Portfolio optimisation with Gaussian and non-Gaussian stochastic volatility

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joint work with

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Plan

- Financial models and Merton's optimisation problem
- Stochastic volatility, Gaussian or with jumps
- Systems with random parameters and multiple scales
- The Hamilton-Jacobi-Bellman approach to Singular Perturbations
  - Tools
  - Assumptions
- Convergence results
- Applications to financial models

Financial models and Merton's optimisation problem

The evolution of the price of a stock S is described by

$$d \log S_s = \gamma \, ds + \sigma \, dW_s, \qquad s = \text{time}, \; W_s = \text{Wiener proc.},$$

whereas a riskless bond B evolves with $\qquad d \log B_s = r \, ds.$

Invest u_s in the stock S_s, $1 - u_s$ in the bond B_s.

Then the wealth x_s evolves as

$$d \, x_s = (1 - u_s) r x_s \, ds + u_s x_s (\gamma \, ds + \sigma \, dW_s)$$

$$= (r + (\gamma - r) u_s) x_s \, ds + x_s u_s \, \sigma \, dW_s$$

and want to maximize the expected utility at time $T > 0$ starting at $t < T$, so the value function is

$$V(t, x) = \sup_{u.} E[g(x_T) \,|\, x_t = x]$$

for some utility g increasing and concave.

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The HJB equation is

$$-\frac{\partial V}{\partial t} - rxV_x - \max_u \left\{ (\gamma - r)uxV_x + \frac{u^2 x^2 \sigma^2}{2} V_{xx} \right\} = 0$$

Assume the utility g has $g' > 0$ and $g'' < 0$. Then expect a value function strictly increasing and concave in x, i.e., $V_x^\varepsilon > 0$, $V_{xx}^\varepsilon < 0$.
If $\max_u$ in HJB equation is attained in the interior of the constraint for u the equation becomes

$$-\frac{\partial V}{\partial t} - rxV_x + \frac{(\gamma - r)^2 V_x^2}{2\sigma^2 V_{xx}} = 0 \quad \text{in } (0, T) \times \mathbf{R}.$$

If $g(x) = ax^\delta/\delta$ with $a > 0$, $0 < \delta < 1$, a HARA function, the problem has the explicit solution

$$V(t,x) = a\frac{x^\delta}{\delta}e^{c(T-t)}, \quad c = \delta\left(r + \frac{(\gamma - r)^2}{4(1-\delta)\sigma}\right), \quad u^* = \frac{(\gamma - r)^2}{2(1-\delta)\sigma}$$

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Stochastic volatility

In reality the parameters of such models are not constants.
In particular, the volatility σ is not a constant, it rather looks like an ergodic mean-reverting stochastic process, see next slide.

Therefore it has been modeled as $\sigma = \sigma(y_s)$

with y_s either an Ornstein-Uhlenbeck diffusion process,

$$dy_s = -y_s\, ds + \tau\, d\tilde{W}_s$$

with $\tilde{W}_s$ a Wiener process possibly correlated with W_s,
Refs.: Hull-White 87, Heston 93, Fouque-Papanicolaou-Sircar 2000,...

or by a non-Gaussian process

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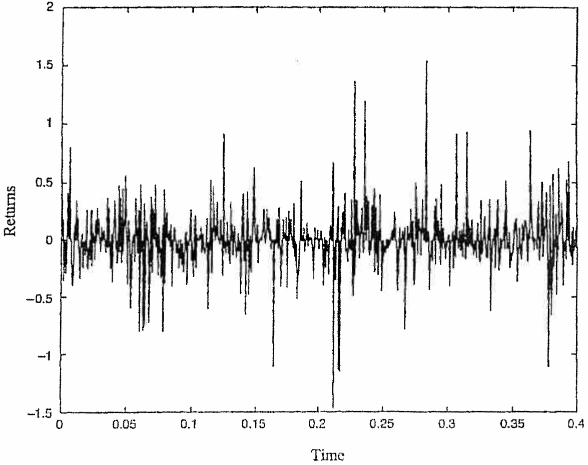

Figure 3.7. 1996 S&P 500 returns computed from half-hourly data.

The diffusion model (Gaussian) was used for many papers in finance, see the refs. in the book by Fleming - Soner, 2nd ed., 2006,

for Merton's problem it was studied by Fleming - Hernandez 03.

The non-Gaussian model was used for option pricing (Nicolato - Venerdos 03, Hubalek - Sgarra 09, 11)

and for portfolio optimisation by Benth - Karlsen - Reikvam 03.

Fast stochastic volatility

It is argued in the book

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that the process y_s also evolves on a faster time scale than the stock prices: this models better the typical bursty behavior of volatility, see previous picture.

