# LECTURE 5 (by A. Agrachev)

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We introduced germs of distributions and we try to classify generic cases

generic = open and dense

property which holds for open and dense set of distributions.

$\Delta$ distribution, we work on a neighborhood of a point $q_0 \in M$, $\dim \Delta_q = k$ and $\Delta$ is bracket-generating. $2 \leq k < n$

What we explained (without complete proofs)

- two germs of distributions are equivalent if there exists a local diffeo $\Phi : \Delta \mapsto \tilde{\Delta}$

Notice that singular curves are preserved by equivalence relation above (check if you wish!)

? In which cases singular curves determine the distribution up to equivalence?

We try to answer in generic situation.

The theorem of Jacubczyk–Montgomery says:

- if $n-k \geq 3$ then the property that

$$\Delta_q = \text{span}\{ \dot{\gamma}(q) \mid \gamma \text{ singular} \} \qquad (P)$$

is open non empty.

- if moreover $(n,k)$ is not fat pair of indices, then the property (P) is generic
cf last lecture.

next cases which are missing: codimension $n-k \leq 2$.

Case $n-k=1$ (corank 1 distr.)

We have two subclasses

(1) $k=2m$, $n=2m+1$

$\rightsquigarrow$ normal form of Darboux (in this case Pfaff $H \neq 0$).

all such generic distribution are equivalent to Heisenberg distribution locally.

To describe $\Delta$ is better to use equations (which means $\Delta^\perp$ in $T^*M$)

$\Delta^\perp_q = \text{span}\{\omega(q)\}$   $\omega =$ 1-form
non vanishing.

$\Delta_q = \ker \omega(q)$

exercise   $\text{Pfaff } H \neq 0 \iff \omega \wedge \underbrace{d\omega \wedge \dots \wedge d\omega}_{m \text{ times}} \neq 0$

normalization of $\omega$

Darboux theorem $\Longrightarrow$ $\omega \cong \sum_{i=1}^{m} x_i \, dx_{i+m} + dx_{2m+1}$

in $M = \mathbb{R}^n$   $n = 2m+1$.

(2) $n = 2m$ $k = 2m-1$ ( quasi contact)

generic condition means $\boxed{\text{rank } H = 2m-2}$

the Pfaff is zero so at least dim drop by 2.

In this case through each point we have one abnormal curve.

Here of course the tangent vectors of sing curves cannot span the distribution but still all such distributions are equivalent

Normal form $\omega \cong \sum_{i=1}^{m} x_i \, dx_{i+m}$ normal form but not at zero (ie $q_0 \neq 0$)

Singular curves are generated by the vector field

$$\sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i}$$ recall that $q_0 \neq 0$ check! on this

RANK 2 DISTRIBUTIONS : $k = 2$

maybe open question

there is no a general result, we say something in small dimension, ie $n$ small.

start with $k=2$, $n$ any

$\Delta = \text{span}\{X_1, X_2\}$ $h_i(p,q) = \langle p, X_i(q) \rangle$

$\Delta^\perp = \{ (p,q) \mid h_1(p,q) = h_2(p,q) = 0 \}$

We denote $h_{12} := \{h_1, h_2\}$ associated to $[X_1, X_2]$

then we can iterate and build

$$h_{i_1 \dots i_\ell}(p,q) = \langle p, [X_{i_1}, \dots, [X_{i_{\ell-1}}, X_{i_\ell}]](q) \rangle .$$

and we have $\{h_j, h_{i_1 \dots i_\ell}\} = h_{j i_1 \dots i_\ell}$.

The matrix $H = \begin{pmatrix} 0 & h_{12} \\ -h_{12} & 0 \end{pmatrix}$ recall $H$ is $k \times k$ so here $2\times 2$.

$\det H = h_{12}^2$ $\qquad$ $\mathrm{Pfaff}\, H = h_{12}$

Here we also have that

$$\mathrm{Char}_\Delta = \{(p,q) \mid h_1 = h_2 = 0,\ h_{12} = 0\}$$

All equations are linear eq. wrt $p$.

