Uniqueness and non-uniqueness in Mean-Field Games systems of PDEs

Martino Bardi

Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova

INdAM workshop "New Trends in Control Theory and PDEs"

Dedicated to Piermarco Cannarsa on his 60th birthday

Roma, July 3-7, 2017

Plan

- A review of uniqueness of solutions for evolutive MFG PDEs:
  1. the "monotonicity" regime, as in Lasry-Lions
  2. the "small time-horizon" regime, as in a lecture of Lions (2009) revisited by M.B. and M. Fischer
- Non-uniqueness of solutions for evolutive MFG PDEs: explicit examples for
  1. any time horizon $T > 0$, non-smooth Hamiltonian H
  2. T not too small, smooth H

  joint work with Markus Fischer (Padova)
- MFGs with several populations
  joint work with Marco Cirant (Padova)

Mean Field differential games

Consider a large population of identical players, a representative agent has dynamics

$$dX_s = \alpha_s \, ds + \sigma \, dW_s, \quad X_t = x \in \mathbf{R}^d$$

with W_s a Brownian motion, $\alpha_s =$ control, $\sigma > 0$ volatility.

We are given running and terminal costs $F, G : \mathbf{R}^d \times \mathcal{P}(\mathbf{R}^d) \to \mathbf{R}$ and the finite horizon cost functional is

$$E\left[\int_t^T L(X_s, \alpha_s) + F(X_s, m(s, \cdot))ds + G(X_T, m(T, \cdot))\right],$$

with $L(x, \alpha)$ a convex Lagrangian superlinear in α, and $m(s, \cdot)$ is the density of the whole population of players at time s.

MFGs describe the equilibrium configuration where all players behave optimally and then the overall density m coincides with the density of a representative agent using an optimal feedback.

The backward-forward HJB - KFP system of PDEs

$$\begin{cases} -v_t + H(x, Dv) = \frac{\sigma^2(x)}{2}\Delta v + F(x, m(t,\cdot)) \text{ in } (0,T) \times \mathbf{R}^d, \\ v(T,x) = G(x, m(T,\cdot)) \\ m_t - div(D_pH(x, Dv)m) = \Delta\left(\frac{\sigma^2(x)}{2} m\right) \quad \text{ in } (0,T) \times \mathbf{R}^d, \\ m(0,x) = \nu(x), \end{cases}$$

where

- v is the value function of a representative agent,
- $m(\cdot, t) \in \mathcal{P}(\mathbf{R}^d)$ is the density of the population of agents,
- the Hamiltonian H is the convex conjugate of the Lagrangian L,
- $D_pH(x, Dv)$ is the optimal feedback for the representative agent.

- Boundary conditions:

  - most theory deals with periodic B.C.,

  we are more interested in

  - problem in all $\mathbf{R}^d$ with growth conditions or integrabilty conditions at infinity, or

  - Neumann boundary conditions in a bounded smooth domain.

- Existence results and regularity:

  - Lasry - Lions (2006 -...),

  - Cardaliaguet, Porretta,

  - Gomes and coworkers

- Boundary conditions:

  - most theory deals with periodic B.C.,

  we are more interested in

  - problem in all $\mathbf{R}^d$ with growth conditions or integrabilty conditions at infinity, or

  - Neumann boundary conditions in a bounded smooth domain.

- Existence results and regularity:

  - Lasry - Lions (2006 -...),

  - Cardaliaguet, Porretta,

  - Gomes and coworkers

The Lasry-Lions monotonicity condition

A sufficient condition for uniqueness of classical solutions is

$$p \to H(x,p) \ \ \textit{convex}$$

$$\int_{\mathbf{R}^n} [F(x,m) - F(x,\bar{m})] d(m-\bar{m})(x) > 0, \ \forall\, m \neq \bar{m} \in \mathcal{P}(\mathbf{R}^d)$$

$$\int_{\mathbf{R}^n} [G(x,m) - G(x,\bar{m})] d(m-\bar{m})(x) \geq 0, \ \forall\, m, \bar{m} \in \mathcal{P}(\mathbf{R}^d)$$

the costs are "increasing with the density" in L^2.
(See Cardaliaguet's notes for the proof)

Example

F is "local", i.e., $F(\cdot,m)(x) = f(x,m(x))$ and f is increasing in $m(x)$: the more is crowded the place where I am, the more I pay.

