Mean-Field Games with non-convex Hamiltonian

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Plan

- Risk-sensitive and robust control
- Robust MFGs
- Uniqueness of the solutions: the classical conditions and the "small data" regime
- Examples of non-uniqueness [M.B. and M. Fischer]
- Well-posedness with Neumann B.C. and non-convex H [M.B. and M. Cirant]
- MFGs with several populations [Achdou - M.B.- Cirant]
- Remarks and perspectives

Risk-sensitive control

Consider a stochastic control system

$$dX_s = f(X_s, \alpha_s)\, ds + \sigma(X_s)\, dW_s, \quad X_t = x \in \mathbf{R}^d, \quad 0 \le t \le T$$

with W_s a Brownian motion, α_s = control (adapted to W_s), σ a volatility matrix, and a finite horizon loss functional

$$C_T(t, x, \alpha.) := \int_t^T L(X_s, \alpha_s) ds + G(X_T).$$

The usual cost functional is $J_T(t, x, \alpha.) := E\,[C_T(t, x, \alpha.)]$.

The Risk-sensitive cost functional is

$$I_T(t, x, \alpha.) := \delta \log E \left[e^{\frac{1}{\delta} C_T(t,x,\alpha.)} \right]$$

$\delta > 0$ = risk sensitivity index (small δ = great sensitivity).

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Note:

$$I_T = E\,[C_T] + \frac{1}{2\delta} Var(C_T) + O(\frac{1}{\delta}) \qquad as \quad \delta \to \infty,$$

and in general I_T takes into account all moments of the cost C_T, not only E.

The risk-sensitive H-J equation

The risk-sensitive value function is $\quad v(x) := \inf_{\alpha.} I_T(t, x, \alpha.)$.

From the Hamilton-Jacobi-Bellman equation for $\quad \inf_{\alpha.} E\left[e^{\frac{1}{\delta}C_T}\right]$
easy calculations give for v

$$-v_t + \tilde{H}(x, Dv) - \frac{1}{2\delta}|\sigma(x)^T Dv|^2 = tr\left(\frac{\sigma\sigma^T(x)}{2} D^2 v\right) \quad \text{in } (0, T) \times \mathbf{R}^d,$$

$$\tilde{H}(x, p) := \sup_{a \in A}[-f(x, a) \cdot p - L(x, a)]$$

with the terminal condition $v(T, x) = G(x)$.

This is a H-J equation with a non-convex Hamiltonian

$$H = \tilde{H} - |\sigma(x)^T p|^2,$$

perhaps a H-J-Isaacs equation.

A classical Verification Theorem holds, see Fleming and Soner's book.

Robust control

Now consider the stochastic control system with an additional disturbance β

$$dX_s = [f(X_s, \alpha_s) + \tau(X_s)\beta_s]\, ds + \sigma(X_s)\, dW_s, \quad X_t = x \in \mathbf{R}^d, \quad 0 \le t \le T$$

with W_s, $\alpha_s =$, and σ as before, τ a given matrix and β an UNKNOWN disturbance (typically unbounded).

Following Fleming (1960) we can perform a worst case analysis by modelling β as an adversary playing strategies (adapted to W_s) in a 0-sum differential game.
The value function is, for $\delta > 0$

$$V(x) := \inf_{\alpha.} \sup_{\beta.} E\left[\int_t^T L(X_s, \alpha_s) - \frac{\delta}{2}|\beta_s|^2 ds + G(X_T)\right]$$

H-J-Isaacs equation for robust control

Th H-J-I equation associated to V by Dynamic Programming is

$$-v_t + H(x, Dv) = tr\left(\frac{\sigma\sigma^T(x)}{2} D^2 v\right) \quad \text{in } (0,T)\times \mathbf{R}^d,$$

$$H(x,p) := \sup_{a\in A} \inf_{b\in\mathbf{R}^m}[-(f(x,a)+\tau(x)b)\cdot p - L(x,a) + \frac{\delta}{2}|b|^2]$$

$$= \tilde{H}(x,p) - |\tau(x)^T p|^2$$

It is the same PDE as in risk-sensitive control if $\tau = \sigma$!

