LECTURE 3 (by A. Agrachev)

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We formulated a test / criterion to understand whether a curve $\lambda(t)$ in T^*M is abnormal or not

symplectic form

annihilator of Δ, living in T^*M

Theorem Consider $\sigma|_{\Delta^\perp}$ restriction of σ to $\Delta^\perp \subset T^*M$

$\lambda(t)$ is abnormal extremal $\iff \begin{cases} \lambda(t) \in \Delta^\perp \\ \dot\lambda(t) \in \operatorname{Ker} \sigma|_{\Delta^\perp} . \end{cases}$

Kernel as bilinear form

We have $\sigma_\lambda : T_\lambda(T^*M) \times T_\lambda(T^*M) \longrightarrow \mathbb{R}$

$$T_\lambda(T^*M) \supseteq \operatorname{Ker} \sigma|_{\Delta^\perp} = \{ \xi \in T_\lambda \Delta^\perp \mid \sigma(\xi, T_\lambda \Delta^\perp) = 0 \}$$

recall σ is non deg $\Rightarrow$ no kernel, but its restriction might have

Recall that here $\perp$ means orthogonal with respect to σ, which is treated as a "skew-symm. inner product"

Some terminology from symplectic geometry.

RK (Darboux theorem) locally every symplectic manifold is equivalent to $(\mathbb{R}^{2n}, \sigma)$

$$\sigma = \sum_{i=1}^{n} dp_i \wedge dq_i$$

A symplectic space (Σ, σ) ↙ only linear algebra.

Σ vector space, σ bilinear skew sym. form non degenerate.

A symplectic manifold (N, σ) is given by

σ bilinear skew sym. bil. for on every tangent spaces. non degenerate

$\boxed{d\sigma = 0}$ ↖ global on manifold

we add a strong condition

not present when you mimic this in Riem geometry.

this requirement is the key for Darboux theorem.

Still the good properties do not depend on the choice of local coordinates. This is an example.

let $h: T^*M \to \mathbb{R}$ smooth, then one can prove that $\vec{h} \in Vec(T^*M)$ satisfy

intrinsic charact. of Ham. vect field

$$\boxed{d_\lambda h = \sigma(\cdot\,, \vec{h}(\lambda))}$$

$\vec{h}$ is the unique vector field satisfying this

no need to be sym.

notice that a bilinear not degenerate form

$$B : V \times V \to \mathbb{R}$$

we can identify it with an isomorphism.

$$A : V \to V^* \qquad A(v) := B(v, \cdot)$$

above $B = \sigma$ and A maps dh to $\vec{h}$

then one can introduce also coordinate independent def of Poisson brackets

$h, g : T^*M \to \mathbb{R}$ $$\boxed{\{h,g\}(\lambda) \overset{def}{=} \sigma(\vec{h}(\lambda), \vec{g}(\lambda))}$$

Given a subset of a symplectic space (Σ, σ) one can define its skew orth. complement

$S \subset \Sigma$ $\qquad S^{\angle} = \{ \xi \in \Sigma \mid \sigma(\xi, S) = 0 \}$

subspace.

If σ non degenerate $\dim S + \dim S^{\perp} = \dim \Sigma$.

But notice that the situation is different from usual orthogonality. For instance $\sigma(v,v) = 0$ so every 1d subspace is orthogonal to itself.

Still $(S^{\angle})^{\angle} = S$ $\leftarrow$ start with S subspace otherwise $(S^{\perp})^{\perp} = \mathrm{span}(S)$.

then we have all ingredients to understand the meaning of

$$\boxed{\dot{\lambda}(t) \in \operatorname{Ker} \sigma|_{\Delta^{\perp}}} \qquad (*)$$

Recall that every abnormal $\lambda(t) \in \Delta^{\perp}_{\lambda(t)}$
(due to the conditions $h_i(\lambda(t)) \equiv 0$ $i = 1 \dots k$)
hence it makes sense to ask $(*)$

So we have that

$\langle \lambda, v \rangle = 0$
on M
annihilator

show orthogonal.
on T^*M.
$\sigma(\xi, \cdot) = 0$.