A non-local example

Notation: Mean of $\mu \in \mathcal{P}_1(\mathbb{R}^n)$, $\quad M(\mu) := \int_{\mathbb{R}} y\, \mu(dy)$.
Variant of the L-L uniqueness result: replace the strict monotonicity of F with: F and G depend on m only via $M(m)$ and

$$\int_{\mathbf{R}^n} [F(x,m) - F(x,\bar{m})] d(m-\bar{m})(x) > 0, \ \forall\, M(m) \neq M(\bar{m})$$

Example

$$F(x,\mu) = \beta x M(\mu), \quad G(x,\mu) = \gamma x M(\mu)$$

$\beta, \gamma \in \mathbf{R}$. Then

$\int_{\mathbf{R}^n} [F(x,m) - F(x,\bar{m})] d(m-\bar{m})(x) = \beta (M(m) - M(\bar{m}))^2 \geq 0,$

and the condition above is satisfied if $\quad \beta > 0, \ \gamma \geq 0$.

A non-local example

Notation: Mean of $\mu \in \mathcal{P}_1(\mathbb{R}^n)$, $\quad M(\mu) := \int_{\mathbb{R}} y\,\mu(dy)$.
Variant of the L-L uniqueness result: replace the strict monotonicity of F with: F and G depend on m only via $M(m)$ and

$$\int_{\mathbf{R}^n} [F(x,m) - F(x,\bar{m})] d(m - \bar{m})(x) > 0, \ \forall\, M(m) \neq M(\bar{m})$$

Example

$$F(x,\mu) = \beta x M(\mu), \quad G(x,\mu) = \gamma x M(\mu)$$

$\beta, \gamma \in \mathbf{R}$. Then

$\int_{\mathbf{R}^n} [F(x,m) - F(x,\bar{m})] d(m - \bar{m})(x) = \beta (M(m) - M(\bar{m}))^2 \geq 0,$

and the condition above is satisfied if $\beta > 0, \ \gamma \geq 0$.

Short time uniqueness, Lions at C.d.F. 1.9.2009

Assume for simplicity $H = H(p)$ only.
For two solutions $(v_1, m_1), (v_2, m_2)$ take $v := v_1 - v_2$, $m := m_1 - m_2$, write the PDEs for (v, m): the 1st is

$$\begin{cases} -v_t + B(t,x) \cdot Dv = \Delta v + F(x, m_1) - F(x, m_2) \quad \text{in } (0,T) \times \mathbf{R}^d, \\ v(T,x) = G(x, m_1(T)) - G(x, m_2(T)). \end{cases}$$

with $B(t,x) := \int_0^1 DH(Dv_2 + s(Dv_1 - Dv_2))ds \in L^\infty((0,T) \times \mathbf{R}^d)$.

Then by parabolic estimates one can get

$$\|Dv(t,\cdot)\|_{L^p_x} \leq C_1 \int_t^T \|F(\cdot, m_1(s)) - F(\cdot, m_2(s))\|_{L^r_x} ds + C_2 \|DG(\cdot, m_1(T)) - DG(\cdot, m_2(T))\|_{L^r_x}.$$

Similarly, from the 2nd equation can estimate

$$\|m(t,\cdot)\|_{L^q_x} \le C_3 \int_0^t \|Dv(s,\cdot)\|_{L^p_x} ds$$

A Lipschitz assumption on F and $DG : L^q \to L^r$ implies

$$\|Dv(t,\cdot)\|_{L^p_x} \le C_1 L_F \int_t^T \|m(s,\cdot)\|_{L^q_x} ds + C_2 L_G \|m(T,\cdot)\|_{L^q_x}$$

Now set $\phi(t) := \|Dv(t,\cdot)\|_{L^p_x}$ and combine the inequalities to get

$$\phi(t) \le C_4 \int_t^T \int_0^\tau \phi(s)ds\, d\tau + C_5 \int_0^T \phi(s)ds$$

and $\Phi := \sup_{0\le t\le T} \phi(t)$ satisfies

$$\Phi \le \Phi(C_4 T^2/2 + C_5 T)$$

so $\Phi = 0$ for T small enough.