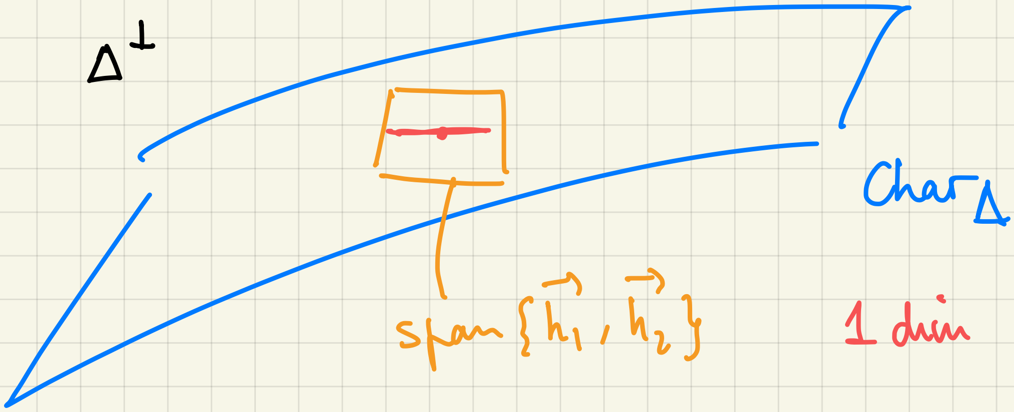
generic condition: $\mathrm{Char}_\Delta \cap T_q^* M$ is a codim 3 subspace in $T_q^* M$ (codim 1 in $\Delta^\perp$)

exercise: prove that this is eq. $\mathrm{Pfaff} = 0$ is a regular equation (with nonzero diff).

The genericity condition is equivalent to

$$\boxed{X_1(q),\ X_2(q),\ [X_1, X_2](q_0) \text{ are lin ind.}}$$

the same generic condition




$u_1 \vec{h}_1 + u_2 \vec{h}_2$ is tangent to $\mathrm{Char}_\Delta$
iff it vanish when applied to $h_{12}$

the function that defines $\mathrm{Char}_\Delta$.

int curve $u_1\vec{h}_1 + u_2\vec{h}_2$

$$h_{12}(\lambda_t) = 0 \implies \{u_1 h_1 + u_2 h_2, h_{12}\} = 0$$

$$\implies \boxed{u_1 h_{112} + u_2 h_{212} = 0} \qquad \boxed{\begin{aligned} u_1 &= -h_{212} \\ u_2 &= +h_{112} \end{aligned}} \ (\#)$$

The generic condition says that this is a non trivial equation $(h_{112}, h_{212}) \neq (0,0)$

This says that if $\lambda \perp \{X_1, X_2, [X_1, X_2]\}$
then $\lambda$ is not orthogonal to at least one among $[X_1, [X_1, X_2]]$, $[X_2, [X_1, X_2]]$.

$$\mathrm{Char}_\Delta = \{\lambda \in \Delta^\perp \mid \lambda \perp \mathrm{span}\{X_1, X_2, [X_1, X_2]\}\}$$

$$\widehat{\mathrm{Char}}_\Delta = \{\lambda \in \Delta^\perp \mid \lambda \not\perp \mathrm{span}\{[X_1,[X_1,X_2]], [X_2,[X_1,X_2]]\}\}$$

Nice abnormals

look to (*) and $h_{221} = -h_{212}$

We get $u_1 = h_{221}$ $\quad u_2 = h_{112}$

and $\boxed{\dot{\lambda} = h_{221}\vec{h}_1 + h_{112}\vec{h}_2}$

nice abnormal are solutions to this equation with $h_1 = h_2 = h_{12} = 0$ and $h_{221}^2 + h_{112}^2 \neq 0$.

For $\underline{n=3}$ we have that no abnormal.

$\underline{n=4}$ simplest codim 2 case

In this case the generic condition says.

$4 = \dim \operatorname{span} \{ \underbrace{X_1, X_2, [X_1, X_2]}_{3\,d.}, [X_1,[X_1,X_2]], [X_2,[X_2,X_1]]\}$

This is called <u>Engel distribution</u>

$T_q^*M$ is 4d, $\mathrm{Char}_\Delta \cap T_q^*M$ is 1 dim

up to multiplier just one $\lambda$ $\Rightarrow$ one abnormal through each pt.