$$\dot{\lambda}(t) \in \ker \sigma|_{\Delta^\perp} \iff \dot{\lambda}(t) \in T_\lambda(\Delta^\perp)^\angle$$

Let us compute now $\Delta = \text{span}\{X_1, \dots, X_k\}$

$h_i(\lambda) = \langle p, X_i(q) \rangle$ linear hamiltonians.

using usual coordinates $(p,q) \in T^*M$.

$$\Delta^\perp = \{ \lambda \in T^*M \mid h_i(\lambda) = 0 \quad i = 1 \dots k \}$$

then

$$T_\lambda \Delta^\perp = \{ \xi \in T_\lambda(T^*M) \mid d_\lambda h_i = 0 \quad i = 1 \dots k \}$$ linearization

$$= \bigcap_{i=1}^{k} \ker d_\lambda h_i$$

but recall that for every function we have

$\ker d_\lambda h = \vec{h}^\angle$ since $d_\lambda h = \sigma(\cdot, \vec{h})$

$$T_\lambda \Delta^\perp = \bigcap_{i=1}^{k} \vec{h}_i(\lambda)^\angle$$

check as an exercise!

$$(T_\lambda \Delta^\perp)^\angle = \text{span}\{ \vec{h}_i(\lambda) \mid i = 1 \dots k \}$$

In the end we have.

$$\dot{\lambda}(t) \in \ker \sigma|_{\Delta^\perp} \iff \dot{\lambda}(t) \in (T_\lambda \Delta^\perp)^\angle$$

$$\iff \dot{\lambda}(t) \in \text{span}\{ \vec{h}_i(\lambda) \mid i = 1 \dots k \}$$

this is our differential equation!

$$\iff \dot{\lambda}(t) = \sum_{i=1}^{k} u_i(t)\, \vec{h}_i(\lambda(t))$$

COMPUTING ABNORMALS.

Let us consider our $\Delta^\perp$ and the σ_λ at any point $\lambda \in \Delta^\perp$.

Recall that $\sigma_\lambda|_{\Delta^\perp}$ may be degenerate or not

Abnormals live only on the set where $\sigma_\lambda|_{\Delta^\perp}$ is degenerate

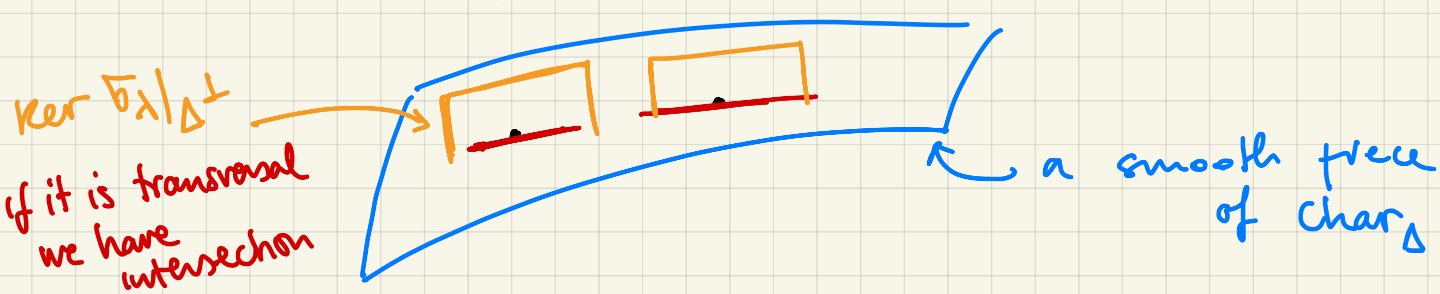
$$\boxed{\mathrm{Char}_\Delta = \{ \lambda \in \Delta^\perp \mid \ker \sigma_\lambda|_{\Delta^\perp} \neq \{0\} \} \subset \Delta^\perp}$$

Characteristic variety

In general it can be a complicated set and not a smooth manifold. Sometimes we can see that it is stratified into smooth pieces.

