Portfolio optimisation with Gaussian and non-Gaussian stochastic volatility

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joint work with

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Analysis and Geometry in Control Theory and its Applications
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Financial models and Merton’s optimisation problem

Stochastic volatility, Gaussian or with jumps

Systems with random parameters and multiple scales

The Hamilton-Jacobi-Bellman approach to Singular Perturbations
  ▶ Tools
  ▶ Assumptions

Convergence results

Applications to financial models
Financial models and Merton’s optimisation problem

The evolution of the price of a stock $S$ is described by

$$d \log S_s = \gamma ds + \sigma dW_s, \quad s = \text{time}, \quad W_s = \text{Wiener proc.},$$

whereas a riskless bond $B$ evolves with

$$d \log B_s = r ds.$$

Invest $u_s$ in the stock $S_s$, $1 - u_s$ in the bond $B_s$. Then the wealth $x_s$ evolves as

$$d x_s = (1 - u_s)rx_s ds + u_s x_s (\gamma ds + \sigma dW_s)$$

$$= (r + (\gamma - r)u_s)x_s ds + x_s u_s \sigma dW_s$$

and want to maximize the expected utility at time $T > 0$ starting at $t < T$, so the value function is

$$V(t, x) = \sup_{u} E[g(x_T) | x_t = x]$$

for some utility $g$ increasing and concave.
Financial models and Merton’s optimisation problem

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$$V(t, x) = \sup_u E[g(x_T) | x_t = x]$$

for some utility $g$ increasing and concave.
The HJB equation is

\[-\frac{\partial V}{\partial t} - rxV_x - \max_u \left\{ (\gamma - r)uxV_x + \frac{u^2x^2\sigma^2}{2} V_{xx} \right\} = 0\]

Assume the utility $g$ has $g' > 0$ and $g'' < 0$. Then expect a value function strictly increasing and concave in $x$, i.e., $V_x^\epsilon > 0$, $V_{xx}^\epsilon < 0$.

If $\max_u$ in HJB equation is attained in the interior of the constraint for $u$ the equation becomes

\[-\frac{\partial V}{\partial t} - rxV_x + \frac{(\gamma - r)^2 V_x^2}{2\sigma^2 V_{xx}} = 0 \quad \text{in } (0, T) \times \mathbb{R}.

If $g(x) = ax^\delta/\delta$ with $a > 0$, $0 < \delta < 1$, a HARA function, the problem has the explicit solution

$$V(t, x) = a\frac{x^\delta}{\delta} e^{c(T-t)}, \quad c = \delta(r + \frac{(\gamma - r)^2}{4(1-\delta)\sigma}), \quad u^* = \frac{(\gamma - r)^2}{2(1-\delta)\sigma}.$$
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Stochastic volatility

In reality the parameters of such models are not constants. In particular, the volatility $\sigma$ is not a constant, it rather looks like an ergodic mean-reverting stochastic process, see next slide. Therefore it has been modeled as $\sigma = \sigma(y_s)$ with $y_s$ either an Ornstein-Uhlenbeck diffusion process,

$$dy_s = -y_s \, ds + \tau \, d\tilde{W}_s$$

with $\tilde{W}_s$ a Wiener process possibly correlated with $W_s$, Refs.: Hull-White 87, Heston 93, Fouque-Papanicolaou-Sircar 2000,... or by a non-Gaussian process

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where $Z_s$ is a pure jump Lévy process with positive increments, Refs.: Barndorff-Nielsen and Shephard 2001.
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3.4 Scales in the Returns Process

Figure 3.7. 1996 S&P 500 returns computed from half-hourly data.
The diffusion model (Gaussian) was used for many papers in finance, see the refs. in the book by Fleming - Soner, 2nd ed., 2006, for Merton’s problem it was studied by Fleming - Hernandez 03.

The non-Gaussian model was used for option pricing (Nicolato - Venerdos 03, Hubalek - Sgarra 09, 11) and for portfolio optimisation by Benth - Karlsen - Reikvam 03.
Fast stochastic volatility

It is argued in the book

Fouque, Papanicolaou, Sircar: Derivatives in financial markets with stochastic volatility, 2000,

that the process $y_s$ also evolves on a faster time scale than the stock prices: this models better the typical bursty behavior of volatility, see previous picture.

