PDE methods for multiscale control and differential games

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Plan

- Two-scale systems: examples and motivations
- The Hamilton-Jacobi approach to Singular Perturbations
- Representation of the limit control problem:
  - Uncontrolled fast variables
  - Homogenization in low dimensions

Two-scale systems

Dynamical systems with two groups of variables evolving on different time-scales (x_s, y_τ), $\tau = s/\varepsilon$, $0 < \varepsilon << 1$, governed by ODEs

$$\dot{x}_s = f(x_s, y_s) \qquad x_s \in \mathbf{R}^n,$$
$$\dot{y}_s = \frac{1}{\varepsilon} g(x_s, y_s) \qquad y_s \in \mathbf{R}^m,$$

or by Stochastic DEs

$$dx_s = f(x_s, y_s)\, ds + \sigma(x_s, y_s)\, dW_s$$
$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s)\, dW_s$$

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Hope to simplify the model in the limit $\varepsilon \to 0$:

a Singular Perturbation problem.

The theory is classical for Ordinary Differential Equations, see Levinson and Tychonov 1952, O'Malley's book 1974, and has a large literature also for systems with controls,

$$\dot{x}_s = f(x_s, y_s, \alpha_s)$$
$$\dot{y}_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s),$$

α_s a measurable control function taking values in a given set A, see Kokotović - Khalil - O'Reilly book 1986 (deterministic case)
Bensoussan 1988, Kushner 1990 also stochastic case:

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There are many different models in Physics, Engineering, Finance,....

Example 1: Mechanical system with Large Damping

The large time behavior of

$$\ddot{X} = F(X,t) - \frac{\dot{X}}{\varepsilon}$$

is described by $x(s) := X(s/\varepsilon)$ that solves

$$\varepsilon^2 \ddot{x} = F(x, s/\varepsilon) - \dot{x}.$$

In the autonomous case ($F = F(x)$) this writes

$$\dot{x}_s = y_s$$
$$\dot{y}_s = \frac{F(x_s) - y_s}{\varepsilon^2},$$

The limit is the Quasi-Static approximation

$$\dot{x} = F(x).$$

F. Hoppensteadt, "Quasi-Static state analysis...", Courant L. N. 2010

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Example 2: Control systems with stable fast variables

In Example 1 the formal limit

$$\dot{x}_s = f(x_s, y_s), \qquad 0 = g(x_s, y_s)$$

is correct: it fits in the Reduced Order Method. For control system the ROM gives in the limit the DIFFERENTIAL-ALGEBRAIC system

$$\dot{x}_s = f(x_s, y_s, \alpha_s), \qquad 0 = g(x_s, y_s, \alpha_s),$$

provided that (roughly speaking) the "fast subsystem" (with frozen x)

$$\dot{y}_\tau = g(x, y_\tau, \alpha_\tau)$$

has an equilibrium regime with an asymptotically stabilizing feedback, see Kokotović - Khalil - O'Reilly book 1986.

Examples in Engineering: high gain feedback, cheap control,....

BUT, many other models do NOT have this stability property!

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Example 3: Control systems in oscillating media

The homogenization problem

$$\dot{x}_s = f\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s\right), \quad J = \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right)$$

by setting $y_s = x_s/\varepsilon$ can be written as

$$\dot{x}_s = f(x_s, y_s, \alpha_s)$$
$$\dot{y}_s = \frac{1}{\varepsilon} f(x_s, y_s, \alpha_s)$$
$$J = \int_0^t l\,(x_s, y_s, \alpha_s)\, ds + h\,(x_t, y_t)$$

that is a Singular Perturbation problem with $g = f$.

Example 4: Financial models

The evolution of a stock S with stochastic volatility σ is

$$d \log S_s = \gamma\, ds + \sigma(y_s)\, dW_s$$
$$dy_s = \frac{1}{\varepsilon}(m - y_s) + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

see Fouque - Papanicolaou - Sircar, book 2000, for empirical data and many examples.