This variety can be empty also (recall that $\lambda \neq 0$ so when we write $\Delta^\perp$ we should remove the zero section)

In the "smooth part" of Char_Δ we can determine geometrically the abnormals.



(P)

In most examples we will see this intersection is 1d and define a line field on $\Delta^\perp$ — in most points.
It means that abnormals are integral curves of some field.
Then we may have points where
- dimension of the intersection increases
- Points where $Char_\Delta$ is not smooth

this increases the difficulty.

Now we make computations on a basis.
Let $X_1 \dots X_k$ our basis for the distribution

$$\Delta = \mathrm{span}\{X_1 \dots X_k\}$$

$$h_i(\lambda) = \langle p, X_i(q) \rangle$$

The abnormal condition gives

$$\begin{cases} \dot{\lambda}(t) = \sum_{i=1}^{k} u_i(t) \vec{h}_i(\lambda(t)) \\ h_i(\lambda(t)) = 0 \end{cases}$$

we can differentiate these conditions.

$$\boxed{\frac{d}{dt} h_i(\lambda(t)) = -\sum_{j=1}^{n} u_j(t) \{h_j, h_i\}(\lambda(t)) = 0 . \qquad i = 1 \dots k}$$

We can introduce the matrix

$$H(\lambda) = \Big(\{h_i, h_j\}(\lambda) \Big)_{i,j=1\dots k}$$

skew sym. matrix $k \times k$ at every k.

So we have

$$\boxed{H(\lambda(t))\, u(t) = 0 \iff u(t) \in \operatorname{Ker} H(\lambda(t))}$$

Notice that if k odd then H necessarily have a kernel, k even it might not have.
(for instance in 1st part we have studied $k=2$).

We can now write equations for Char_Δ.

$$\mathrm{Char}_\Delta = \{ \lambda \mid h_i(\lambda) = 0 \underset{i=1\dots k}{,} \det H(\lambda) = 0 \}$$

equations which are linear in p!

<u>Assume k odd.</u> Then $\mathrm{Char}_\Delta = \Delta^\perp$
indeed $\det H(\lambda) = 0$ for every λ due to matrix skew-symm.

As skew-symm. matrix of odd variables, generically it has 1-dim kernels.

Let us consider the "good" part of $\Delta^\perp$

$$\hat{\Delta}^\perp = \{ \lambda \in \Delta^\perp \mid \dim \operatorname{Ker} H(\lambda) = 1 \}$$

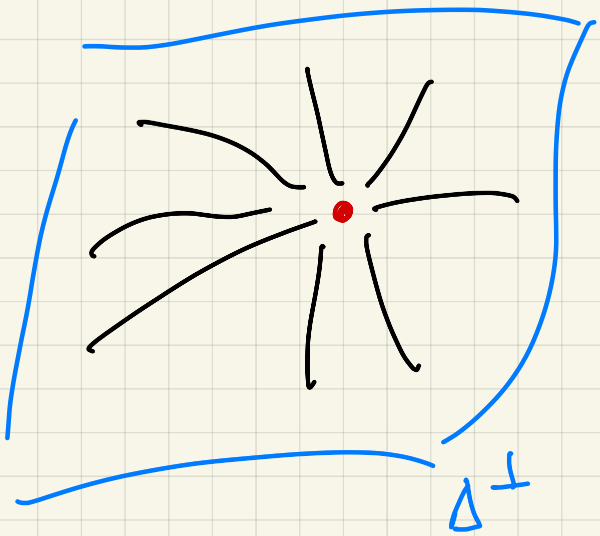
where in the picture (P) above intersect. is 1 dim

This subset is foliated by line distribution.

whose integral curves are exactly the abnormal extremals. Through every point of $\hat{\Delta}^\perp$ we have a unique abnormal.

<u>RK</u> Any reparam of abnormal is abnormal. (here no metric so no length param).

