LECTURE 4 (by A. Agrachev)

Notes by
D. Barilari
20/04/2021

We start by recalling some facts from the previous lecture.

$k = \dim \Delta$

$\Delta = \mathrm{span}\{X_1, \dots, X_k\}$ distribution in TM.
bracket-generating.

$n = \dim M$

$h_i(\lambda) = h_i(p,q) = \langle p, X_i(q)\rangle \qquad \lambda \in T^*M$

$(q \in M,\ p \in T_q^*M)$

Then we can consider

$$H(\lambda) = H(p,q) = \Big(\{h_i, h_j\}(\lambda) \Big)_{i,j=1\dots k}$$

$k \times k$ matrix

Recall that $\{h_i, h_j\}(\lambda) = \langle p, [X_i, X_j](q)\rangle$.
so that H is linear wrt p.

We introduce the charact. variety in T^*M

$$\mathrm{Char}_\Delta = \{ (p,q) \mid p \in \Delta_q^\perp \setminus \{0\},\ \ker H(p,q) \neq 0 \}$$

or $\det = 0$.

all h_i are zero

$p \neq 0$ we always have to put this but maybe in previous lect. is omitted.

FIRST CASE

If k odd then $\mathrm{Char}_\Delta = \Delta^\perp$ since it is always true that $\det H(\lambda) = 0$.

Denote by

$$\widehat{\mathrm{Char}}_\Delta = \{ (p,q) \in \mathrm{Char}_\Delta \mid \dim \ker H(q,p) = 1 \}$$

here we can compute all abnormals.

On $\widehat{\mathrm{Char}}_\Delta$ we have $\dim \ker H(p,q) = 1$ and the kernel is a vector, but let us realize it as a tangent vector to Char_Δ.

$$H(p,q)\, u = 0$$

u unique up to multiplier.

Then the vector $\sum_{i=1}^{k} u_i \vec{h}_i$ is tangent to $\Delta^\perp = \mathrm{Char}\,\Delta$.

As a function of p we have that minors are polynomial in p (for fixed q)

But given q the set of p where the rank is maximal is Zariski open in the fiber.

$$\{ p \in T^*_{\bar{q}} M \mid \operatorname{rank} H(p,\bar{q}) = \max_p \operatorname{rank} H(p,\bar{q}) \}$$

If $\operatorname{rank} H(p,\bar{q}) = k-1$ for at least one p then the set of such p is open dense in the fiber.

The set of (p,q) where the rank is maximal is also open in T^*M.

$$\dim \widehat{Char}_\Delta = 2n-k = n + (n-k)$$

as submanifold of T^*M.

Then in this case we have 1d intersection and a line field which generates flow

$$t \longmapsto \lambda_t = (p(t), q(t))$$

horiz proj are singular curves.

Remark let $s \in \mathbb{R} \setminus \{0\}$. Notice that

$$(p(t), q(t)) \text{ abnormal } \iff (s\,p(t), q(t)) \text{ abnormal}$$

this is a homogeneity property in the fiber. (we can reduce the dimension by 1).

So the "nice" $\widehat{Char}_\Delta \cap T^*_q M \approx n-k$ dimensional.

$\boxed{n-k-1}$ if we keep into account homogen.

We can "projectivize" the set since our curves pass to the quotient.

For each q (in the proj of Char_Δ) we have a $\boxed{n-k-1}$ dim family of such nice abnormal

SECOND CASE k is even

In this case (cf. last lecture)

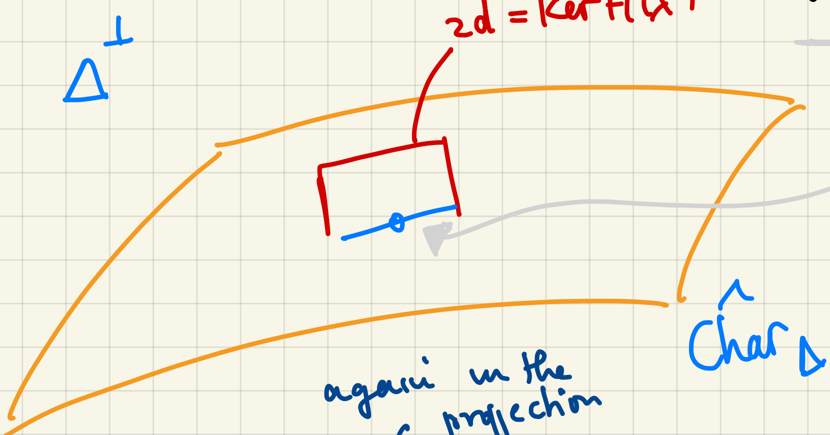
$$\mathrm{Char}_\Delta = \Delta^\perp \cap \mathrm{Pfaff}^{-1}(0)$$

We want to find the "nice" part of this set.

regular part of

$$\widehat{\mathrm{Char}}_\Delta = \left\{ \lambda \in \mathrm{Char}_\Delta : \begin{array}{l} d_\lambda \mathrm{Pfaff} \neq 0 \\ \dim(T_\lambda \mathrm{Char}_\Delta \cap \ker H(\lambda)) = 1 \end{array} \right\}$$

open but in general is a generic condition.



notice: Char_Δ might be singular

$\widehat{\mathrm{Char}}_\Delta$ is C^∞.

For every $q \in M$ (again in the projection) we have $\boxed{n-k-2}$ dim family of singular curves that are "nice"

staying in the smooth set.