Merton portfolio optimization problem with stochastic volatility: invest β_s in the stock S_s, $1 - \beta_s$ in a bond with interest rate r. Then the wealth x_s evolves as

$$dx_s = (r + (\gamma - r)\beta_s)x_s\, ds + x_s \beta_s\, \sigma(y_s)\, dW_s$$
$$dy_s = \frac{1}{\varepsilon}(m - y_s)\, ds + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

Problem: maximize the expected utility at time t, $E[h(x_t)]$, h increasing concave function.

Example 5: Control systems on thin structures

State constraint on the z variables

$$\dot{x}_s = f(x_s, z_s, \alpha_s)$$
$$\dot{z}_s = g(x_s, z_s, \alpha_s) \qquad |z_s| \le \varepsilon,$$

by setting $y_s = z_s/\varepsilon$ becomes

$$\dot{x}_s = f(x_s, \varepsilon y_s, \alpha_s)$$
$$\dot{y}_s = \frac{1}{\varepsilon} g(x_s, \varepsilon y_s, \alpha_s) \qquad |y_s| \le 1.$$

This can be used to justify models of optimal control on graphs or networks.

The H-J approach to Singular Perturbations

General control system with TWO controllers

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s)\,ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\,dW_s,\;\; x_s \in \mathbf{R}^n,\; \alpha_s \in A,\; \beta_s \in B,$$

$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s, \beta_s)\,ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s, \alpha_s, \beta_s)\,dW_s, \qquad y_s \in \mathbf{R}^m,$$

$$x_0 = x, \quad y_0 = y$$

Cost-payoff functional (α minimizes, β_s maximizes)

$$J^\varepsilon(t, x, y, \alpha, \beta) := \int_0^t l(x_s, y_s, \alpha_s, \beta_s)\,ds + h(x_t, y_t)$$

Lower value function, where $\Gamma(t)$ are the nonanticipating strategies, $\mathcal{B}(t)$ the open-loop controls

$$u^\varepsilon(t, x, y) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E\left[J^\varepsilon(t, x, y, \alpha[\beta], \beta)\right]$$

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H-J-Bellman-Isaacs equation for the SP problem

Dynamic Programming method:
in the deterministic case $\sigma, \nu \equiv 0$, the value function solves

$$
(\mathrm{CP}_\varepsilon) \quad \begin{cases} \frac{\partial u^\varepsilon}{\partial t} + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0 & \text{in } (0, +\infty) \times \mathbf{R}^n \times \mathbf{R}^m, \\ u^\varepsilon(0, x, y) = h(x, y) & \text{in } \mathbf{R}^n \times \mathbf{R}^m, \end{cases}
$$

$$
H(x, y, p, q) = \min_{\beta \in B} \max_{\alpha \in A} \{-p \cdot f(x, y, \alpha, \beta) - q \cdot g(x, y, \alpha, \beta) - l(x, y, \alpha, \beta)\}
$$

in the viscosity sense.

In the stochastic case $\sigma \neq 0$ or $\nu \neq 0$ the H-J-B-I equation is

$$\frac{\partial u^\varepsilon}{\partial t} + \min_{\beta \in B} \max_{\alpha \in A} \left[\mathcal{L}^\varepsilon_{\alpha,\beta} u^\varepsilon - l(x, y, \alpha, \beta) \right] = 0 \quad \text{in } (0, +\infty) \times \mathbf{R}^n \times \mathbf{R}^m,$$

$\mathcal{L}^\varepsilon_{\alpha,\beta}$ = infinitesimal generator of the process with constant controls α, β

it is of 2*nd* order involving also $D^2_{xx}, D^2_{yy}/\varepsilon, D^2_{xy}/\sqrt{\varepsilon}$.

This H-J-B-I PDE is (degenerate) parabolic.

Again, u^ε is the unique viscosity solution of the Cauchy problem.