Question: which are good L^p norms to do this? with which precise assumptions on the data?

Similarly, from the 2nd equation can estimate

$$\|m(t,\cdot)\|_{L^q_x} \le C_3 \int_0^t \|Dv(s,\cdot)\|_{L^p_x} ds$$

A Lipschitz assumption on F and $DG : L^q \to L^r$ implies

$$\|Dv(t,\cdot)\|_{L^p_x} \le C_1 L_F \int_t^T \|m(s,\cdot)\|_{L^q_x} ds + C_2 L_G \|m(T,\cdot)\|_{L^q_x}$$

Now set $\phi(t) := \|Dv(t,\cdot)\|_{L^p_x}$ and combine the inequalities to get

$$\phi(t) \le C_4 \int_t^T \int_0^\tau \phi(s)ds\, d\tau + C_5 \int_0^T \phi(s)ds$$

and $\Phi := \sup_{0\le t\le T} \phi(t)$ satisfies

$$\Phi \le \Phi(C_4 T^2/2 + C_5 T)$$

so $\Phi = 0$ for T small enough.

Question: which are good L^p norms to do this? with which precise assumptions on the data?

Question: which are good L^p norms to do this? with which precise assumptions on the data?

Lions chooses $\|Dv(t,\cdot)\|_{L^\infty_x}$ and $\|m(t,\cdot)\|_{L^1_x}$,

works with periodic BC and assumes

- $G = G(x)$ independent of m,
- F "regularizing", i.e., Lipschitz $L^1 \to C^1$.

Our choice is simpler: work in L^2 by energy estimates.

Question: which are good L^p norms to do this? with which precise assumptions on the data?

Lions chooses $\|Dv(t,\cdot)\|_{L^\infty_x}$ and $\|m(t,\cdot)\|_{L^1_x}$,

works with periodic BC and assumes

- $G = G(x)$ independent of m,
- F "regularizing", i.e., Lipschitz $L^1 \to C^1$.

Our choice is simpler: work in L^2 by energy estimates.

Theorem [B.-Fischer]: uniqueness for short horizon

Assume $H \in C^2(\mathbf{R}^d)$, $\nu \in \mathcal{P} \cap L^\infty(\mathbf{R}^d)$,

$$\|F(\cdot,\mu) - F(\cdot,\bar{\mu})\|_2 \le L_F \|\mu - \bar{\mu}\|_2,$$

$$\|DG(\cdot,\mu) - DG(\cdot,\bar{\mu})\|_2 \le L_G \|\mu - \bar{\mu}\|_2$$

$(v_1, m_1), (v_2, m_2)$ two classical solutions of the MFG PDEs with $v_1 - v_2, m_1, m_2$ and their derivatives in $L^2([0,T] \times \mathbf{R}^d)$, and

$$|DH(Dv_i)|, \ |D^2 H(Dv_i)| \le C_H.$$

Then $\exists\, \bar{T} = \bar{T}(d, L_F, L_G, \|\nu\|_\infty, C_H) > 0$ such that $\forall\, T < \bar{T}$,

$v_1(\cdot,t) = v_2(\cdot,t)$ and $m_1(\cdot,t) = m_2(\cdot,t)$ for all $t \in [0,T]$.

Corollary (Uniqueness for "small data")

Uniqueness remains true for all $T > 0$ if either L_F, L_G are small, or $\sup |D^2 H(Dv_i)|$ is small.