So risk-sensitive control can be interpreted as robust control, and both as 0-sum stochastic differential games.

Large engineering literature on these subjects and on the related H^∞ control, see, e.g., Basar and Bernhard's book.

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Saddle feedback trajectory for the 0-sum game

A Verification Theorem holds for the 0-sum stochastic differential game if the H-J-I equation + terminal condition $v(T,x) = G(x)$ has a smooth solution V. It produces a saddle point in feedback form.

If $\tilde{H}$ is smooth and $\inf_a[...]$ is attained at a single point it is also known that the trajectory associated to the saddle strategies satisfies

$$dX_s = D_p H(X_s, DV(X_s))$$

as in the case of a single player!

This allows to derive, at least formally, the MFG system of PDEs for a large population of identical agents with independent Brownian noises and independent deterministic disturbances, see

Tembine, Zhu, Basar 2014 for risk sensitive and

Bauso, Tembine, Basar 2016 for robust control.

Robust Mean Field Games

Assume now the cost functional of a representative agent in a population is of the form

$$E\left[\int_t^T L(X_s,\alpha_s) + F(X_s, m(s,\cdot)) - \frac{\delta}{2}|\beta_s|^2 ds + G(X_T, m(T,\cdot))\right],$$

where $F, G : \mathbf{R}^d \times \mathcal{P}(\mathbf{R}^d) \to \mathbf{R}$ depend on the distribution of the population of agents $m(\cdot,\cdot)$.
If the agent and the disturbance β "behave optimally", i.e., choose a feedback saddle of their 0-sum game, the probability distribution $\mu(t,x)$ of the agent solves the Kolmogorov-Fokker-Plank equation

$$\mu_t - div(D_pH(x, Dv)\mu) = tr\, D^2\left(\frac{\sigma\sigma^T(x)}{2}\mu\right) \quad \text{in } (0,T)\times\mathbf{R}^d$$

Assume also that we are given the initial distribution of the representative agent $\mu(0,x) = \nu(x)$

The PDE system for robust MFG

We have an MFG equilibrium if all players are identical and "behave optimally", so $\mu(t,x) = m(t,x)$ and it satisfies

$$\begin{cases} -v_t + H(x, Dv) = tr\left(\frac{\sigma\sigma^T(x)}{2} D^2 v\right) + F(x, m(t,\cdot)) \text{ in } (0,T)\times \mathbf{R}^d, \\ v(T,x) = G(x, m(T,\cdot)) \\ m_t - div(D_pH(x,Dv)m) = \sum_{i,j} \partial_{ij}\left(\frac{\sigma\sigma^T(x)}{2}\mu\right)_{i,j} \quad \text{in } (0,T)\times \mathbf{R}^d, \\ m(0,x) = \nu(x), \end{cases}$$

where

- $H(x,p) := \sup_a[-f(x,a)\cdot p - L(x,a)] - |\tau(x)^T p|^2$
- v is the value function of a representative agent.

Known results

- Tembine, Zhu, Basar 2014: model and numerical simulations for risk-sensitive MFG
- Bauso, Tembine, Basar 2016: same for robust MFG
- Moon, Basar 2017: LQ risk-sensitive and robust MFG
- Tran preprint 2017: existence and uniqueness (small data) for toy model with periodic BC
- Existence for periodic BC and regularity: Some results can be adapted from general theory Lasry - Lions (2006 -...), Cardaliaguet, Porretta, Gomes and coworkers, etc....
- We are more interested in
  - Neumann boundary conditions in a bounded smooth domain, or
  - problem in all $\mathbf{R}^d$ with growth conditions or integrabilty conditions at infinity.
- Main difference in the non-convex case: Uniqueness ?

From now on, for simplicity, $\sigma > 0$ scalar constant.

