LECTURE 6 (by A. Agrachev)

Notes by D. Barilari' 22/04/2020

THREE-DIM. CASE

M is a 3D man. $M \simeq \mathbb{R}^3$, $\dim \Delta_q = 2$

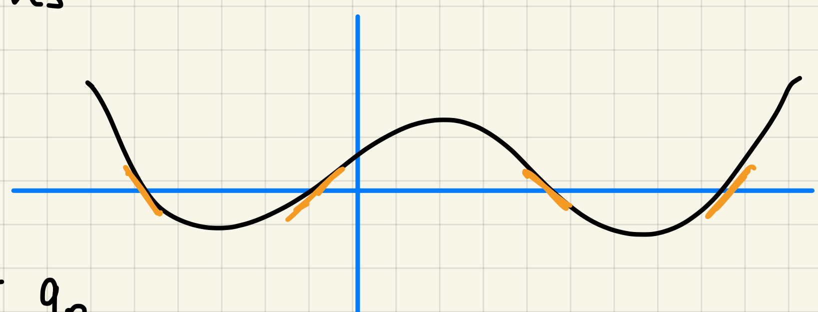
Given $q_0 \in \mathbb{R}^3$ generically Δ is contact distrib.

no abnormals $\to$ get out of generic distrib.

"generic distributions $\neq$ generic germs"

Think to functions

$\varphi : \mathbb{R} \to \mathbb{R}$



germ of φ

generic germ of φ at q_0

$\Rightarrow$ $\varphi(q_0) \neq 0$ — center of points of the germ.

In other words $\varphi(q_0) \neq 0$ is a generic property

φ

generic globally defined functions will have zero

we cannot destroy that by small perturbations

but we can perturb in such a way

if $\varphi(q) = 0 \Rightarrow \varphi'(q) \neq 0$. (at a point)

Difference between generic for germ of functions

for functions.

In 3D if we restrict to generic germs of distributions we have only contact

Now we study germs of generic distributions at a point

$$\Delta = \operatorname{span}\{X_1, X_2\} \qquad \Delta = \operatorname{Ker}\omega$$

1-form. unique up to mult of $f \neq 0$.

def Δ contact $\iff \operatorname{span}\{X_1, X_2, [X_1,X_2]\} = \mathbb{R}^3$

$\iff \omega \wedge d\omega|_q = b(q)\,\mathrm{vol}$, $b \neq 0$

Martinet set $\quad N := b^{-1}(0) \subset M$

For generic b we have $db \neq 0$ on N so that N is actually a surface C^∞.

By def the germ of Δ at points of N are not contact.

At those points $\quad [X_1, X_2]_q \in \Delta_q \qquad q \in N$

At most points of N we can expect that brackets of higher order are lin ind.

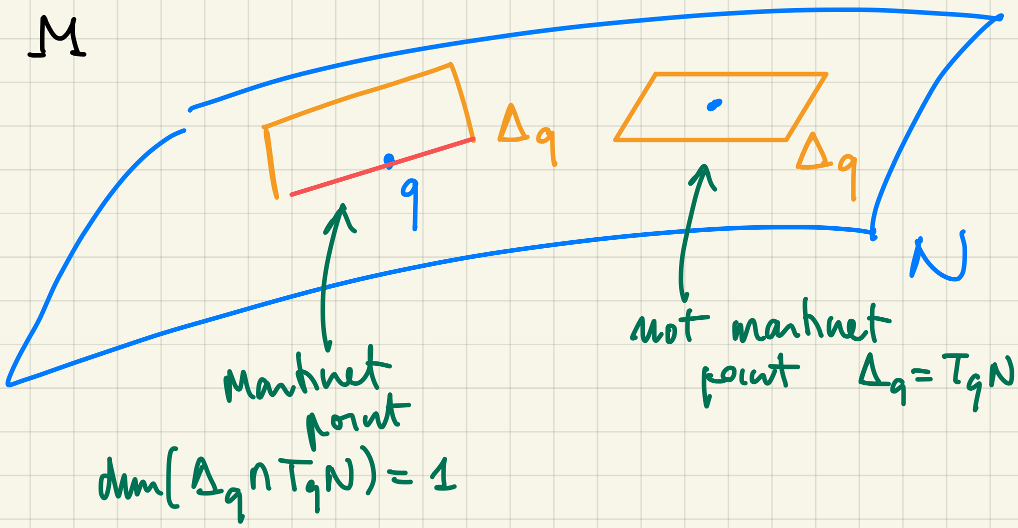
We consider different type of points on the Martinet set/surface N.