Theorem [B.-Fischer]: uniqueness for short horizon

Assume $H \in C^2(\mathbf{R}^d)$, $\nu \in \mathcal{P} \cap L^\infty(\mathbf{R}^d)$,

$$\|F(\cdot,\mu) - F(\cdot,\bar{\mu})\|_2 \leq L_F \|\mu - \bar{\mu}\|_2,$$

$$\|DG(\cdot,\mu) - DG(\cdot,\bar{\mu})\|_2 \leq L_G \|\mu - \bar{\mu}\|_2$$

$(v_1, m_1), (v_2, m_2)$ two classical solutions of the MFG PDEs with $v_1 - v_2, m_1, m_2$ and their derivatives in $L^2([0,T] \times \mathbf{R}^d)$, and

$$|DH(Dv_i)|,\ |D^2 H(Dv_i)| \leq C_H.$$

Then $\exists\ \bar{T} = \bar{T}(d, L_F, L_G, \|\nu\|_\infty, C_H) > 0$ such that $\forall\ T < \bar{T}$,

$v_1(\cdot,t) = v_2(\cdot,t)$ and $m_1(\cdot,t) = m_2(\cdot,t)$ for all $t \in [0,T]$.

Corollary (Uniqueness for "small data")

Uniqueness remains true for all $T > 0$ if either L_F, L_G are small, or $\sup |D^2 H(Dv_i)|$ is small.

Theorem [B.-Fischer]: uniqueness for short horizon

Assume $H \in C^2(\mathbf{R}^d)$, $\nu \in \mathcal{P} \cap L^\infty(\mathbf{R}^d)$,

$$\|F(\cdot,\mu) - F(\cdot,\bar\mu)\|_2 \le L_F \|\mu - \bar\mu\|_2,$$

$$\|DG(\cdot,\mu) - DG(\cdot,\bar\mu)\|_2 \le L_G \|\mu - \bar\mu\|_2$$

$(v_1, m_1), (v_2, m_2)$ two classical solutions of the MFG PDEs with $v_1 - v_2, m_1, m_2$ and their derivatives in $L^2([0,T] \times \mathbf{R}^d)$, and

$$|DH(Dv_i)|,\ |D^2 H(Dv_i)| \le C_H.$$

Then $\exists\, \bar T = \bar T(d, L_F, L_G, \|\nu\|_\infty, C_H) > 0$ such that $\forall\, T < \bar T$,

$v_1(\cdot,t) = v_2(\cdot,t)$ and $m_1(\cdot,t) = m_2(\cdot,t)$ for all $t \in [0,T]$.

Corollary (Uniqueness for "small data")

Uniqueness remains true for all $T > 0$ if either L_F, L_G are small, or $\sup |D^2 H(Dv_i)|$ is small.

Remark: a crucial estimate is

$$\|m_i(t,\cdot)\|_\infty \le C(T, \|DH(Dv_i)\|_\infty)\|\nu\|_\infty, \quad i = 1,2,\ \forall\, t \in [0,T],$$

that we prove by probabilistic methods.

Example (Regularizing costs)

$$F(x,\mu) = F_1\left(x, \int_{\mathbf{R}^d} k_1(x,y)\mu(y)dy\right),$$

with $k_1 \in L^2(\mathbf{R}^d \times \mathbf{R}^d)$, $|F_1(x,r) - F_1(x,s)| \le L_1|r-s|$;

$$G(x,\mu) = g_1(x)\int_{\mathbf{R}^d} k_2(x,y)\mu(y)dy + g_2(x)$$

with $g_1, g_2 \in C^1(\mathbf{R}^d)$, Dg_1 bounded, $k_2, D_x k_2 \in L^2(\mathbf{R}^d \times \mathbf{R}^d)$.

Remark: a crucial estimate is

$$\|m_i(t,\cdot)\|_\infty \le C(T, \|DH(Dv_i)\|_\infty)\|\nu\|_\infty, \quad i = 1,2,\ \forall\, t \in [0,T],$$

that we prove by probabilistic methods.