It is still an open question to determine if the measure of set reached by abnormals is zero (starting from an arbitrary point).

FAT DISTRIBUTIONS

def We say that a distribution Δ is *fat* if it holds $\underline{\mathrm{Char}}_\Delta = 0$.

First observation: Δ fat $\Rightarrow$ k even.

(by previous observations.)

Moreover, either $n = k+1$ or $\frac{k}{2}$ even ($k \equiv 0 \bmod 4$)

When $\boxed{n = k+1}$ we have contact distribution.

↳ indeed $k = 2m$ $n = 2m+1$

$\mathrm{Pfaff}(p,q) \neq 0 \quad \forall p \neq 0$.

In this case no singular curves (beyond constants)

Looking back $\Delta^\perp$ is a line and up to multiplication it is a single point. So the Pfaffian is indeed a scalar,

$$\mathrm{Pfaff}\, H(p,q) = \alpha(q)\, p^{\frac{k}{2}}.$$

contact distribution if $\alpha(q) \neq 0 \quad \forall q \in M$.

Notice that if $n-k > 1$ then.

to not to have zeros $\Rightarrow \frac{k}{2}$ even.

$p \longmapsto \mathrm{Pfaff}(p,q)$ homog polyn. of degree $\frac{k}{2}$.

If Δ is fat $\Rightarrow$ $\boxed{\begin{array}{l}\Delta \text{ contact} \\ k = 2m \qquad n = 2m+1.\end{array}}$

or

$\boxed{\begin{array}{l}\Delta \text{ is a distrib st.} \\ k = 4m \qquad n \leq 2k-1\end{array}}$

SOME GENERICITY ARGUMENTS

We work locally so we consider

M smooth manifold

$q_0 \in M$, Δ_q for q close to q_0 (say in O_{q_0})

We are studying the germ of Δ at q_0

is an equivalence class of distribution

which coincide in a neighborhood of q_0

$$\Delta \sim \bar{\Delta} \quad \text{iff} \quad \exists\, O_{q_0} \subset M \; : \; \Delta_q = \bar{\Delta}_q \quad \forall q \in O_{q_0}$$

" equal in some neighborhood "

For local classification it is convenient to do it on germs. For instance bracket-generating at q_0 a property of the germ, not only of a distr.

Similarly, to be fat or not is a property which is well-def for the germ.

def We say that a property (P) of a germ of a distribution at q_0 is open if $\forall \Delta$ $\exists \, \mathcal{O}_{q_0}, \varepsilon > 0 , \nu \in \mathbb{N}$: property (P) is true for every distribution ε-close to Δ at q_0 with ν-deriv.

repr. neigh.

(P) is true for distr. that are ε-close.

more formally smth like this

this means $\Delta = \operatorname{span}\{X_1, \dots, X_k\}$

$\tilde{\Delta} = \operatorname{span}\{\tilde{X}_1, \dots, \tilde{X}_k\}$

$$\Rightarrow \quad \| X_i - \tilde{X}_i \|_\infty \leq \varepsilon \quad \text{on } \mathcal{O}_{q_0}.$$

$$\| \partial_j X_i - \partial_j \tilde{X}_i \|_\infty \leq \varepsilon \quad \forall j : |j| \leq \nu.$$

Similarly we say that (P) is dense (sometimes everywhere dense) if $\forall \Delta$, $\forall \varepsilon > 0, \nu \in \mathbb{N}$ $\exists \, \mathcal{O}_{q_0}$ and $\tilde{\Delta}$ such that Δ and $\tilde{\Delta}$ are ε-close for ν derivatives and the property (P) holds for $\tilde{\Delta}$.

$\boxed{\text{(P) is called generic if (P) is open and dense}}$

if some Δ satisfies then small pert. still satisfy

if some Δ do not satisfy then by small perturbation we find one that satisfy.

Q: What are properties that happens generically?

these are to be thought as typical properties

theorem (Jacubczyk) Montgomery also?

Consider the property (P) defined as.

$$\boxed{\Delta_{q_0} = \operatorname{span}\{\dot\gamma(q_0) \mid \gamma \text{ is a nice singular curve}\}}$$

If $n-k \geq 3 \Rightarrow$ (P) is open (non empty)

If (k,n) not fat $\Rightarrow$ (P) is generic.

Proof: go back to computations and have a look :)

In many cases, even if few/no abnormals then they determine your distributions. In the follow. sense.

def Two germs of distributions are equivalent if there exists a diffeo $\Phi : O_{q_0} \longrightarrow V_{q_0}$ such that $\Phi_* \Delta\big|_{O_{q_0}} = \tilde\Delta\big|_{V_{q_0}}$ with $\Phi(q_0) = q_0$.

If two distributions have the same abnormals then the distribution are equivalent

we go back tomorrow

Consider corank 1 distribution $n = k+1$
(may be contact or quasi contact)
$\underbrace{}_{k \text{ even}}$ $\underbrace{}_{k \text{ odd.}}$

In this case $\Delta^{\perp}_{q_0}$ has dim 1
(a point up to multiplication)

$H(p, q_0)$ has maximal rank (' generic)
as a matrix.

quasi contact $\to$ one abnormal through a pt
contact $\to$ no abnormal " " "

they do not satisfy the (P) Jacubzyk.
but these distribution are locally equivalent
(it is the Darboux theorem). No rigid structure
when looking locally.

⚠ Be careful: fat have no abnormals but
they are not all equivalent