The equations for the evolution of a stock $S$ with fast stochastic volatility $\sigma$ proposed in [FPS] are Gaussian, with $\varepsilon > 0$,

\[
\begin{align*}
    d \log S_s &= \gamma \, ds + \sigma(y_s) \, dW_s \\
    dy_s &= -\frac{1}{\varepsilon} \, y_s + \frac{\tau}{\sqrt{\varepsilon}} \, d\tilde{W}_s
\end{align*}
\]

and they study the asymptotics $\varepsilon \to 0$ for many option pricing problems. We’ll study also the non-Gaussian volatility

\[
    dy_s = -\frac{1}{\varepsilon} \, y_s \, ds + dZ_{s/\varepsilon}
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and they study the asymptotics $\varepsilon \to 0$ for many option pricing problems. We’ll study also the non-Gaussian volatility

$$dy_s = -\frac{1}{\varepsilon} y_s - ds + dZ_{s/\varepsilon}$$
Two-scale control systems with random parameters

We consider control systems either of the form (Gaussian volatility)

\[ dx_s = f(x_s, y_s, u_s) \, ds + \sigma(x_s, y_s, u_s) \, dW_s \quad x_s \in \mathbb{R}^n, \]
\[ dy_s = \frac{1}{\varepsilon} b(x_s, y_s) \, ds + \frac{1}{\sqrt{\varepsilon}} \tau(x_s, y_s) \, dW_s \quad y_s \in \mathbb{R}^m, \]

or of the form (Jump volatility)

\[ dx_s = f(x_s, y_{s-}, u_s) \, ds + \sigma(x_s, y_{s-}, u_s) \, dW_s \quad x_s \in \mathbb{R}^n, \]
\[ dy_s = -\frac{1}{\varepsilon} y_{s-} \, ds + dZ_{s/\varepsilon} \quad y_s \in \mathbb{R} \]

Basic assumptions

- \( f, \sigma, b, \tau \) Lipschitz in \((x, y)\) (unif. in \(u\)) with linear growth
- \( Z \). 1-dim. pure jump Lévy process, independent of \( W \).
  + conditions (later).
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Value function and HJB

\[
V^\varepsilon(t, x, y) := \sup_u E[e^{c(t-T)}g(x_T) \mid x_t = x, y_t = y]
\]

with \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) continuous, \( g(x) \leq K(1 + |x|^2) \), \( c \geq 0 \).

1. Gaussian case.
The value \( V^\varepsilon \) solves the (backward) HJB equation in \((0, T) \times \mathbb{R}^n \times \mathbb{R}^m\)

\[
-\frac{\partial V^\varepsilon}{\partial t} + \mathcal{H} \left( x, y, D_x V^\varepsilon, D_{xx}^2 V^\varepsilon, \frac{1}{\sqrt{\varepsilon}} D_{xy}^2 V^\varepsilon \right) - \frac{1}{\varepsilon} \mathcal{L} V^\varepsilon + c V^\varepsilon = 0
\]

\[
\mathcal{H} (x, y, p, M, Y) := \min_{u \in U} \left\{ -\frac{1}{2} \text{tr}(\sigma \sigma^T M) - f \cdot p - \text{tr}(\sigma \tau Y^T) \right\}
\]

\[
\mathcal{L} := \text{tr}(\tau \tau^T D_{yy}^2) + b \cdot D_y = \text{generator of } dy_s = b \, ds + \tau \, dW_s,
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and the terminal condition \( V^\varepsilon(T, x, y) = g(x) \).
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and the terminal condition \( V^{\varepsilon}(T, x, y) = g(x) \).
2. Non-Gaussian case.

The value $V^\varepsilon$ solves the **integro-differential** HJB equation in $(0, T) \times \mathbb{R}^n \times \mathbb{R}$

$$- \frac{\partial V^\varepsilon}{\partial t} + \mathcal{H} \left( x, y, D_x V^\varepsilon, D_{xx}^2 V^\varepsilon, 0 \right) - \frac{1}{\varepsilon} \mathcal{L}[y, V^\varepsilon] + c V^\varepsilon = 0,$$

$$\mathcal{L}[y, v] := -yv_y(y) + \int_0^{+\infty} (v(z + y) - v(y) - v_y(y)z1_{z \leq 1})d\nu(z)$$

is the **generator** of the unscaled volatility process $dy_s = -y_s ds + dZ_s$,

$\nu$ is the Lévy measure associated to the jump process $Z$.