Example (Regularizing costs)

$$F(x,\mu) = F_1\left(x, \int_{\mathbf{R}^d} k_1(x,y)\mu(y)dy\right),$$

with $k_1 \in L^2(\mathbf{R}^d \times \mathbf{R}^d)$, $|F_1(x,r) - F_1(x,s)| \le L_1|r-s|$;

$$G(x,\mu) = g_1(x)\int_{\mathbf{R}^d} k_2(x,y)\mu(y)dy + g_2(x)$$

with $g_1, g_2 \in C^1(\mathbf{R}^d)$, Dg_1 bounded, $k_2, D_x k_2 \in L^2(\mathbf{R}^d \times \mathbf{R}^d)$.

Example (Local costs)

$G = G(x)$ independent of $m(T)$ and F of the form

$$F(x, \mu) = f(x, \mu(x))$$

with $f : \mathbf{R}^d \times [0, +\infty) \to \mathbf{R}$ such that

$$|f(x, r) - f(x, s)| \le L_f |r - s| \quad \forall x \in \mathbf{R}^d,\ r, s \ge 0.$$

Then F is Lipschitz in L^2 with $L_F = L_f$.

Examples of non-uniqueness

The stationary MFG PDEs:

$$\text{(MFE)} \qquad \begin{cases} -\Delta v + H(x, \nabla v) + \lambda = F(x, m) \quad \text{in } \mathbb{T}^d, \\ \Delta m + div(\nabla_p H(x, \nabla v) m) = 0 \quad \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} m(x)dx = 1, \quad m > 0, \quad \int_{\mathbb{T}^d} v(x)dx = 0, \end{cases}$$

has uniqueness for F monotone increasing and H convex. Otherwise:

- Lasry-Lions for $H(x, p) = |p|^2$ via a Hartree equation of Quantum Mechanics,
- Gueant 2009 for (local) logarithmic utility $\quad F = -\log m$
- M.B. 2012 and M.B. - F. Priuli 2014 for LQG models in $\mathbf{R}^d$
- M. Cirant 2015 and Y. Achdou - M.B. - M. Cirant 2016 for systems of two populations with Neumann boundary conditions.

Question: counter-examples for the evolutive case?

How far from the monotonicity condition? Also for T small?

Existence of two solutions

Theorem (Any $T > 0$)

Assume $d = 1$, $H(p) = |p|$, $F, G \in C^1$, $\sigma > 0$ *and* C^2, $M(\nu) = 0$, *and*

$$\frac{\partial F}{\partial x}(x,\mu) \begin{cases} \le 0 & \text{if } M(\mu) > 0, \\ \ge 0 & \text{if } M(\mu) < 0. \end{cases}$$

$$\frac{\partial G}{\partial x}(x,\mu) \begin{cases} \le 0 \quad \text{and not } \equiv 0 & \text{if } M(\mu) > 0, \\ \ge 0 \quad \text{and not } \equiv 0 & \text{if } M(\mu) < 0, \end{cases}$$

$\Longrightarrow \quad \exists$ *solutions* (v, m) , $(\bar{v}, \bar{m})$ *with*

$$v_x(t,x) < 0, \quad \bar{v}_x(t,x) > 0 \quad \text{for all } 0 < t < T.$$

- $T > 0$ can also be small: H convex but not C^1.
- No assumption on the monotonicity of F, G w.r.t. μ.
- We have also a probabilistic formulation and proof of non-uniqueness under less assumptions on σ.

Existence of two solutions

Theorem (Any $T > 0$)

Assume $d = 1$, $H(p) = |p|$, $F, G \in C^1$, $\sigma > 0$ *and* C^2, $M(\nu) = 0$, *and*

$$\frac{\partial F}{\partial x}(x,\mu) \begin{cases} \leq 0 & \text{if } M(\mu) > 0, \\ \geq 0 & \text{if } M(\mu) < 0. \end{cases}$$

$$\frac{\partial G}{\partial x}(x,\mu) \begin{cases} \leq 0 \quad \text{and not } \equiv 0 & \text{if } M(\mu) > 0, \\ \geq 0 \quad \text{and not } \equiv 0 & \text{if } M(\mu) < 0, \end{cases}$$