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Plan of the method:

1. pass to the limit as $\varepsilon \to 0$ in the PDE
2. associate to the limit PDE a "limit control problem"

1. was developed in the papers

O. ALVAREZ - M. B. : SIAM J. Cont. '01, Arch. Rat. Mech. Anal. '03, Memoir A.M.S. '10;

O. A. - M. B. - C. MARCHI : J. D. E. '07, '08 (more than 2 scales)

under boundedness assumptions on the fast state variables,
and for unbounded but uncontrolled fast variables in

M. B. - A. CESARONI - L. MANCA : SIAM J. Financial Math. 2010

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Methods are related to HOMOGENIZATION of H- J equations:

P.L.Lions- Papanicolaou- Varadhan 1986, L.C.Evans 1989, ...

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1. Search *effective Hamiltonian* $\overline{H}$ and effective initial data $\overline{h}$ s. t.

$$u^\varepsilon(t,x,y) \to u(t,x) \quad \text{as } \varepsilon \to 0,$$

u solution of

$$(\overline{\text{CP}}) \qquad \begin{cases} \frac{\partial u}{\partial t} + \overline{H}\left(x, D_x u, D^2_{xx} u\right) = 0 & \text{in} \quad (0,+\infty) \times \mathbf{R}^n, \\ u(0,x) = \overline{h}(x) & \text{in } \mathbf{R}^n \end{cases}$$

2. Intepret the effective Hamiltonian $\overline{H}$ as the Bellman-Isaacs Hamiltonian for a new *effective system*

$$\dot{x}_s = \overline{f}(x_s, \eta_s, \theta_s) \qquad x_s \in \mathbf{R}^n,\ \eta_s \in E(x_s),\ \theta_s \in \Theta(x_s)$$

and effective cost functional

$$\overline{J}(t,x,\eta,\theta) := \int_0^t \overline{l}(x_s,\eta_s,\theta_s)\, ds + \overline{h}(x_t).$$

This is a variational limit of the initial $n+m$-dimensional problem.
Step 2 is largely OPEN !

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Definition of $\overline{H}$: deterministic case $\sigma \equiv 0, \nu \equiv 0$

Consider the fast subsystem with frozen x and $\varepsilon = 1$

(FS) $$\dot{y}_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau), \qquad y_0 = y$$

and the family of value functions in $\mathbf{R}^m$ with parameters $x, p \in \mathbf{R}^n$

$$w(t, y; x, p) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \int_0^t L(y_\tau, \alpha[\beta]_\tau, \beta_\tau; x, p)\, d\tau,$$

$$L(y, \alpha, \beta; x, p) := p \cdot f(x, y, \alpha, \beta) + l(x, y, \alpha, \beta)$$

Say (FS) is ERGODIC if, for all x, p,

$$\lim_{t \to +\infty} \frac{w(t, y; x, p)}{t} = \text{constant (in } y\text{), uniformly in } y$$

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- Example: (FS) uncontrolled, i.e.

$$\dot{y}_\tau = g(x, y_\tau),$$

and ergodic in the classical sense with a UNIQUE INVARIANT MEASURE μ_x. Then

$$-\overline{H}(x,p) = \int_{\mathbf{R}^m} \min_{\alpha \in A} L(y, \alpha; x, p)\, d\mu_x(y).$$

- NO explicit formula for $\overline{H}$ in general!
- Definition of $\overline{H}$ in general non-deterministic case: fast subsystem

$$dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau)\, d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau)\, dW_\tau, \qquad y_0 = y,$$

$L = \text{trace}\left(M\sigma\sigma^T(x,y,\alpha,\beta)\right)/2 + \ldots$ (M a $n \times n$ symmetric matrix)

$$w(t, y; x, p, M) = \inf_{\alpha} \sup_{\beta} E\left[\int_0^t L\, d\tau\right], \quad \lim_{t \to +\infty} w/t =: -\overline{H}(x,p,M)$$