The equations for the evolution of a stock S with fast stochastic volatility σ proposed in [FPS] are Gaussian, with $\varepsilon > 0$,

$$d \log S_s = \gamma \, ds + \sigma(y_s) \, dW_s$$

$$dy_s = -\frac{1}{\varepsilon} y_s + \frac{\tau}{\sqrt{\varepsilon}} d\tilde{W}_s$$

and they study the asymptotics $\varepsilon \to 0$ for many option pricing problems. We'll study also the non-Gaussian volatility

$$dy_s = -\frac{1}{\varepsilon} y_{s^-} ds + dZ_{s/\varepsilon}$$

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Two-scale control systems with random parameters

We consider control systems either of the form (Gaussian volatility)

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\begin{aligned}
dx_s &= f(x_s, y_s, u_s)\, ds + \sigma(x_s, y_s, u_s) dW_s \qquad x_s \in \mathbf{R}^n,\\
dy_s &= \tfrac{1}{\varepsilon} b(x_s, y_s)\, ds + \tfrac{1}{\sqrt{\varepsilon}} \tau(x_s, y_s)\, dW_s \qquad y_s \in \mathbf{R}^m,
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or of the form (Jump volatility)

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Basic assumptions

- f, σ, b, τ Lipschitz in (x, y) (unif. in u) with linear growth
- $Z.$ 1-dim. pure jump Lévy process, independent of $W.$, + conditions (later).

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Value function and HJB

$$V^\varepsilon(t,x,y) := \sup_{u.} E[e^{c(t-T)} g(x_T) \,|\, x_t = x, y_t = y]$$

with $\quad g : \mathbf{R}^n \to \mathbf{R}$ continuous, $\quad g(x) \le K(1+|x|^2), \quad c \ge 0.$

1. Gaussian case.

The value V^ε solves the (backward) HJB equation in $(0,T)\times\mathbf{R}^n\times\mathbf{R}^m$

$$-\frac{\partial V^\varepsilon}{\partial t} + \mathcal{H}\left(x, y, D_x V^\varepsilon, D^2_{xx} V^\varepsilon, \frac{1}{\sqrt{\varepsilon}} D^2_{xy} V^\varepsilon\right) - \frac{1}{\varepsilon}\mathcal{L} V^\varepsilon + cV^\varepsilon = 0$$

$$\mathcal{H}(x,y,p,M,Y) := \min_{u\in U}\left\{-\frac{1}{2}\mathrm{tr}(\sigma\sigma^T M) - f\cdot p - \mathrm{tr}(\sigma\tau Y^T)\right\}$$

$$\mathcal{L} := \mathrm{tr}(\tau\tau^T D^2_{yy}) + b\cdot D_y = \text{generator of } dy_s = b\,ds + \tau\,dW_s,$$

and the terminal condition $\quad V^\varepsilon(T,x,y) = g(x).$

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$$\mathcal{L}[y, v] := -y v_y(y) + \int_0^{+\infty} (v(z+y) - v(y) - v_y(y) z 1_{z \le 1}) d\nu(z)$$

is the generator of the unscaled volatility process $dy_s = -y_{s^-} ds + dZ_s$,

ν is the Lévy measure associated to the jump process $Z_\cdot$:

$$\nu(B) = E(\#\{s \in [0,1],\ Z_s - Z_{s^-} \neq 0,\ Z_s - Z_{s^-} \in B\})$$

$=$ expected number of jumps of a certain height in a unit-time interval.

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PDE approach to the singular limit $\varepsilon \to 0$

Search an *effective Hamiltonian* $\overline{H}$ such that

$$V^\varepsilon(t,x,y) \to V(t,x) \quad \text{as } \varepsilon \to 0,$$

V solution of

$$(\overline{\text{CP}}) \qquad \begin{cases} -\frac{\partial V}{\partial t} + \overline{H}\left(x, D_x V, D^2_{xx} V\right) + cV = 0 & \text{in } (0,T) \times \mathbf{R}^n, \\ V(T,x) = g(x) \end{cases}$$

Then, if possible, intepret the effective Hamiltonian $\overline{H}$ as the Bellman Hamiltonian for a new effective optimal control problem in $\mathbf{R}^n$,

which is therefore a variational limit of the initial $n+m$-dimensional problem.

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Tools

1. Ergodicity of the unscaled volatility process, or fast subsystem, i.e., of

$$dy_s = b(x, y_s)\, ds + \tau(x, y_s)\, dW_s, \; x \text{ frozen}, \quad \text{in the Gaussian case,}$$

$$dy_s = -y_{s^-}\, ds + dZ_s\ , \quad \text{in the non-Gaussian case.}$$

Assume conditions such that this process has a unique invariant probability measure μ_x and it is uniformly ergodic.