$\Longrightarrow \quad \exists$ *solutions* (v, m) , $(\bar{v}, \bar{m})$ *with*

$$v_x(t,x) < 0, \quad \bar{v}_x(t,x) > 0 \quad \text{for all } 0 < t < T.$$

- $T > 0$ can also be small: H convex but not C^1.
- No assumption on the monotonicity of F, G w.r.t. μ.
- We have also a probabilistic formulation and proof of non-uniqueness under less assumptions on σ.

Explicit example of non-uniqueness

$$F(x,\mu) = \beta x M(\mu) + f(\mu), \quad G(x,\mu) = \gamma x M(\mu) + g(\mu),$$

with $\beta, \gamma \in \mathbf{R}$, $f, g : \mathcal{P}_1(\mathbb{R}) \to \mathbf{R}$, e.g., f, g depend only on the moments of μ .

There are two different solutions if

$$\beta \leq 0, \quad \gamma < 0,$$

By the L-L monotonicity result there is uniqueness if $f = g \equiv 0$ and

$$\beta > 0, \quad \gamma \geq 0.$$

If $\beta < 0, \gamma < 0$ F and G are not decreasing in $M(\mu)$, but an agents has a negative cost, i.e., a reward, for having a position x with the same sign as the average position $M(m)$ of the whole population. Conversely, the conditions for uniqueness express aversion to crowd.

Explicit example of non-uniqueness

$$F(x,\mu) = \beta x M(\mu) + f(\mu), \quad G(x,\mu) = \gamma x M(\mu) + g(\mu),$$

with $\beta, \gamma \in \mathbf{R}$, $f, g : \mathcal{P}_1(\mathbb{R}) \to \mathbf{R}$, e.g., f, g depend only on the moments of μ .

There are two different solutions if

$$\beta \le 0, \quad \gamma < 0,$$

By the L-L monotonicity result there is uniqueness if $f = g \equiv 0$ and

$$\beta > 0, \quad \gamma \ge 0.$$

If $\beta < 0, \gamma < 0$ F and G are not decreasing in $M(\mu)$, but an agents has a negative cost, i.e., a reward, for having a position x with the same sign as the average position $M(m)$ of the whole population.
Conversely, the conditions for uniqueness express aversion to crowd.

Explicit example of non-uniqueness

$$F(x,\mu) = \beta x M(\mu) + f(\mu), \quad G(x,\mu) = \gamma x M(\mu) + g(\mu),$$

with $\beta, \gamma \in \mathbf{R}$, $f, g : \mathcal{P}_1(\mathbb{R}) \to \mathbf{R}$, e.g., f, g depend only on the moments of μ .

There are two different solutions if

$$\beta \leq 0, \quad \gamma < 0,$$

By the L-L monotonicity result there is uniqueness if $f = g \equiv 0$ and

$$\beta > 0, \quad \gamma \geq 0.$$

If $\beta < 0, \gamma < 0$ F and G are not decreasing in $M(\mu)$, but an agents has a negative cost, i.e., a reward, for having a position x with the same sign as the average position $M(m)$ of the whole population.
Conversely, the conditions for uniqueness express aversion to crowd.

Existence of two solutions - 2

Theorem (H smooth and $T > \varepsilon$)

Same assumptions as previous Thm., BUT, for some $\delta, \varepsilon > 0$,

$$H(p) = |p|, \text{ for } |p| \geq \delta$$

$$\frac{\partial G}{\partial x}(x,\mu) \begin{cases} \leq -\delta & \text{if } M(\mu) \geq \varepsilon, \\ \geq \delta & \text{if } M(\mu) \leq -\varepsilon, \end{cases}$$

$\Longrightarrow$ *for* $T \geq \varepsilon$ $\exists$ *solutions* (v,m) , $(\bar{v},\bar{m})$ *with*

$$v_x(t,x) \leq -\delta, \quad \bar{v}_x(t,x) \geq \delta \quad \text{for all } 0 < t < T.$$