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$$w(t, y; x, p, M) = \inf_{\alpha} \sup_{\beta} E\left[\int_0^t L d\tau\right], \quad \lim_{t \to +\infty} w/t =: -\overline{H}(x, p, M)$$

- Example: (FS) uncontrolled, i.e.

$$\dot{y}_\tau = g(x, y_\tau),$$

and ergodic in the classical sense with a UNIQUE INVARIANT MEASURE μ_x. Then

$$-\overline{H}(x,p) = \int_{\mathbf{R}^m} \min_{\alpha\in A} L(y,\alpha;x,p)\, d\mu_x(y).$$

- NO explicit formula for $\overline{H}$ in general!
- Definition of $\overline{H}$ in general non-deterministic case: fast subsystem

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Weak convergence theorem

Meta-Theorem

Fast subsystem (FS) ergodic $\Longrightarrow$

$$u^\varepsilon(t,x,y) \to u(t,x) \quad \text{as } \varepsilon \to 0,$$

(in the sense of relaxed semilimits, or weak viscosity limits), and u solves (in viscosity sense)

$$\frac{\partial u}{\partial t} + \overline{H}\left(x, D_x u, D^2_{xx} u\right) = 0$$

This is in fact a Theorem if the fast variables y live on the torus $\mathbb{T}^M$ (i.e., all data are $\mathbb{Z}^m$- periodic in y), or in all $\mathbf{R}^M$ but the process

$$dy_\tau = g(x, y_\tau)\, d\tau + \nu(x, y_\tau)\, dW_\tau$$

is an UN-controlled NON-degenerate diffusion.

The effective initial data $\overline{h}$

For the Fast Subsystem with frozen x

(FS) $$dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau)\, d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau)\, dW_\tau, \qquad y_0 = y,$$

consider now the value function

$$v(t, y; x) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E\,[h(x, y_t)]$$

Definition: (FS) is STABILIZING (to a constant) for the cost h if, $\forall\, x$,

$$\lim_{t \to +\infty} v(t, y; x) = \text{constant (in } y\text{), uniformly in } y \;=: \overline{h}(x)$$

Convergence Theorem at $t = 0$

Fast subsystem (FS) stabilizing for the cost $h \implies$

$$\lim_{t \to 0} \lim_{\varepsilon \to 0} u^\varepsilon(t, x, y) = \overline{h}(x)$$

(in the sense of relaxed semilimits, or weak viscosity limits).

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Main convergence theorem

Corollary

Fast subsystem (FS) ERGODIC and STABILIZING $\Longrightarrow$

$\exists\ \overline{H}$ and $\overline{h}$ continuous, $\overline{H}$ degenerate elliptic, such that

$u^\varepsilon(t,x,y) \to u(t,x)$ as $\varepsilon \to 0$, u solution of

$$(\overline{\text{CP}}) \qquad \begin{cases} \frac{\partial u}{\partial t} + \overline{H}\left(x, D_x u, D^2_{xx} u\right) = 0 & \text{in} \quad (0,+\infty)\times \mathbf{R}^n, \\ u(0,x) = \overline{h}(x) & \text{in } \mathbf{R}^n. \end{cases}$$

If moreover, $\overline{H}$ is regular enough w.r.t. x, then $(\overline{\text{CP}})$ has a unique solution and

$$u^\varepsilon \to u \qquad \text{locally uniformly.}$$

Conclusion

The initial $(n+m)$-dimensional H-J-B-I equation is split into

- two *m*-dimensional ergodic-type problems (one for $\overline{H}$ and one for $\overline{h}$),
- a *n*-dimensional "effective" PDE

$\Longrightarrow$ we got the desired SEPARATION OF SCALES for the H-J-B-I equation.