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The Lasry-Lions monotonicity condition

A sufficient condition for uniqueness of classical solutions is

$$p \to H(x,p) \ \ convex$$

$$\int_{\mathbf{R}^n} [F(x,m) - F(x,\bar{m})] d(m - \bar{m})(x) > 0, \ \forall\, m \neq \bar{m} \in \mathcal{P}(\mathbf{R}^d)$$

$$\int_{\mathbf{R}^n} [G(x,m) - G(x,\bar{m})] d(m - \bar{m})(x) \geq 0, \ \forall\, m, \bar{m} \in \mathcal{P}(\mathbf{R}^d)$$

the costs are "increasing with the density" in L^2.
(See Cardaliaguet's notes for the proof)

Example

F is "local", i.e., $F(\cdot, m)(x) = f(x, m(x))$ and f is increasing in $m(x)$: the more is crowded the place where I am, the more I pay.

A non-local example

Notation: Mean of $\mu \in \mathcal{P}_1(\mathbb{R}^n)$, $\quad M(\mu) := \int_{\mathbb{R}} y\, \mu(dy)$.
Variant of the L-L uniqueness result: replace the strict monotonicity of F with: F and G depend on m only via $M(m)$ and

$$\int_{\mathbf{R}^n} [F(x,m) - F(x,\bar{m})] d(m - \bar{m})(x) > 0, \ \forall\, M(m) \neq M(\bar{m})$$

Example

$$F(x,\mu) = \beta x M(\mu), \quad G(x,\mu) = \gamma x M(\mu)$$

$\beta, \gamma \in \mathbf{R}$. Then

$\int_{\mathbf{R}^n} [F(x,m) - F(x,\bar{m})] d(m - \bar{m})(x) = \beta (M(m) - M(\bar{m}))^2 \geq 0,$

and the condition above is satisfied if $\beta > 0, \ \gamma \geq 0$.

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Uniqueness for non-convex H?

Lions lecture at College de France 2009: uniqueness for short horizon T.
It turns that variants of that unpublished idea work for

- non-convex but smooth H
- non-monotone costs F and G,
- boundary conditions different from periodic
- smallness of some other data instead of T
- MFG with several populations of agents, i.e. systems of n HJB and n KFP equations.

Thm. [B.-Fischer]: uniqueness for short horizon in $\mathbf{R}^d$

Assume $H \in C^2$ with respect to p, $\nu \in \mathcal{P} \cap L^\infty(\mathbf{R}^d)$,

$$\|F(\cdot,\mu) - F(\cdot,\bar\mu)\|_2 \le L_F \|\mu - \bar\mu\|_2,$$

$$\|DG(\cdot,\mu) - DG(\cdot,\bar\mu)\|_2 \le L_G \|\mu - \bar\mu\|_2$$

$(v_1, m_1), (v_2, m_2)$ two classical solutions of the MFG PDEs with $D(v_1 - v_2) \in L^2([0,T] \times \mathbf{R}^d)$, and

$$|D_pH(x, Dv_i)|, \; |D_p^2H(x, Dv_i)| \le C_H.$$

Then $\exists\, \bar T = \bar T(d, L_F, L_G, \|\nu\|_\infty, C_H) > 0$ such that $\forall\, T < \bar T$,

$v_1(\cdot,t) = v_2(\cdot,t)$ and $m_1(\cdot,t) = m_2(\cdot,t)$ for all $t \in [0,T]$.

Corollary (Uniqueness for "small data")

Uniqueness remains true for all $T > 0$ if either L_F, L_G are small, or $\sup |D_p^2H(x, Dv_i)|$ is small.