Example

$$H(p) := \max_{|\gamma|\leq 1}\left\{-p\gamma + \frac{1}{2}\delta(1-\gamma^2)\right\} = \begin{cases} \frac{p^2}{2\delta} + \frac{\delta}{2}, & \text{if } |p| \leq \delta, \\ |p|, & \text{if } |p| \geq \delta, \end{cases}$$

Idea of proof

$$\begin{cases} -v_t + |v_x| = \frac{\sigma^2(x)}{2} v_{xx} + F(x, m(t,\cdot)), & v(T,x) = G(x, M(m(T))), \\ m_t - (sign(v_x) m)_x = \left(\frac{\sigma^2(x)}{2} m\right)_{xx}, & m(0,x) = \nu(x). \end{cases}$$

Ansatz: $sign(v_x) = -1$ and m solves

$$m_t + m_x = \left(\frac{\sigma^2(x)}{2} m\right)_{xx}, \quad m(0,x) = \nu(x).$$

Then m is the law of the process

$$X(t) = X(0) + t + \int_0^t \sigma(X(s)) dW(s)$$

with $X(0) \sim \nu$, so $M(m(t)) = \mathbf{E}[X(t)] = M(\nu) + t = t > 0 \quad \forall t$.

(E-) $$-v_t - v_x = \frac{\sigma^2(x)}{2} v_{xx} + F(x, m), \quad v(T,x) = G(x, m(T)).$$

Then $w = v_x$ satifies

$$-w_t - w_x - \sigma\sigma_x w_x - \frac{\sigma^2}{2} w_{xx} = \frac{\partial F}{\partial x}(x, m) \leq 0$$

$$w(T, x) = \frac{\partial G}{\partial x}(x, m(T)) \leq 0 \text{ and not } \equiv 0,$$

Similarly we can build a solution with $sign(\bar{v}_x) = 1$ and $\bar{m}$ solving

$$\bar{m}_t - \bar{m}_x = \frac{\sigma^2(x)}{2} \bar{m}_{xx}, \quad \bar{m}(0, x) = \nu(x),$$

so that $M(\bar{m}(t, \cdot)) = -t < 0$ and $\frac{\partial F}{\partial x}(x, \bar{m}(t, \cdot)), \frac{\partial G}{\partial x}(x, \bar{m}(T)) \geq 0$.

Other examples of non-uniqueness in finite horizon MFGs

All very recent:

- A. Briani, P. Cardaliaguet 2016: for a potential MFG
- M. Cirant, D. Tonon 2017: for a focusing MFG

Several populations, Neumann boundary conditions

Motivation for 2 population: models of segregation phenomena in urban settlements, inspired by the Nobel laureate T. Schelling: Y. Achdou - M. B. - M. Cirant , M^3AS 2017.

For Ω bounded and smooth, $k = 1, 2$

$$\begin{cases} -\partial_t v_k + H^k(x, Dv_k) = \Delta v_k + F^k(x, m_1(t,\cdot), m_2(t,\cdot)) \text{ in } (0,T)\times\Omega, \\ v_k(T,x) = G^k(x, m_1(T,\cdot), m_2(T,\cdot)), \quad \partial_n v_k = 0 \quad on\ \partial\Omega\times(0,T), \\ \partial_t m_k - div(D_p H^k(x, Dv_k) m_k) = \Delta m_k \quad \text{in } (0,T)\times\Omega, \\ m_k(0,x) = \nu_k(x), \quad \partial_n m_k + m_k D_p H^k(x, Du_k)\cdot n = 0 \quad on\ \partial\Omega\times(0,T) \end{cases}$$

Some sufficient conditions for existence:

- F^k, G^k continuous in $\overline{\Omega} \times \mathcal{P}(\overline{\Omega})^2$.
- F^k, G^k bounded, respectively, in $C^{1,\beta}(\overline{\Omega}), C^{2,\beta}(\overline{\Omega})$ uniformly w.r.t. $m \in \mathcal{P}(\overline{\Omega})^2$.
- $H^k \in C^1(\overline{\Omega} \times \mathbf{R}^d)$ and $D_p H^k(x,p) \cdot p \geq -C_0(1+|p|^2)$.
- $\nu_k \in C^{2,\beta}(\overline{\Omega})$.
- Compatibility conditions on boundary data

The L-L monotonicity condition on F^k for uniqueness becomes:
$\exists\ \lambda_i > 0 : \forall (m_1, m_2) \neq (\bar{m}_1, \bar{m}_2)$

$$\int_{\mathbf{R}^d} \sum_{i=1}^{2} \lambda_i [F^i(x, m_1, m_2) - F^i(x, \bar{m}_1, \bar{m}_2] d(m_i - \bar{m}_i)(x) > 0$$

But in the simplest models $F^1 = F^1(x, m_2), F^2 = F^2(x, m_1)$, so, e.g.,

$$[F^1(x, m_2) - F^1(x, \bar{m}_2)](m_1 - \bar{m}_1)(x)$$

cannot have a sign!

Theorem [M.B. - M. Cirant]: uniqueness for small data

Assume $H^k \in C(\Omega \times \mathbf{R}^d)$, C^2 in p, $\nu_k \in \mathcal{P} \cap L^\infty(\Omega)$,

$$\|F^k(\cdot, \mu_1, \mu_2) - F^k(\cdot, \nu_1, \nu_2)\|_2 \leq L_F(\|\mu_1 - \nu_1\|_2 + \|\mu_2 - \nu_2\|_2),$$

$$\|DG^k(\cdot, \mu_1, \mu_2) - DG(\cdot, \mu_1, \mu_2)\|_2 \leq L_G(\|\mu_1 - \nu_1\|_2 + \|\mu_2 - \nu_2\|_2)$$

$(v_1, v_2, m_1, m_2), (\bar{v}_1, \bar{v}_2, \bar{m}_1, \bar{m}_2)$ two classical solutions

with $v_k - \bar{v}_k, m_k, \bar{m}_k$ and their derivatives in $L^2([0, T] \times \Omega)$, and

$$|D_p H^k(x, Dv_k)|, |D_p H^k(x, D\bar{v}_k)| \leq C_1,$$

$$|D_p^2 H^k(x, Dv_k)|, |D_p^2 H^k(x, D\bar{v}_k)| \leq C_2.$$

If either T is small, or L_F and L_G are small, or C_2 is small,

then $v_k(\cdot, t) = \bar{v}_k(\cdot, t)$ and $m_k(\cdot, t) = \bar{m}_k(\cdot, t)$ $\forall\, t \in [0, T]$, $k = 1, 2$.

Remarks and perspectives

- Can build examples of non-uniqueness, as for 1 population, e.g., $H^i(x,p) = |p|$, $M(\nu_i) = 0$,

$$F_i(x,\mu_1\mu_2) = \alpha_i x M(\mu_1) + \beta_i x M(\mu_2) + f_i(\mu_1,\mu_2), \quad i = 1,2,$$

$$G_i(x,\mu_1,\mu_2) = \gamma_i x M(\mu_1) + \delta_i x M(\mu_2) + g_i(\mu_1,\mu_2), \quad i = 1,2,$$

  with $\alpha_i, \beta_i, \gamma_i, \delta_i \leq 0, \quad \gamma_i + \delta_i < 0, \; i = 1,2, \; f_i, g_i : \mathcal{P}_1(\mathbb{R})^2 \to \mathbf{R}.$
- The proof of uniqueness for small data is flexible: can use other assumptions with different norms,
  a hard point is the L^∞ estimate for $m(t,\cdot)$,
- it can be used if $H(x,p) - F(x,m)$ is replaced by $\mathcal{H}(x,p,m)$, under smoothness conditions on $\mathcal{H}$,
- in principle it can be used for mean-field control, i.e., control of McKean-Vlasov stochastic differential equations.