TOPICS IN SUB-RIEMANNIAN GEOMETRY

References

Book Agrachev, Barilari, Boscain

A Comprehensive Introduction to sub-Riemannian Geometry

Cambridge Univ Press, 2020.

Lecture notes Course "Differential Geometry"

ongoing → preliminary

Draft of both available on my webpage.

Book

Teaching → webpage of the course

Schedule

I	Wed	17/03	12^h30
	Tue	23/03	14^h30
II	Wed	24/03	12^h30
	Tue	30/03	14^h30
III	Wed	31/03	12^h30
	Wed	7/04	12^h30
IV	Thu	8/04	11^h30

7 lectures 2^h each

+

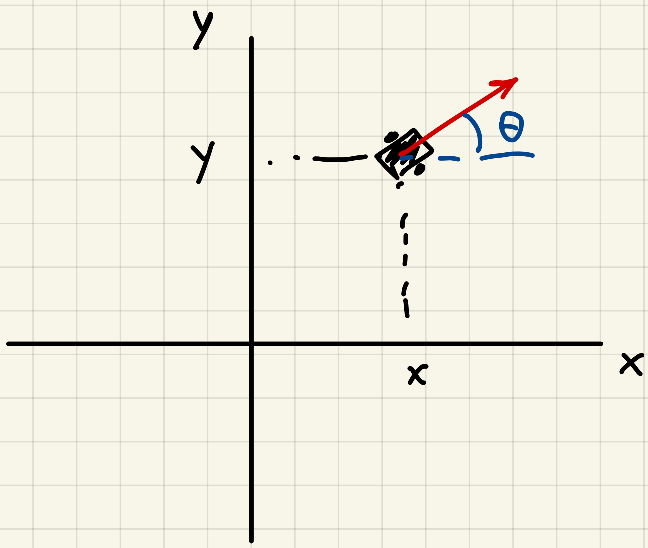
1^h somewhere

=

15 hours.