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Sketch of proof

Assume for simplicity $H = H(p)$ only and $\sigma = 1$.
For two solutions $(v_1, m_1), (v_2, m_2)$ take $v := v_1 - v_2$, $m := m_1 - m_2$, write the PDEs for (v, m): the 1st is

$$\begin{cases} -v_t + B(t,x) \cdot Dv = \Delta v + F(x, m_1) - F(x, m_2) & \text{in } (0,T) \times \mathbf{R}^d, \\ v(T,x) = G(x, m_1(T)) - G(x, m_2(T)). \end{cases}$$

with $B(t,x) := \int_0^1 DH(Dv_2 + s(Dv_1 - Dv_2))ds \in L^\infty((0,T) \times \mathbf{R}^d)$.

Then by energy estimates we get

$$\|Dv(t,\cdot)\|_{L^2_x} \leq C_1 \int_t^T \|F(\cdot, m_1(s)) - F(\cdot, m_2(s))\|_{L^2_x} ds + C_2 \|DG(\cdot, m_1(T)) - DG(\cdot, m_2(T))\|_{L^2_x}.$$

The Lipschitz assumption on F and DG implies

$$\|Dv(t,\cdot)\|_{L^2_x} \leq C_1 L_F \int_t^T \|m(s,\cdot)\|_{L^2_x} ds + C_2 L_G \|m(T,\cdot)\|_{L^2_x}$$

Similarly, from the 2nd equation can estimate

$$\|m(t,\cdot)\|_{L^2_x} \leq C_3 \int_0^t \|Dv(s,\cdot)\|_{L^2_x} ds$$

Now set $\phi(t) := \|Dv(t,\cdot)\|_{L^2_x}$ and combine the inequalities to get

$$\phi(t) \leq C_4 \int_t^T \int_0^\tau \phi(s) ds\, d\tau + C_5 \int_0^T \phi(s) ds$$

and $\Phi := \sup_{0\leq t\leq T} \phi(t)$ satisfies

$$\Phi \leq \Phi(C_4 T^2/2 + C_5 T)$$

so $\Phi = 0$ for T small enough.

Remark: a crucial estimate is

$$\|m_i(t,\cdot)\|_\infty \le C(T, \|DH(Dv_i)\|_\infty)\|\nu\|_\infty, \quad i = 1, 2, \; \forall\, t \in [0, T],$$

that we prove by probabilistic methods.

Example (Regularizing costs)

$$F(x,\mu) = F_1\left(x, \int_{\mathbf{R}^d} k_1(x,y)\mu(y)dy\right),$$

with $k_1 \in L^2(\mathbf{R}^d \times \mathbf{R}^d)$, $|F_1(x,r) - F_1(x,s)| \le L_1|r-s|$;

$$G(x,\mu) = g_1(x)\int_{\mathbf{R}^d} k_2(x,y)\mu(y)dy + g_2(x)$$

with $g_1, g_2 \in C^1(\mathbf{R}^d)$, Dg_1 bounded, $k_2, D_x k_2 \in L^2(\mathbf{R}^d \times \mathbf{R}^d)$.

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Example (Local costs)

$G = G(x)$ independent of $m(T)$ and F of the form

$$F(x,\mu) = f(x,\mu(x))$$

with $f : \mathbf{R}^d \times [0,+\infty) \to \mathbf{R}$ such that

$$|f(x,r) - f(x,s)| \le L_f |r-s| \quad \forall\, x \in \mathbf{R}^d,\; r,s \ge 0.$$

Then F is Lipschitz in L^2 with $L_F = L_f$.

Remark: no convexity assumption on H, nor monotonicity of F and G, but the minimal regularity of H is $C^{1,1}$ w.r.t. p.

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More on uniqueness?

Questions: are the conditions for uniqueness merely technical or close to necessary? how far from optimal?

Related very recent papers (with periodic BC):

- Cirant, Gianni, Mannucci preprint 2018: short-time existence and uniqueness for parabolic systems more general than MFG
- Cirant, Goffi preprint 2018: short-time existence and uniqueness for MFG with non-local terms.

Nonetheless, are there examples of multiple solutions? even for short time horizon?