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The next steps

Main remaining questions:

1. when is (FS) ergodic and stabilizing ?
2. can we find effective dynamics $\bar{f}, \bar{\sigma}$, running cost $\bar{l}$, and control constraints E, Θ associated to $\overline{H}$?

Some answers:

1. for bounded fast variables (related to homogenization):
   - uniformly nondegenerate fast subsystem (FS),
   - deterministic fast subsystem (FS) controllable by one player from each point to any other point in a uniformly bounded time,
   - under nonresonance conditions on the torus;

   for unbounded fast variables: uncontrolled diffusion processes with a unique invariant measure;
2. several examples but no general recipe.

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The case of uncontrolled fast variables

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, y_s) dW_s$$

$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s)\, dW_s,$$

$$J^\varepsilon(t, x, y, \alpha, \beta) := \int_0^t l(x_s, y_s, \alpha_s, \beta_s)\, ds + h(x_t, y_t)$$

Assume there exists a unique invariant measure μ_x of

(FS) $$dy_\tau = g(x, y_\tau)\, d\tau + \nu(x, y_\tau)\, dW_\tau,$$

Denote $\langle \phi \rangle(x) := \int \phi(x, y) d\mu_x(y)$. Then effective H and h are

$$\overline{h}(x) = \langle h \rangle(x)$$

$$\overline{H}(x, p, M) = \langle \min_{\beta \in B} \max_{\alpha \in A} \left\{ -\text{trace}(M\sigma\sigma^T)/2 - f \cdot p - l \right\} \rangle$$

Corollary

For split systems and cost, i.e.,

$$f = f_0(x, y) + f_1(x, \alpha, \beta), \quad l = l_0(x, y) + l_1(x, \alpha, \beta),$$

the linear averaging of the data is the correct limit, i.e.,

$$\lim_{\varepsilon \to 0} u^\varepsilon(t, x, y) = u(t, x) :=$$

$$\inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E\left[\int_0^t \langle l \rangle (x_s, \alpha[\beta]_s, \beta_s)\, ds + \langle h \rangle (x_t)\right],$$

$$dx_s = \langle f \rangle (x_s, \alpha[\beta]_s, \beta_s)\, ds + \langle \sigma\sigma^T \rangle^{1/2}(x_s)\, dW_s$$

Proof: $\overline{H}(x, p, M) = -\mathrm{trace}(M \langle \sigma\sigma^T \rangle)/2 + \min_B \max_A \{-\langle f \rangle \cdot p - \langle l \rangle\}$.

A similar result was proved by Kushner (book, 1990) for a single controller by probabilistic methods.

In general, for system or cost NOT split,

$$\overline{H}(x,p,M) = \langle \min_B \max_A \{...\} \rangle \neq \min_B \max_A \langle \{...\} \rangle$$

and the limit control problem is not obvious.
For some classical problems we derived the explicit form of the effective control problem:

- Merton portfolio optimization with stochastic volatility,
- Ramsey model of optimal economic growth with (fast) random parameters,
- Vidale - Wolfe advertising model with random parameters,
- advertising game in a duopoly with Lanchester dynamics and random parameters.

Often they involve a nonlinear average of some parameter.
E.g., the limit of Merton problem is still a Merton problem with constant volatility the harmonic average $\overline{\sigma} := \langle \sigma^{-2} \rangle^{-1/2}$.

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We can give a general representation formula for the effective control problem by enlarging the control set.
Assume for simplicity $\sigma \equiv 0$ (deterministic slow subsystem) and a single controller (B singleton).
Define $E_x := L^1((\mathbf{R}^m, \mu_x); A) \supset A$ and for $\tilde{\alpha} \in E_x$

$$\bar{f}(x, \tilde{\alpha}) := \int_{\mathbf{R}^m} f(x, y, \tilde{\alpha}(y))\, d\mu_x(y), \ \bar{l}(x, \tilde{\alpha}) := \int_{\mathbf{R}^m} l(x, y, \tilde{\alpha}(y))\, d\mu_x(y).$$