INTRODUCTION

Model of a car in a plane

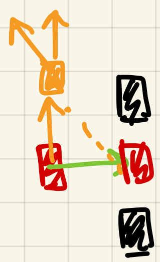


$(x, y, \theta) \in \mathbb{R}^2 \times S^1$

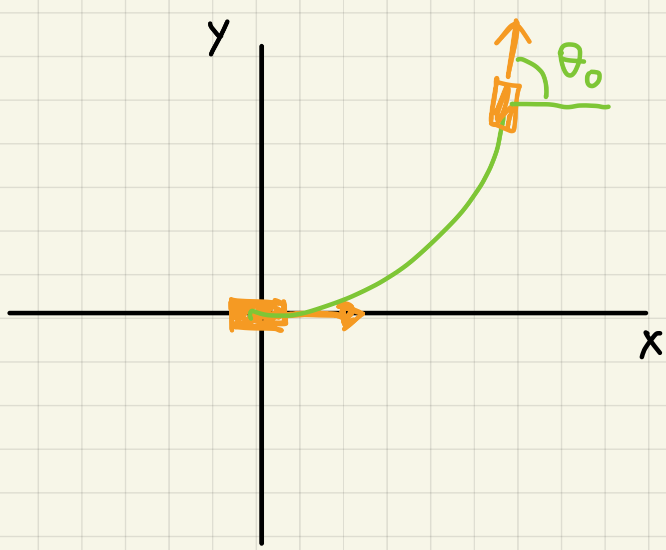
dimension 3

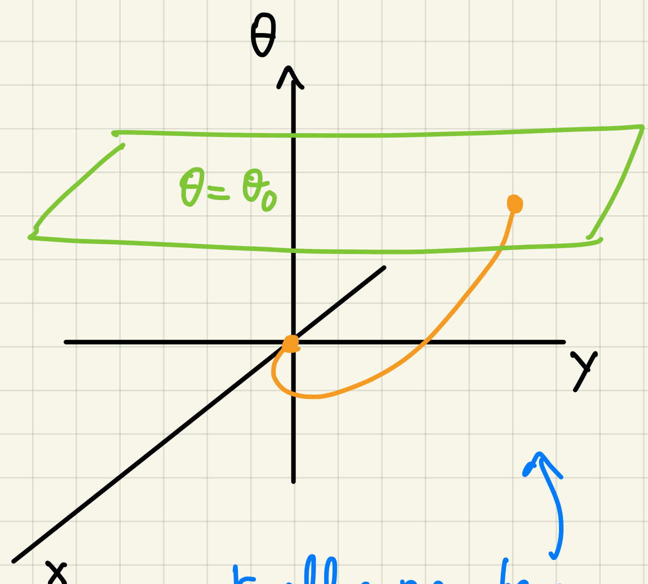
$= 3$ degree of freedom

allowed movements for the car : only 2



We can think in this way





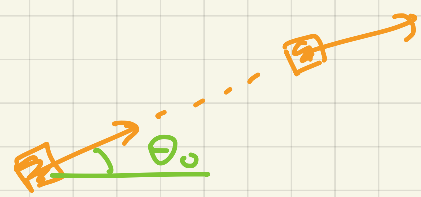
not all curves here corresponds to "true" movements of the car.

Infinitesimal movements $\rightsquigarrow$ differential equations (ODE)

$\cap$ vector fields.

1st movement : go straight.

$$\begin{cases} \dot{x} = \cos\theta \\ \dot{y} = \sin\theta \\ \dot{\theta} = 0 \end{cases} \rightsquigarrow \begin{aligned} x(t) &= x_0 + (\cos\theta_0)\, t \\ y(t) &= y_0 + (\sin\theta_0)\, t \\ \theta(t) &= \theta_0 \end{aligned}$$



$$X = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

2nd movement : turn on your place.

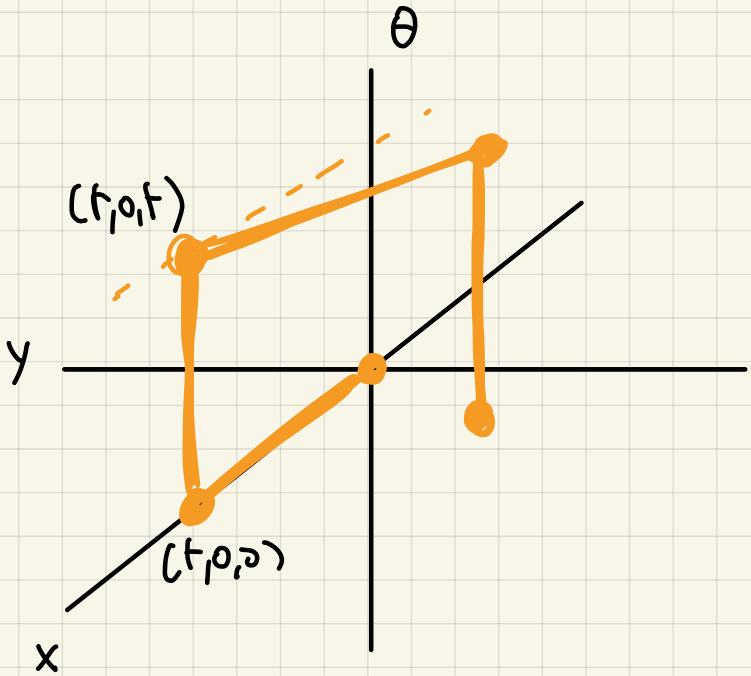
$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \\ \dot{\theta} = 1 \end{cases} \qquad \begin{aligned} x(t) &= x_0 \\ y(t) &= y_0 \\ \theta(t) &= \theta_0 + t \end{aligned}$$

$$Y = \frac{\partial}{\partial\theta}$$

3rd movem : not allowed

$$\begin{cases} \dot{x} = \sin\theta \\ \dot{y} = -\cos\theta \\ \dot{\theta} = 0 \end{cases} \qquad Z = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}.$$