Examples of non-uniqueness: stationary MFGs

The stationary MFG PDEs:

$$(MFE) \qquad \begin{cases} -\Delta v + H(x, \nabla v) + \lambda = F(x, m) \quad \text{in } \mathbb{T}^d, \\ \Delta m + div(\nabla_p H(x, \nabla v) m) = 0 \quad \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} m(x)dx = 1, \quad m > 0, \quad \int_{\mathbb{T}^d} v(x)dx = 0, \end{cases}$$

has uniqueness for F monotone increasing and H convex. Otherwise:

- Lasry-Lions for $H(x, p) = |p|^2$ via a Hartree equation of Quantum Mechanics,
- Gueant 2009 for (local) logarithmic utility $\quad F = -\log m$
- M.B. 2012 and M.B. - F. Priuli 2014 for LQG models in $\mathbf{R}^d$
- M. Cirant 2015 and Y. Achdou - M.B. - M. Cirant 2016 for systems of two populations with Neumann boundary conditions.

Question: counter-examples for the evolutive case?

How far from the monotonicity condition? Also for T small?

Existence of two solutions

Theorem (Any $T > 0$)

Assume $d = 1$, $H(p) = |p|$, $F, G \in C^1$, $\sigma > 0$ *and* C^2, $M(\nu) = 0$, *and*

$$\frac{\partial F}{\partial x}(x,\mu) \begin{cases} \leq 0 & \text{if } M(\mu) > 0, \\ \geq 0 & \text{if } M(\mu) < 0. \end{cases}$$

$$\frac{\partial G}{\partial x}(x,\mu) \begin{cases} \leq 0 \quad \text{and not } \equiv 0 & \text{if } M(\mu) > 0, \\ \geq 0 \quad \text{and not } \equiv 0 & \text{if } M(\mu) < 0, \end{cases}$$

$\implies \quad \exists$ *solutions* (v, m) , $(\bar{v}, \bar{m})$ *with*

$$v_x(t,x) < 0, \quad \bar{v}_x(t,x) > 0 \quad \text{for all } 0 < t < T.$$

- $T > 0$ can also be small: H convex but not C^1.
- No assumption on the monotonicity of F, G w.r.t. μ.
- We have also a probabilistic formulation and proof of non-uniqueness under less assumptions on σ.

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Explicit example of non-uniqueness

$$F(x,\mu) = \beta x M(\mu) + f(\mu), \quad G(x,\mu) = \gamma x M(\mu) + g(\mu),$$

with $\beta, \gamma \in \mathbf{R}$, $f, g : \mathcal{P}_1(\mathbb{R}) \to \mathbf{R}$, e.g., f, g depend only on the moments of μ .

There are two different solutions if

$$\beta \le 0, \quad \gamma < 0,$$

By the L-L monotonicity result there is uniqueness if $f = g \equiv 0$ and

$$\beta > 0, \quad \gamma \ge 0.$$

If $\beta < 0, \gamma < 0$ F and G are not decreasing in $M(\mu)$, but an agent has a negative cost, i.e., a reward, for having a position x with the same sign as the average position $M(m)$ of the whole population. Conversely, the conditions for uniqueness express aversion to crowd.

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$$\beta > 0, \quad \gamma \geq 0.$$

If $\beta < 0, \gamma < 0$ F and G are not decreasing in $M(\mu)$, but an agent has a negative cost, i.e., a reward, for having a position x with the same sign as the average position $M(m)$ of the whole population. Conversely, the conditions for uniqueness express aversion to crowd.

Explicit example of non-uniqueness

$$F(x,\mu) = \beta x M(\mu) + f(\mu), \quad G(x,\mu) = \gamma x M(\mu) + g(\mu),$$

with $\beta, \gamma \in$ **R** , $f, g : \mathcal{P}_1(\mathbb{R}) \to$ **R** , e.g., f, g depend only on the moments of μ .