Here also abnormal do not span the distrib.

But the set of all abnormal curves are

$\rightsquigarrow$ Engel distributions are all equivalent

again in generic situation. (no proof).

When $\underline{n=5}$ Cartan Distribution.

Generic condition is that the first 5 brackets are lin ind.

$$5 = \dim \operatorname{span} \{ X_1, X_2, [X_1, X_2], [X_1 [X_1 X_2]], [X_2, [X_1 X_2]] \}$$

Here $\mathrm{Char}_\Delta = \widehat{\mathrm{Char}}_\Delta$

$\mathrm{Char}_\Delta \cap T^*_q M$ is a 2d. subspace

In this case we have a 1 par. family of abnormal curves (up to multiplication) and the projection is a linear isomorphism with the distribution

$$\mathrm{Proj} : \mathrm{Char}_\Delta \cap T^*_q M \longrightarrow \Delta_q \quad \text{isom}.$$
$$\lambda_0 \longmapsto \dot{\gamma}(0)$$

solution of the equation $\boxed{\dot{\lambda} = h_{221} \vec{h}_1 + h_{112} \vec{h}_2}$

Let us now observe the case

$\underline{n=5}$ x $\underline{k=3}$

We claim that here no chance to have a normal form. Let us discuss such a possibility By an heuristic reasoning.

Indeed if there exists diffeo transforming singular into singular then we have also a diffeo which sends distr. to distrib.

distribution $\quad q \longmapsto \Delta_q \in G_k(\mathbb{R}^n)$ GRASSMANNIAN Smooth

$\uparrow$ $k(n-k)$ dim manifold

GRASSMANNIAN

$$G_k(\mathbb{R}^n) = \{ W \mid W \subseteq \mathbb{R}^n, \dim W = k \}$$

we can describe every $W$ as the graph of a linear map from $\mathbb{R}^k \longrightarrow \mathbb{R}^{n-k}$.

So the space of germs of rank $k$ distributions is $\approx$ $k(n-k)$ real functions def on $\mathbb{R}^n$

A change of variable $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a vector funct. $(\Phi_1, \dots \Phi_n)$ given by $n$ scalar functions.

If $n=5$ $\quad k=3$ $\quad$ then $\quad n = 5$

$$k(n-k) = 3 \cdot 2 = 6$$

$$\boxed{\text{We have "less functions than distributions"!}}$$

We have $\quad k(n-k) - n \quad$ functional invariants at least

We have a hope to normalize if $\boxed{k(n-k)-n \leq 0}$

$\cancel{k=1}$ $\boxed{k=n-1}$

$\hookrightarrow$ $(n-1)\cdot 1 - n \leq 0$ ok !

and few other cases. (as $k=2, n=4$).

For $k=2$ $\quad 2(n-2)-n = n-4 \leq 0 \iff n \leq 4$

Going back to the case $n=5$ $k=3$

$\mathrm{Char}_\Delta = \Delta^\perp$, $\mathrm{Char}_\Delta \cap T^*_{q_0}M$ is 2d.

$\dim \mathrm{span}\{X_i, [X_j, X_k]\} = 5 \Rightarrow \widehat{\mathrm{Char}}_\Delta = \mathrm{Char}_\Delta$

here $\mathrm{span}\{\dot\gamma(q_0) \mid \gamma \text{ singular}\} = \Delta^0_{q_0}$

can remove the span here

rank 2 sub distribution

$\Delta^0_{q_0} \subset \Delta_{q_0}$ we have abnormal sub distrib.

seems not clear if we can recover $\Delta$ from $\Delta^0$

It turns out that yes ! Indeed it holds

$(\Delta^0)^2 := \Delta^0 + [\Delta^0, \Delta^0] = \Delta$

EQUIVALENCE OF ABN

$\Rightarrow$ EQUIV OF DISTR

( given $D$, $D^2 = \mathrm{span}\{X_i, [X_j, X_k] \mid X_\alpha \in D\}$ ).