By solving an auxiliary (linear) PDE called cell problem we find that the candidate effective Hamiltonian is

$$\overline{H}(x, p, M) = \int_{\mathbf{R}^m} \mathcal{H}(x, y, p, M)\, d\mu_x(y).$$

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2. The generator $\mathcal{L}$ has the Liouville property
(based on the Strong Maximum Principle), i.e.

any bounded sub- or supersolution of $-\mathcal{L}[y, v] = 0$ is constant.

Then the relaxed semilimits

$$\underline{V}(t,x,y) := \liminf_{\varepsilon \to 0, t' \to t, x' \to x, y' \to y} V^{\varepsilon}(t', x', y'),$$

$\overline{V}(t,x,y) :=$ lim sup of the same, do not depend on y.

3. Perturbed test function method,
evolving from Evans (periodic homogenisation) and
Alvarez-M.B. (singular perturbations with bounded fast variables),
allows to prove that

$\underline{V}(t,x)$ is supersol., $\overline{V}(t,x)$ is subsol. of limit PDE in $(\overline{CP})$.

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4. Comparison principle

between a subsolution and a supersolution of the Cauchy problem ($\overline{\text{CP}}$) satisfying

$$|V(t,x)| \leq C(1+|x|^2),$$

see Da Lio - Ley 2006. It gives

- uniqueness of solution V of ($\overline{\text{CP}}$)
- $\underline{V}(t,x) \geq \overline{V}(t,x)$, then $\underline{V} = \overline{V} = V$ and, as $\varepsilon \to 0$,

$$V^\varepsilon(t,x,y) \to V(t,x) \quad \text{locally uniformly.}$$

Assumptions: 1. Gaussian case

The generator $\mathcal{L} = \text{tr}(\tau\tau^T D^2_{yy}) + b \cdot D_y$ of the volatility process satisfies

Ellipticity: $\exists\, \Lambda(y) > 0$ s.t. $\forall\, x \quad \tau(x,y)\tau^T(x,y) \geq \Lambda(y)\mathbb{I}$

Lyapunov condition: $\exists w \in \mathcal{C}(\mathbf{R}^m)$, $k > 0$, $R_0 > 0$ s.t.

$$-\mathcal{L}w \geq k \text{ for } |y| > R_0, \forall x, \qquad w(y) \to +\infty \text{ as } |y| \to +\infty.$$

Example: Ornstein-Uhlenbeck process $dy_s = -y_s ds + \nu(x) dW_s$, ν bounded, by choosing as Lyapunov function $w(y) = |y|^2$.

Then, $\forall x \in \mathbf{R}^n$ frozen, the unscaled volatility process is uniformly ergodic and $\mathcal{L}$ has the Liouville property.

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Assumptions: 2. Non-Gaussian case

The Lévy measure ν of the jump process $Z_\cdot$ satisfies

- $\exists\, C > 0,\ 0 < p < 2,\ 0 < \delta \le 1\ :\quad \int_{|z|\le\delta} |z|^2 \nu(dz) \ge C\,\delta^{2-p}$
- $\exists\, q > 0\ :\quad \int_{|z|>1} |z|^q \nu(dz) < +\infty.$

Then the unscaled volatility $dy_s = -y_{s^-}\,ds + dZ_s$ is uniformly ergodic (Kulik 2009). If, moreover,

- either $p > 1$,
- or $0 \in \text{int supp}(\nu)$,

then the integro-differential generator $\mathcal{L}$ of the process $y_\cdot$ has the Liouville property.

Examples: α-stable Lévy processes

- $\nu(dz) = |z|^{-1-\alpha} dz,\ 0 < \alpha < 2,\quad \mathcal{L} = (-\Delta)^{\alpha/2}$ fractional Lapl.
- $\nu(dz) = 1_{\{z \ge 0\}}(z)|z|^{-1-\alpha} dz,\ 1 < \alpha < 2$, no negative jumps.

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Assumptions: 2. Non-Gaussian case

The Lévy measure ν of the jump process $Z_\cdot$ satisfies

- $\exists\, C > 0,\ 0 < p < 2,\ 0 < \delta \le 1\ :\quad \int_{|z|\le\delta} |z|^2 \nu(dz) \ge C\,\delta^{2-p}$
- $\exists\, q > 0\ :\quad \int_{|z|>1} |z|^q \nu(dz) < +\infty.$

Then the unscaled volatility $dy_s = -y_{s^-}\,ds + dZ_s$ is uniformly ergodic (Kulik 2009). If, moreover,

- either $p > 1$,
- or $0 \in \text{int}\ \text{supp}(\nu)$,

then the integro-differential generator $\mathcal{L}$ of the process $y_\cdot$ has the Liouville property.