There are two different solutions if

$$\beta \le 0, \quad \gamma < 0,$$

By the L-L monotonicity result there is uniqueness if $f = g \equiv 0$ and

$$\beta > 0, \quad \gamma \ge 0.$$

If $\beta < 0, \gamma < 0$ F and G are not decreasing in $M(\mu)$, but an agent has a negative cost, i.e., a reward, for having a position x with the same sign as the average position $M(m)$ of the whole population. Conversely, the conditions for uniqueness express aversion to crowd.

Existence of two solutions - 2

Theorem (H smooth and $T > \varepsilon$)

Same assumptions as previous Thm., BUT, for some $\delta, \varepsilon > 0$,

$$H(p) = |p|, \text{ for } |p| \geq \delta$$

$$\frac{\partial G}{\partial x}(x,\mu) \begin{cases} \leq -\delta & \text{if } M(\mu) \geq \varepsilon, \\ \geq \delta & \text{if } M(\mu) \leq -\varepsilon, \end{cases}$$

$\Longrightarrow$ *for* $T \geq \varepsilon$ $\exists$ *solutions* (v,m) , $(\bar{v},\bar{m})$ *with*

$$v_x(t,x) \leq -\delta, \quad \bar{v}_x(t,x) \geq \delta \quad \text{for all } 0 < t < T.$$

Example

$$H(p) := \max_{|\gamma|\leq 1}\left\{-p\gamma + \frac{1}{2}\delta(1-\gamma^2)\right\} = \begin{cases} \frac{p^2}{2\delta} + \frac{\delta}{2}, & \text{if } |p| \leq \delta, \\ |p|, & \text{if } |p| \geq \delta, \end{cases}$$

Idea of proof

$$\begin{cases} -v_t + |v_x| = \frac{\sigma^2(x)}{2} v_{xx} + F(x, m(t,\cdot)), & v(T,x) = G(x, M(m(T))), \\ m_t - (sign(v_x)m)_x = \left(\frac{\sigma^2(x)}{2} m\right)_{xx}, & m(0,x) = \nu(x). \end{cases}$$

Ansatz: $sign(v_x) = -1$ and m solves

$$m_t + m_x = \left(\frac{\sigma^2(x)}{2} m\right)_{xx}, \quad m(0,x) = \nu(x).$$

Then m is the law of the process

$$X(t) = X(0) + t + \int_0^t \sigma(X(s))dW(s)$$

with $X(0) \sim \nu$, so $M(m(t)) = \mathbf{E}[X(t)] = M(\nu) + t = t > 0 \quad \forall t$.

(E-) $$-v_t - v_x = \frac{\sigma^2(x)}{2} v_{xx} + F(x, m), \quad v(T,x) = G(x, m(T)).$$

Then $w = v_x$ satifies

$$-w_t - w_x - \sigma\sigma_x w_x - \frac{\sigma^2}{2} w_{xx} = \frac{\partial F}{\partial x}(x, m) \leq 0$$

$$w(T, x) = \frac{\partial G}{\partial x}(x, m(T)) \leq 0 \text{ and not } \equiv 0,$$

Similarly we can build a solution with $sign(\bar{v}_x) = 1$ and $\bar{m}$ solving

$$\bar{m}_t - \bar{m}_x = \frac{\sigma^2(x)}{2}\bar{m}_{xx}, \quad \bar{m}(0, x) = \nu(x),$$

so that $M(\bar{m}(t, \cdot)) = -t < 0$ and $\frac{\partial F}{\partial x}(x, \bar{m}(t, \cdot)), \frac{\partial G}{\partial x}(x, \bar{m}(T)) \geq 0$.

Other examples of non-uniqueness in finite horizon MFGs

For periodic boundary conditions:

- A. Briani, P. Cardaliaguet 2016: for a potential MFG
- M. Cirant, D. Tonon 2017: for a focusing MFG
- M. Cirant 2018: bifurcation of periodic solutions from stationary one.