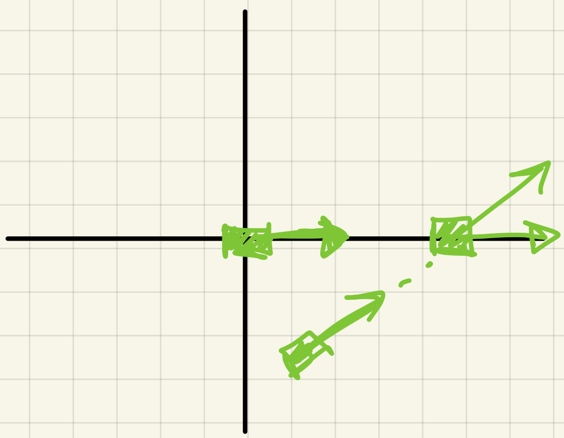
$$Z(0,0,0) = -\frac{\partial}{\partial y} \simeq (0,-1,0)$$



The 1st mov (X)
2nd mov (Y)
do not commute.

$$AB = BA$$

$$A^{-1}B^{-1}AB = id$$



$$(0,0,0) \xrightarrow{e^{tX}} (t,0,0)$$

$$\xrightarrow{e^{tY}} (t, 0, t)$$

$$\xrightarrow{e^{-tX}} (t - t\cos t, -t\sin t, t)$$

$$\xrightarrow{e^{-tY}} (t - t\cos t, -t\sin t, 0)$$

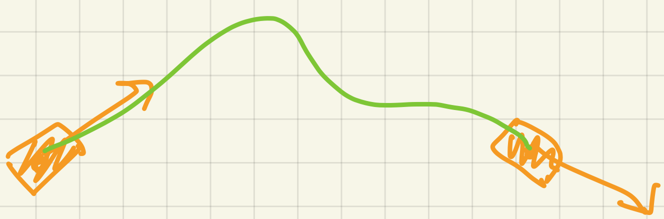
Using the exp. notation.

$$e^{-tY} e^{-tX} e^{tY} e^{tX}(0) = t^2 \underbrace{(0,-1,0)}_{Z(0)} + o(t^2)$$

We can model all this as follows: in $\mathbb{R}^3$ (or better $\mathbb{R}^2 \times S^1$) we limit ourselves to curves satisfying a differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = u_1(t) \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + u_2(t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Given any initial (x_0, y_0, θ_0) and final (x_1, y_1, θ_1) configuration, we can find a choice of $u_1(t)$, $u_2(t)$ (also piecewise const) such that the solution joins the two points