$$\nu(B) = E(\#\{s \in [0, 1], Z_s - Z_{s^-} \neq 0, Z_s - Z_{s^-} \in B\})$$

$=$ **expected number of jumps of a certain height in a unit-time interval.**
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$$\nu(B) = E(\#\{s \in [0, 1], Z_s - Z_{s^-} \neq 0, Z_s - Z_{s^-} \in B\})$$

$$= \text{expected number of jumps of a certain height in a unit-time interval.}$$
Search an *effective Hamiltonian* $\overline{H}$ such that

$$V^\varepsilon(t, x, y) \to V(t, x) \quad \text{as} \quad \varepsilon \to 0,$$

$V$ solution of

$$\begin{cases} 
- \frac{\partial V}{\partial t} + \overline{H}(x, D_x V, D_{xx}^2 V) + cV = 0 & \text{in} \ (0, T) \times \mathbb{R}^n, \\
V(T, x) = g(x) \end{cases}$$

(\text{CP})

Then, if possible, interpret the effective Hamiltonian $\overline{H}$ as the Bellman Hamiltonian for a new *effective optimal control problem* in $\mathbb{R}^n$, which is therefore a variational limit of the initial $n + m$-dimensional problem.
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Then, if possible, interpret the effective Hamiltonian $\overline{H}$ as the Bellman Hamiltonian for a new effective optimal control problem in $\mathbb{R}^n$, which is therefore a variational limit of the initial $n + m$-dimensional problem.
1. Ergodicity of the unscaled volatility process, or fast subsystem, i.e., of

\[ dy_s = b(x, y_s) \, ds + \tau(x, y_s) \, dW_s, \quad x \text{ frozen,} \]  

in the \textbf{Gaussian} case,

\[ dy_s = -y_s \, ds + dZ_s, \]  

in the \textbf{non-Gaussian} case.

Assume conditions such that this process has a unique invariant probability measure \( \mu_x \) and it is \textbf{uniformly ergodic}.

By solving an auxiliary (linear) PDE called \textbf{cell problem} we find that the candidate effective Hamiltonian is

\[
\overline{H}(x, p, M) = \int_{\mathbb{R}^m} H(x, y, p, M) \, d\mu_x(y).
\]
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Assume conditions such that this process has a unique invariant probability measure \( \mu_x \) and it is uniformly ergodic.

By solving an auxiliary (linear) PDE called cell problem we find that the candidate effective Hamiltonian is

\[ \overline{H}(x, p, M) = \int_{\mathbb{R}^m} H(x, y, p, M) \, d\mu_x(y). \]
2. The generator $\mathcal{L}$ has the Liouville property (based on the Strong Maximum Principle), i.e.

any bounded sub- or supersolution of $-\mathcal{L}[y, v] = 0$ is constant.

Then the relaxed semilimits

$$\underline{V}(t, x, y) := \liminf_{\varepsilon \to 0, t' \to t, x' \to x, y' \to y} V^\varepsilon(t', x', y'),$$

$$\overline{V}(t, x, y) := \limsup \text{ of the same, do not depend on } y.$$

3. Perturbed test function method, evolving from Evans (periodic homogenisation) and Alvarez-M.B. (singular perturbations with bounded fast variables), allows to prove that

$\underline{V}(t, x)$ is supersol., $\overline{V}(t, x)$ is subsol. of limit PDE in (CP).
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$$\underline{V}(t, x, y) := \lim_{\varepsilon \to 0, t' \to t, x' \to x, y' \to y} \inf \ V^\varepsilon(t', x', y'),$$
$$\overline{V}(t, x, y) := \lim_{\varepsilon \to 0, t' \to t, x' \to x, y' \to y} \sup \ V^\varepsilon(t', x', y'),$$

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$\underline{V}(t, x)$ is supersol., $\overline{V}(t, x)$ is subsol. of limit PDE in (CP).
4. **Comparison principle**
between a subsolution and a supersolution of the Cauchy problem (CP) satisfying
\[ |V(t, x)| \leq C(1 + |x|^2), \]
see Da Lio - Ley 2006. It gives

- uniqueness of solution \( V \) of (CP)
- \( V(t, x) \geq \overline{V}(t, x) \), then \( V = \overline{V} = V \) and, as \( \varepsilon \to 0 \),
  \[ V^\varepsilon(t, x, y) \to V(t, x) \text{ locally uniformly}. \]
Assumptions: 1. Gaussian case

The generator $\mathcal{L} = \text{tr}(\tau \tau^T D^2_{yy}) + b \cdot D_y$ of the volatility process satisfies

**Ellipticity:** $\exists \Lambda(y) > 0$ s.t. $\forall x \quad \tau(x, y) \tau^T(x, y) \geq \Lambda(y)I$

**Lyapunov condition:** $\exists w \in C(\mathbb{R}^m), \quad k > 0, \quad R_0 > 0$ s.t.

$$-\mathcal{L}w \geq k \quad \text{for } |y| > R_0, \quad \forall x, \quad w(y) \to +\infty \quad \text{as } |y| \to +\infty.$$

Example: Ornstein-Uhlenbeck process  
$$dy_s = -y_s ds + \nu(x) dW_s,$$  
$\nu$ bounded, by choosing as Lyapunov function $w(y) = |y|^2$.