Then

$$\overline{H}(x, p) = \langle \max_{\alpha \in A} \{-f \cdot p - l\} \rangle = \max_{\tilde{\alpha} \in E_x} \left\{ -\bar{f}(x, \tilde{\alpha}) \cdot p - \bar{l}(x, \tilde{\alpha}) \right\}$$

and the effective control problem is

$$\dot{x}_s = \bar{f}(x_s, \tilde{\alpha}_s), \quad x_0 = x, \quad \tilde{\alpha}_s \in E_{x_s}$$
$$\min \quad \bar{J}(t, x, \tilde{\alpha}.) = \int_0^t \bar{l}(x_s, \tilde{\alpha}_s)\, ds + \langle h \rangle (x_t).$$

A one-dimensional homogenization problem

$$\dot{x}_s = g(x_s)\alpha_s, \quad J = \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right), \ -1 \le \alpha_s \le 1, \ g > 0.$$

Here $\dot{y}_s = \frac{1}{\varepsilon} g(x_s)\alpha_s$ depends on the control.

The value function $u^\varepsilon(t,x)$ solves the H-J equation

$$\frac{\partial u^\varepsilon}{\partial t} + g(x)\left|\frac{\partial u^\varepsilon}{\partial t}\right| = l\left(x, \frac{x_s}{\varepsilon}\right), \qquad u^\varepsilon(0,x) = h\left(x, \frac{x_s}{\varepsilon}\right).$$

For $l(x,\cdot), h(x,\cdot)$ 1-periodic and $\min l(x,\cdot) = 0$ the limit PDE is

$$\frac{\partial u}{\partial t} + \left(g(x)\left|\frac{\partial u}{\partial t}\right| - \langle l\rangle(x)\right)^+ = 0, \quad u(0,x) = \min_{y\in[0,1]} h(x,y).$$

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Homogenization of a 2-D differential game

$$\dot{x}_s = g_1(x_s, z_s)\alpha_s, \quad -1 \le \alpha_s \le 1,\ g_1 > 0$$

$$\dot{z}_s = g_2(x_s, z_s)\beta_s, \quad -1 \le \beta_s \le 1,\ g_2 > 0$$

$$J^\varepsilon(t, x, z, \alpha, \beta) := \int_0^t \left[l_1\left(x_s, z_s, \tfrac{x_s}{\varepsilon}\right) + l_2\left(x_s, z_s, \tfrac{z_s}{\varepsilon}\right)\right] ds + h\left(x_t, z_t, \tfrac{x_s}{\varepsilon}, \tfrac{z_s}{\varepsilon}\right)$$

$$\min_{[0,1]} l_1(x, z, \cdot) = 0,\ \max_{[0,1]} l_2(x, z, \cdot) = 0$$

and assume h has a saddle

$$\min_{\xi\in[0,1]} \max_{\eta\in[0,1]} h(x, z, \xi, \eta) = \max_{\eta\in[0,1]} \min_{\xi\in[0,1]} h(x, z, \xi, \eta) =: h_S(x, z)$$

The limit differential game has the same system and controls, and the effective cost-payoff

$$\overline{J} = \int_0^t [|\alpha_s| \langle l_1 \rangle (x_s, z_s) + |\beta_s| \langle l_2 \rangle (x_s, z_s)]\, ds + h_S(x_t, z_t)$$

joint work with G.Terrone.

Work in progress on homogenization of control systems and games:

- the last two examples extend to n-dimensional systems provided the oscillations are at 1-dimensional scale;
- an "abstract" representation of some effective control problems can be obtained by the limit occupational measures studied by Artstein, Gaitsgory, Borkar,..., (joint work with Gabriele TERRONE);
- homogenization of deterministic differential games is wide open: there are easy examples of non-convergence of the value functions and only a few known cases of convergence (see also Cardaliaguet '09)

Thanks for your attention !