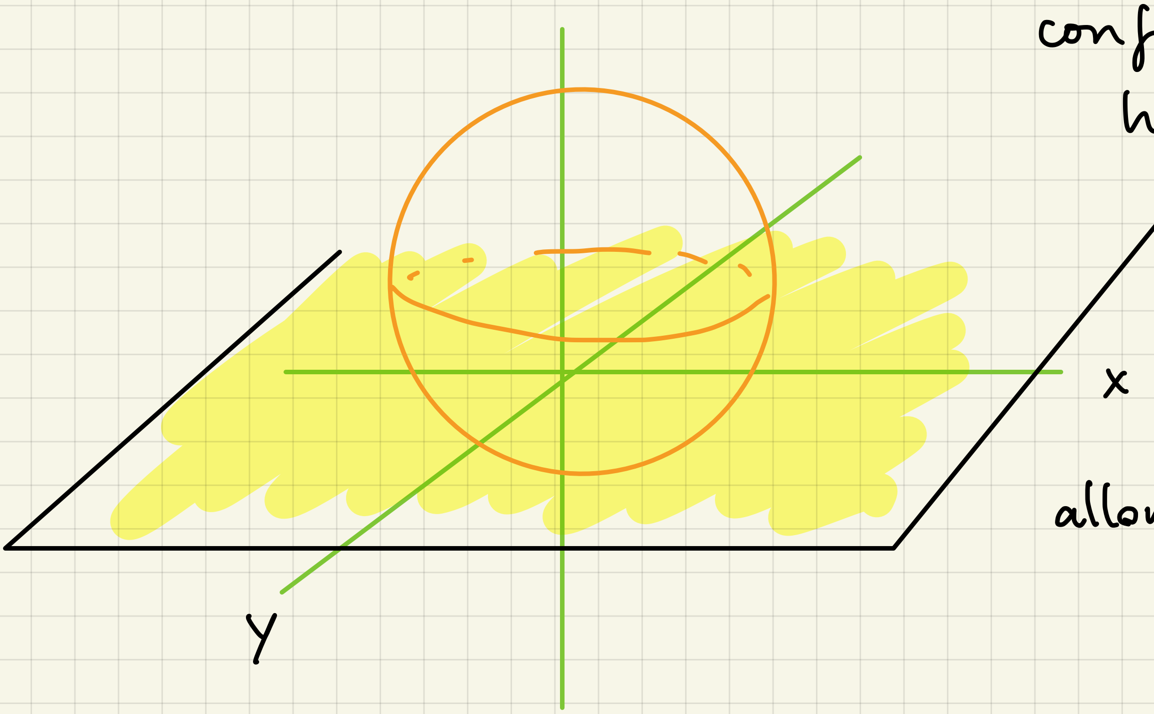


Once we have proved that every final conf. is reachable from every initial one, one can try to do it "optimally"

for example $\quad \min \int_0^T \left(u_1^2(t) + u_2^2(t)\right) dt$

A BALL ROLLING ONTO THE PLANE

(without slipping nor twisting)



configuration space has dim 5

allowed movements 2

fix
$$e_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad e_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad e_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis of $so(3) = \{ A \mid A^T = -A \}$

which generates rotations around axes x, y, z.

meaning: if one considers differential eq.

$$\begin{cases} \dot{M}(t) = M(t) A(t) \\ M(0) = M_0 \in SO(3) \end{cases} \qquad A(t)^T = -A(t)$$

$MM^T = I$

→ the solution $M(t)$ belongs to $SO(3)$ $\forall t$.

$$\frac{d}{dt}\underbrace{M(t)M(t)^T}_{\text{const.}} = \dot{M}(t)M(t)^T + M(t)\dot{M}(t)^T$$

$$= M(t)\left(A(t)+A(t)^T\right)M(t)^T = 0$$

$$M(t)M(t)^T = M(0)M(0)^T = I.$$

the allowed movements $(\overbrace{x, y}^{\in \mathbb{R}^2}, \overset{\in SO(3)}{M})$

$$X_1 = \frac{\partial}{\partial x} - M e_y$$ ← vector field in $\mathbb{R}^2 \times SO(3)$

flow
$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 0 \\ \dot{M} = -M e_y \end{cases}$$ ← this is ok

$$X_2 = \frac{\partial}{\partial y} + M e_x$$

<u>RK</u>

$$T_I SO(3) = so(3) = \{A \mid A + A^T = 0\}$$

$$T_M SO(3) = M \cdot so(3) = \{MA \mid A + A^T = 0\}$$

left translation = left multiplication

for linear group.

Here if we try to build the "commutator" between X_1 and X_2 meaning.

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 0 \\ \dot{M} = -M e_y \end{cases} \qquad \begin{cases} \dot{x} = 0 \\ \dot{y} = 1 \\ \dot{M} = M e_x \end{cases}$$

what you get is a rotation around z axes.

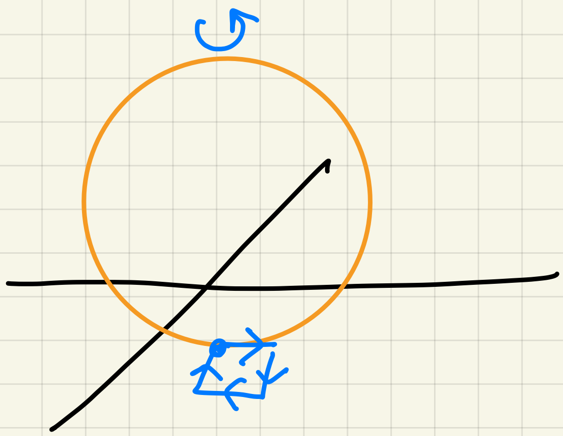
<u>Exercise</u> Compute the commutator as matrices

$$[e_x, e_y] := e_x e_y - e_y e_x = e_z$$

the idea here is that the "Lie bracket"

