p H(x, v, D_x u, D_v u)) = 0 & R^{2N} \times (0, T) \\ m(x, 0) = m_0(x),\ \ u(x, T) = G(x, m(T)) & R^{2N} \end{cases}$$

$$H(x, v, p_x, p_v) = \frac{1}{2}|p_v|^2 - v \cdot p_x, \quad (x, v) \in R^{2N}, p = (p_x, p_v).$$

Thanks for your attention !