Examples: α-stable Lévy processes

- $\nu(dz) = |z|^{-1-\alpha} dz,\ 0 < \alpha < 2, \quad \mathcal{L} = (-\Delta)^{\alpha/2}$ fractional Lapl.
- $\nu(dz) = 1_{\{z\ge 0\}}(z)|z|^{-1-\alpha} dz,\ 1 < \alpha < 2$, no negative jumps.

Convergence Theorems

$$\lim_{\varepsilon \to 0} V^\varepsilon(t, x, y) = V(t, x) \quad \text{locally uniformly,}$$

V solving

$$-\frac{\partial V}{\partial t} + \int_{\mathbf{R}^m} \mathcal{H}\left(x, y, D_x u, D^2_{xx} u, 0\right) \, d\mu_x(y) = 0 \quad \text{in } (0, T) \times \mathbf{R}^n$$

with $\quad V(T, x) = g(x), \quad$ if

- Gaussian volatility case with b, τ independent of x [M.B. - Cesaroni - Manca, SIAM J. Financial Math. 2010],
- Gaussian volatility case with $b, \tau \in \mathcal{C}^{1,\alpha}$ with bounded derivatives [M.B. - Cesaroni, Eur. J. Control 2011],
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Merton's problem with stochastic volatility

Now the wealth x_s evolves with

$$d\, x_s = (r + (\gamma - r)u_s)x_s\, ds + x_s u_s\, \sigma(y_s)\, dW_s, \quad x_t = x,$$

and y_s is either Gaussian

$$dy_s = \frac{1}{\varepsilon} b(y_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \tau(y_s)\, d\tilde{W}_s \qquad y_t = y,$$

with $\rho =$ possible correlation of W_s and $\tilde{W}_s$,

or with jumps

$$dy_s = -\frac{1}{\varepsilon} y_{s^-}\, ds + dZ_{s/\varepsilon} \qquad y_t = y.$$

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$$-\frac{\partial V^\varepsilon}{\partial t} - rxV^\varepsilon_x + \frac{[(\gamma - r)V^\varepsilon_x]^2}{\sigma^2(y)2V^\varepsilon_{xx}} = \frac{1}{\varepsilon}\mathcal{L}[y, V^\varepsilon]$$

Rmk.: in the Gaussian case, if $\rho \neq 0$ there is also a term $+\frac{x\rho\sigma\nu}{\sqrt{\varepsilon}} V^\varepsilon_{xy}$ in the $[\dots]^2$.

By the Theorem, $V^\varepsilon(t,x,y) \to V(t,x)$ as $\varepsilon \to 0$ and V solves

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This is the HJB equation of a Merton problem with constant volatility $\overline{\sigma}$

$$\frac{1}{\overline{\sigma}^2} = \int \frac{1}{\sigma^2(y)} d\mu(y) \quad \Rightarrow \quad \overline{\sigma} = \left(\int \frac{1}{\sigma^2(y)} d\mu(y)\right)^{-1/2}.$$

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A practical consequence

In financial problems without controls, e.g., Black-Scholes formula for option pricing, $\mathcal{H}$ is linear (no $\max_u$ involved), so the effective volatility arising in the limit of fast stochastic volatility is the linear average

$$\tilde{\sigma}^2 := \int \sigma^2(y)\mu(dy) \geq \overline{\sigma}^2 = \left(\int \frac{1}{\sigma^2(y)}\, d\mu(y) \right)^{-1}$$

Then if one uses a constant-parameter model as approximation, the nonlinear average $\overline{\sigma}$ is better, it increases the optimal expected utility.

Other problems with (fast) random parameters where an explicit form of the effective control problem can be computed [M.B. - Cesaroni 11]

- Ramsey model of optimal economic growth
- Vidale - Wolfe advertising model
- advertising game in a duopoly with Lanchester dynamics

Often they involve a nonlinear average of some parameter!

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Further results and perspectives

Can treat also

- limit of the optimal feedback,
- utility depending on y, i.e., $g = g(x, y)$, then the effective terminal condition is $V(T, x) = \int g(x, y) d\mu_x(y)$,
- problems with two conflicting controllers, i.e., two-person, 0-sum, stochastic differential games,
- systems with more than two scales.

Developments under investigation:

- more general jump processes for the volatility (without the Liouville property...), e.g., "inverse Gaussian",
- jump terms in the stocks dynamics,
- large deviations for short maturity asymptotics, done with Cesaroni and Ghilli for option pricing models (no control).

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Thanks for your attention!

Best wishes Halina and Hector!