Neumann boundary conditions

For Ω bounded and smooth,

$$\begin{cases} -\partial_t v + H(x, Dv) = \Delta v + F(x, m(t, \cdot)) \ \text{ in } (0, T) \times \Omega, \\ v(T, x) = G(x, m(T, \cdot)), \quad \partial_n v = 0 \quad on\ \partial\Omega \times (0, T), \\ \partial_t m - div(D_p H(x, Dv)m) = \Delta m \quad \text{ in } (0, T) \times \Omega, \\ m(0, x) = \nu(x), \quad \partial_n m + m D_p H(x, Du) \cdot n = 0 \quad on\ \partial\Omega \times (0, T) \end{cases}$$

Theorem [M.B. - M. Cirant]: uniqueness for small data

Assume $H \in C(\overline{\Omega} \times \mathbf{R}^d)$, C^2 in p, $\nu \in \mathcal{P} \cap L^\infty(\Omega)$,

$$\|F(\cdot,\mu) - F(\cdot,\bar{\mu})\|_2 \le L_F \|\mu - \bar{\mu}\|_2,$$

$$\|DG(\cdot,\mu) - DG(\cdot,\bar{\mu})\|_2 \le L_G(\|\mu - \bar{\mu}\|_2$$

$(v, m), (\bar{v}, \bar{m})$ two classical solutions and

$$|D_p H(x, Dv)|, |D_p H(x, D\bar{v})| \le C_1,$$
$$|D_p^2 H(x, Dv)|, |D_p^2 H(x, D\bar{v})| \le C_2.$$

Then $\exists\ \bar{T} = \bar{T}(d, L_F, L_G, \|\nu\|_\infty, C_1, C_2) > 0$ such that $\forall\ T < \bar{T}$,

$v(\cdot,t) = \bar{v}(\cdot,t)$ and $m(\cdot,t) = \bar{m}(\cdot,t)\ \forall\, t \in [0, T]$.

The same conclusion holds for all $T > 0$ if L_F and L_G are small, or C_2 is small.

Remarks

- 1. A crucial estimate for the proof is, for some $r > 1$, $C > 0$,

$$\|m\|_\infty \leq C[1 + \|\nu\|_\infty + (1 + T)\|D_pH(\cdot, Dv)\|_\infty]^r.$$

- 2. The bound $\bar{T}$ on the horizon length may depend on the two solution via C_1 and C_2.
  It depends only on the data if we have an a-priori bound on $Dv, D\bar{v}$.
  This can be got from classical parabolic regularity under, e.g., the additional assumptions
  - F, G bounded, respectively, in $C^{1,\beta}(\overline{\Omega}), C^{2,\beta}(\overline{\Omega})$ uniformly w.r.t. $m \in \mathcal{P}(\overline{\Omega})$ (regularizing costs)
  - $H \in C^1(\overline{\Omega} \times \mathbf{R}^d)$ with at most quadratic growth

$$|H(x,p))| \leq C_0(1 + |p|^2), \quad |D_pH(x,p)|(1 + |p|) \leq C_0(1 + |p|^2).$$

Remarks

- 1. A crucial estimate for the proof is, for some $r > 1$, $C > 0$,

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$$|H(x,p))| \leq C_0(1 + |p|^2), \quad |D_pH(x,p)|(1 + |p|) \leq C_0(1 + |p|^2).$$

Theorem [Y. Achdou - M.B. - M. Cirant]: existence

Assume

- F, G continuous in $\overline{\Omega} \times \mathcal{P}(\overline{\Omega})$ with the Kantorovich distance,
- F, G regularizing, as in Rmk. 2,
- $H \in C^1(\overline{\Omega} \times \mathbf{R}^d)$ with at most quadratic growth, as in Rmk. 2,
- $\nu \in C^{2,\beta}(\overline{\Omega})$.
- Compatibility conditions of data at the boundary:
  $\forall\ x \in \partial\Omega,\ \mu \in \mathcal{P}(\overline{\Omega})),\ u$ with $\partial_n u(x) = 0$

$$\partial_n G(x,\mu)(x) = 0, \quad \partial_n \nu(x) + \nu\, D_p H(x, Du(x)) \cdot n = 0.$$

Then $\forall\ T > 0$ there exists a classical solution of the MFG system with Neumann conditions.