Then, $\forall x \in \mathbb{R}^n$ frozen, the unscaled volatility process is uniformly ergodic and $\mathcal{L}$ has the Liouville property.
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Then, $\forall x \in \mathbb{R}^n$ frozen, the unscaled volatility process is uniformly ergodic and $\mathcal{L}$ has the Liouville property.
Assumptions: 2. Non-Gaussian case

The Lévy measure $\nu$ of the jump process $Z$ satisfies

- $\exists C > 0$, $0 < p < 2$, $0 < \delta \leq 1$ : $\int_{|z| \leq \delta} |z|^2 \nu(dz) \geq C \delta^{2-p}$
- $\exists q > 0$ : $\int_{|z| > 1} |z|^q \nu(dz) < +\infty$.

Then the unscaled volatility $dy_s = -y_s ds + dZ_s$ is uniformly ergodic (Kulik 2009). If, moreover,

- either $p > 1$,
- or $0 \in \text{int supp}(\nu)$,

then the integro-differential generator $\mathcal{L}$ of the process $y_s$ has the Liouville property.

Examples: $\alpha$-stable Lévy processes

- $\nu(dz) = |z|^{-1-\alpha}dz$, $0 < \alpha < 2$, $\mathcal{L} = (-\Delta)^{\alpha/2}$ fractional Lapl.
- $\nu(dz) = 1_{\{z \geq 0\}}(z)|z|^{-1-\alpha}dz$, $1 < \alpha < 2$, no negative jumps.
Assumptions: 2. Non-Gaussian case

The Lévy measure $\nu$ of the jump process $Z_s$ satisfies
\[\exists C > 0, 0 < p < 2, 0 < \delta \leq 1 : \int_{|z| \leq \delta} |z|^2 \nu(dz) \geq C \delta^{2-p}\]
\[\exists q > 0 : \int_{|z| > 1} |z|^q \nu(dz) < +\infty.\]

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then the integro-differential generator $\mathcal{L}$ of the process $y_s$ has the Liouville property.

Examples: $\alpha$-stable Lévy processes
\[\nu(dz) = |z|^{-1-\alpha} dz, \ 0 < \alpha < 2, \ \mathcal{L} = \left(-\Delta\right)^{\alpha/2} \text{ fractional Lapl.}\]
\[\nu(dz) = 1_{\{z \geq 0\}}(z)|z|^{-1-\alpha} dz, \ 1 < \alpha < 2, \ \text{no negative jumps}.\]
Assumptions: 2. Non-Gaussian case

The Lévy measure $\nu$ of the jump process $Z$ satisfies

- $\exists C > 0, 0 < p < 2, 0 < \delta \leq 1 : \int_{|z|\leq\delta} |z|^2\nu(dz) \geq C \delta^{2-p}$
- $\exists q > 0 : \int_{|z|>1} |z|^q\nu(dz) < +\infty$.

Then the unscaled volatility $dy_s = -y_s ds + dZ_s$ is uniformly ergodic (Kulik 2009). If, moreover,

- either $p > 1$,
- or $0 \in \text{int } \text{supp}(\nu)$,

then the integro-differential generator $\mathcal{L}$ of the process $y$ has the Liouville property.