$$[X_1, X_2] = \left[\frac{\partial}{\partial x} - M e_y \, , \, \frac{\partial}{\partial y} + M e_x \right]$$

$$= - M [e_y, e_x] = M e_z .$$



$$X_1 = \frac{\partial}{\partial x} - M e_y$$

$$X_2 = \frac{\partial}{\partial y} + M e_x$$

$$\left\{\begin{array}{l} [e_x, e_y] = e_z. \\ \text{\& also} \\ \; [e_y, e_z] = e_x \\ \; [e_z, e_x] = e_y \end{array}\right.$$

$$X_3 := [X_1, X_2] = M e_z$$

$$X_4 = [X_1, [X_1, X_2]] = -M e_x .$$

$$X_5 = [X_2, [X_1, X_2]] = -M e_y$$

If we combine the two equations

$$q = (x, y, M) \in \mathbb{R}^2 \times SO(3)$$

$$\left\{\begin{array}{l} \dot{x} = u_1 \\ \dot{y} = u_2 \\ \dot{M} = M(u_2 e_x - u_1 e_y) \end{array}\right.$$

repres. in coord.

$$\boxed{\dot{q} = u_1 X_1(q) + u_2 X_2(q).}$$

in dim 5

<u>Optimal solutions</u>? $\min \int_0^T u_1^2(t) + u_2^2(t)\,dt$.

"Setting" space of dimension n $\mathbb{R}^n$

set of admissible mov.

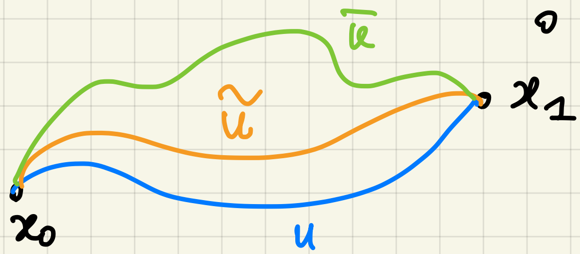
$X_1, \ldots, X_k$

$\dot{x} = F(x,u)$

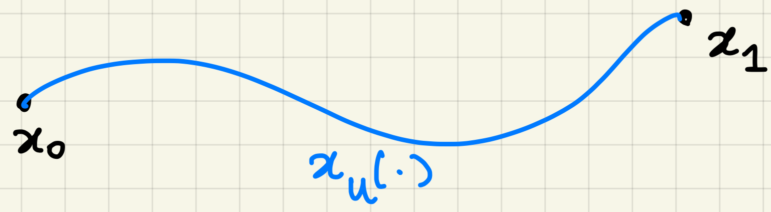
differential eq $\dot{x} = u_1 X_1(x) + \ldots + u_k X_k(x)$ $(*)$

1. condition on $X_1, \ldots, X_k$ such that $\forall\, x_0, x_1 \in M$ $\exists$ a trajectory of $(*)$ joining x_0, x_1.

2. minimize energy $\min \int_0^T \sum u_i^2(t)\, dt = J(u)$



minimizers are divided into two classes $\rightarrow$ $\boxed{\text{normal}}$ / $\boxed{\text{abnormal (singular)}}$



Take $u(\cdot)$ $\rightsquigarrow$ solve $\dot{x} = F(x,u)$
and get x_u

$$\rightsquigarrow \min J(u) = \int_0^T \sum u_i^2(t)\,dt.$$

$$\min \{ J(u) \mid x_u \text{ joining } x_0 \text{ and } x_1 \}$$

minimize a functional

u belongs to some set.

under a constraint.

In $\mathbb{R}^2$ $\min \{ f(x,y) \mid g(x,y) = 0 \}$

if $\underline{\nabla g(x,y) \neq 0}$ for every $(x,y) \in \{ g(x,y) = 0 \}$

we can apply L.M.R and.

(x_0,y_0) solution $\Rightarrow \exists \lambda : \nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0).$

if $\nabla g(x_0,y_0) = 0$ then (x_0,y_0) might be as well a solution of our problem.

For instance $g(x,y) = x^2 - y^2$

$$g(x,y) = 0$$

$$\min \{ f(x,y) \mid x^2 - y^2 = 0 \}$$