We might have sing points ($\hat{\Delta}^\perp$ not closed)



the red point is singular

$\hat{\Delta}^\perp = \Delta^\perp \setminus \{\bullet\}$

migth be a point where dim of int. increases

idea: study behavior in $\hat{\Delta}^\perp$ but close to singular points

If singular points (i.e. points in $\Delta^\perp \setminus \hat{\Delta}^\perp$) are not isolated, that can be complicated

<u>Assume k even</u> in this case

$$\mathrm{Char}_\Delta = \{ \lambda \mid \underset{i=1\dots k}{h_i(\lambda) = 0}, \ \det H(\lambda) = 0 \}$$

not nec. zero. even dim.

digression of anty sym. matrix.

let $A^* = -A$ be skew-symmetry.

we have $|\det(A)| = \mathrm{Pfaff}(A)^2$

degree k degree $k/2$. (with respect to entries).

the equation $\det H(\lambda)=0$ is reduced to $\mathrm{Pfaff}\, H(\lambda)=0$ which is of lower order

Notice that $\boxed{\mathrm{Pfaff}\, H(\lambda)=0}$ is an equation of degree $\frac{k}{2}$ in p. which is homogeneous.

$\lambda = (p,q)$

real solution of this algebraic equation (for q fixed)

Given $A^* = -A$ $\rightsquigarrow$ its spectrum has coupled eigenvalues

treat A as an antsym. bil. form.

$\Omega(x,y) = \langle Ax, y\rangle$ where here $\langle\,,\rangle$ in $\mathbb{R}^n$ eucl. product.

This is not degenerate if and only if

$$\underbrace{\Omega\wedge\ldots\wedge\Omega}_{k/2 \text{ times}} = \mathrm{Pfaff}(\Omega)\,\mathrm{vol}$$

this expression is the pfaffian.

standard dx volume

$A^* = -A$

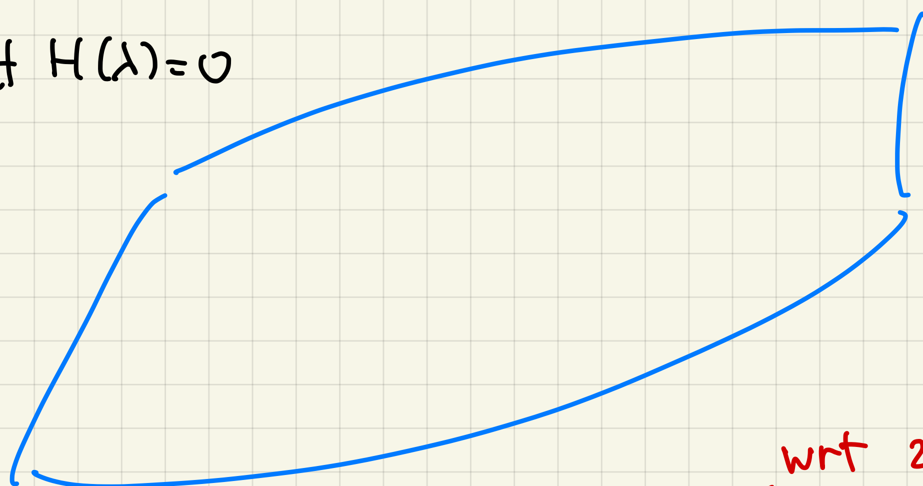
$\operatorname{spec}(A) = \{\pm i\alpha_1, \ldots, \pm i\alpha_k\} \qquad \alpha_i > 0.$

$\det(A) = \alpha_1^2 \alpha_2^2 \cdots \alpha_k^2$ ← product of squares

$\operatorname{Pfaff}(A) = \alpha_1 \cdots \alpha_k$. ← simple product.

In $\Delta^\perp$ we have a $\frac{k}{2}$ degree equation.

$\operatorname{Pfaff} H(\lambda) = 0$



In the smooth part (where $\nabla \operatorname{Pfaff} \neq 0$) wrt $2n$ variables.
roots are simple $\rightsquigarrow$ regular zero.
We denote it $\widehat{\operatorname{Char}}_\Delta$

For q fixed $\quad \dim \Delta_q^\perp = n-k$

$\dim \widehat{\operatorname{Char}}_\Delta = n-k-1$ intersection with one more equation

when $\operatorname{Ker} H(\lambda)$ is transversal to $\operatorname{Char}_\Delta$