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Convergence Theorems

\[ \lim_{\varepsilon \to 0} V^\varepsilon(t, x, y) = V(t, x) \]  locally uniformly,

\[ V \text{ solving} \]

\[-\frac{\partial V}{\partial t} + \int_{\mathbb{R}^m} \mathcal{H} \left( x, y, D_x u, D^2_{xx} u, 0 \right) d\mu_x(y) = 0 \]  in \((0, T) \times \mathbb{R}^n\)

with \(V(T, x) = g(x),\)  if

- Gaussian volatility case with \(b, \tau\) independent of \(x\) [M.B. - Cesaroni - Manca, SIAM J. Financial Math. 2010],
- Gaussian volatility case with \(b, \tau \in C^{1, \alpha}\) with bounded derivatives [M.B. - Cesaroni, Eur. J. Control 2011],
- Non-Gaussian volatility case [M.B. - Cesaroni - Scotti 2014].
Convergence Theorems

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Merton’s problem with stochastic volatility

Now the wealth $x_s$ evolves with

$$d x_s = \left( r + (\gamma - r) u_s \right) x_s \, ds + x_s u_s \sigma(y_s) \, dW_s, \quad x_t = x,$$

and $y_s$ is either Gaussian

$$dy_s = \frac{1}{\varepsilon} b(y_s) \, ds + \frac{1}{\sqrt{\varepsilon}} \tau(y_s) \, d\tilde{W}_s \quad y_t = y,$$

with $\rho = \text{possible correlation of } W_s \text{ and } \tilde{W}_s,$

or with jumps

$$dy_s = -\frac{1}{\varepsilon} y_s - ds + dZ_{s/\varepsilon} \quad y_t = y.$$
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$$dy_s = -\frac{1}{\varepsilon} y_s^{-} \, ds + dZ_{s/\varepsilon} \quad y_t = y.$$
Then \( V^\varepsilon(t, x, y) := \sup_u E[g(x_T)] \) solves
\[
- \frac{\partial V^\varepsilon}{\partial t} - rxV_x^\varepsilon + \frac{[(\gamma - r)V_x^\varepsilon]^2}{\sigma^2(y)2V_{xx}^\varepsilon} = \frac{1}{\varepsilon} \mathcal{L}[y, V^\varepsilon]
\]

Rmk.: in the Gaussian case, if \( \rho \neq 0 \) there is also a term \( +\frac{\chi \rho \sigma \nu}{\sqrt{\varepsilon}} V_{xy}^\varepsilon \) in the \([\ldots]^2\).

By the Theorem, \( V^\varepsilon(t, x, y) \rightarrow V(t, x) \) as \( \varepsilon \rightarrow 0 \) and \( V \) solves
\[
- \frac{\partial V}{\partial t} - rxV_x + \frac{(\gamma - r)^2 V_x^2}{2V_{xx}} \int \frac{1}{\sigma^2(y)} d\mu(y) = 0 \quad \text{in } (0, T) \times \mathbb{R}.
\]

This is the HJB equation of a Merton problem with constant volatility \( \sigma \)
\[
\frac{1}{\sigma^2} = \int \frac{1}{\sigma^2(y)} d\mu(y) \quad \Rightarrow \quad \sigma = \left( \int \frac{1}{\sigma^2(y)} d\mu(y) \right)^{-1/2}.
\]

Then limit control problem has volatility \( \bar{\sigma} = \text{harmonic average of } \sigma(\cdot) \).
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\]

Then limit control problem has volatility \( \overline{\sigma} = \) harmonic average of \( \sigma(\cdot) \).
A practical consequence

In financial problems without controls, e.g., Black-Scholes formula for option pricing, $H$ is linear (no max $_u$ involved), so the effective volatility arising in the limit of fast stochastic volatility is the linear average

$$\tilde{\sigma}^2 := \int \sigma^2(y) \mu(dy) \geq \bar{\sigma}^2 = \left( \int \frac{1}{\sigma^2(y)} d\mu(y) \right)^{-1}$$

Then if one uses a constant-parameter model as approximation, the nonlinear average $\sigma$ is better, it increases the optimal expected utility.

Other problems with (fast) random parameters where an explicit form of the effective control problem can be computed [M.B. - Cesaroni 11]

- Ramsey model of optimal economic growth
- Vidale - Wolfe advertising model
- advertising game in a duopoly with Lanchester dynamics

Often they involve a nonlinear average of some parameter!
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Further results and perspectives

Can treat also

- limit of the optimal feedback,

- utility depending on $y$, i.e., $g = g(x, y)$, then the effective terminal condition is $V(T, x) = \int g(x, y) d\mu_x(y)$,

- problems with two conflicting controllers, i.e., two-person, 0-sum, stochastic differential games,

- systems with more than two scales.

Developments under investigation:

- more general jump processes for the volatility (without the Liouville property...), e.g., "inverse Gaussian",

- jump terms in the stocks dynamics,

- large deviations for short maturity asymptotics, done with Cesaroni and Ghilli for option pricing models (no control).
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Thanks for your attention!

Best wishes Halina and Hector!