Robust MFG with Neumann conditions

For the stochastic system with control α_s and disturbance β d-dimensional

$$dX_s = [f(X_s) + g(X_s)\alpha_s + \tau(X_s)\beta_s]\, ds + dW_s,$$

with g and τ scalar C^1 functions, $f \in C^1$, consider the trajectories that are reflected at the boundary of Ω (Skorokhod problem): this leads to Neumann conditions for the H-J-Isaacs equation.

The functional that α minimizes and β maximizes is

$$\mathbb{E}\left[\int_0^T \left(F(X_s, m(s,\cdot)) + \frac{|\alpha_s|^2}{2} - \delta\frac{|\beta_s|^2}{2}\right) ds + G(X_T, m(T,\cdot))\right]$$

This leads to the Hamiltonian

$$H(x,p) = -f(x)\cdot p + g^2(x)\frac{|p|^2}{2} - \tau^2(x)\frac{|p|^2}{2\delta}.$$

Corollary: well-posedness for robust MFG sysyem

- Take
$$H(x,p) = -f(x)\cdot p + g^2(x)\frac{|p|^2}{2} - \tau^2(x)\frac{|p|^2}{2\delta},$$
- F, G regularizing and "Lip in L^2",
- $\nu \in C^{2,\beta}(\overline{\Omega})$ satisfying the compatibility condition
$$\partial_n \nu(x) - \nu(x) f(x)\cdot n(x) = 0 \quad \forall\, x \in \partial\Omega.$$

Then

- for all $T > 0$ there is a classical solution of of the MFG system with Neumann conditions,
- there exists $\overline{T} > 0$ such that for all $T \in (0, \bar{T}]$ such solution is unique.
- $T < \bar{T}$ Lipschitz dependence on initial data
$$\|m(t,\cdot) - \bar{m}(t,\cdot)\|_2^2 \le \frac{C_T}{\delta}\|\nu - \bar{\nu}\|_2^2$$

MFG with several populations, Achdou - M. B. - Cirant

Motivation for 2 population: models of segregation phenomena in urban settlements, inspired by the Nobel laureate T. Schelling:

$$
\begin{cases}
-\partial_t v_k + H^k(x, Dv_k) = \Delta v_k + F^k(x, m_1(t,\cdot), m_2(t,\cdot)) \text{ in } (0,T)\times\Omega, \\
v_k(T,x) = G^k(x, m_1(T,\cdot), m_2(T,\cdot)), \quad \partial_n v_k = 0 \quad on\ \partial\Omega\times(0,T), \\
\partial_t m_k - div(D_p H^k(x, Dv_k) m_k) = \Delta m_k \quad \text{in } (0,T)\times\Omega, \quad k = 1, 2, \\
m_k(0,x) = \nu_k(x), \quad \partial_n m_k + m_k D_p H^k(x, Du_k)\cdot n = 0 \quad on\ \partial\Omega\times(0,T)
\end{cases}
$$

- Same existence - uniqueness result as for 1 population, under the same structure conditions;
- the monotonicity condition a la Lasry-Lions is very restrictive, makes no sense for the segregation model.

Remarks and perspectives

- We have examples of non-uniqueness for 2 populations
- The proof of uniqueness for small data is flexible: it can be used if $H(x,p) - F(x,m)$ is replaced by $\mathcal{H}(x,p,m)$ smooth, and in principle also for mean-field control (control of McKean-Vlasov SDEs); a hard point is the L^∞ estimate for $m(t,\cdot)$.

Questions

- The existence theory is not complete: results in all $\mathbf{R}^d$ and for local cost F are known only for H convex (Lions, Caradliaguet; Porretta 2016);
- are there examples of non-uniqueness due only to the non-convexity of H?
- For 1st order MFG (no noise) the theory is completely open if H is not convex (and not coercive).

Thanks for your attention !