Martinet point $q \in N$ (in the Martinet set)

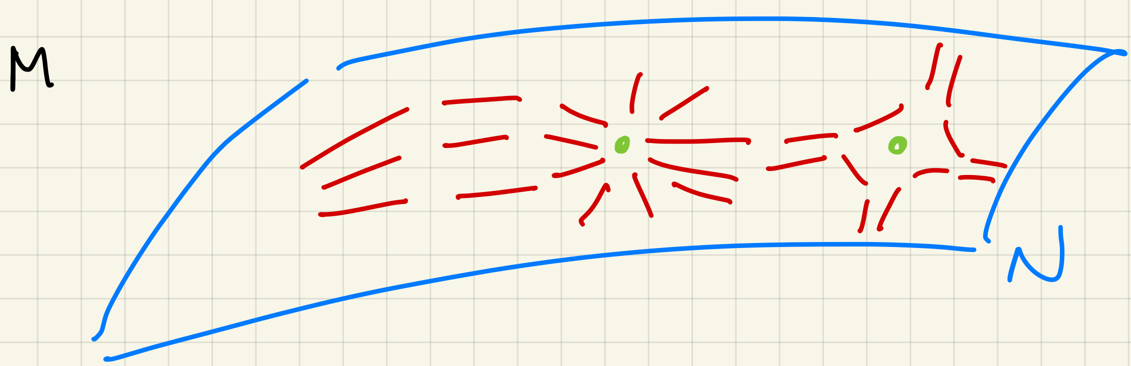
if one of the following equivalent cond.

(a) $\operatorname{span}\{X_1, X_2, [X_1,X_2], [X_1[X_1,X_2]], [X_2,[X_1,X_2]]\} = \mathbb{R}^3$

(b) $\dim \Delta_q \cap T_qN = 1$.



Most points generically are Martinet points.



abnormal curves are contained in N
integral lines of a line field (red one)
the line field has singular points
which are exactly non martinet pts.

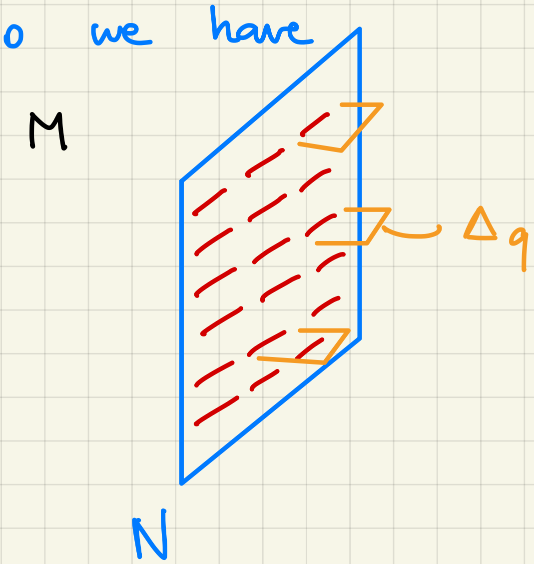
Germs of generic distrib at Martinet pts are all equivalent, we have a normal form. Locally we can describe

$$\Delta = \ker\left(x_1^2\, dx_2 - dx_3\right)$$

recall contact $\ker(x_1\, dx_2 - dx_3)$

$N = \{x_1 = 0\}$ is the Martinet surface in these coordinates.

at $x_1 = 0$ $\Delta = \ker dx_3$ so we have rectified the line field.



<u>def</u> Points of N that are not Martinet points are called <u>tangency points</u>.

at those points

$$\operatorname{span}\{X_1,\ X_2,\ \underbrace{[X_1\ X_2]}_{=X_3}\ [X_1, X_3]\ [X_2, X_3]\} = 2$$

this means three equations

$$\det(X_1, X_2, [X_1, X_2]) = 0$$

one equation of N

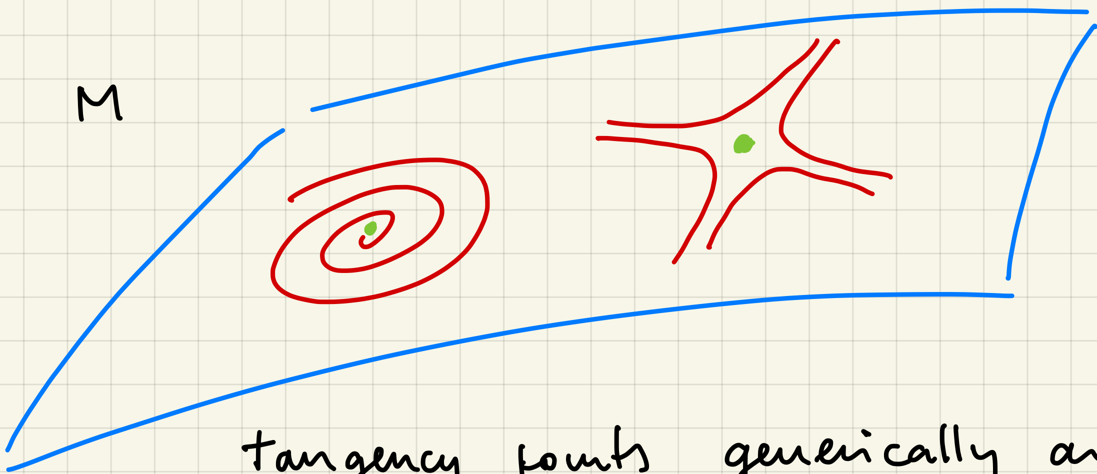
$$\det(X_1, X_2, [X_1, [X_1, X_2]]) = 0$$

$$\det(X_1, X_2, [X_2, [X_1, X_2]]) = 0$$

three equations for tangency points.

Three eq. in dim 3 give generically isolated points as solutions.

We can have different situations



(*)

tangency points generically are either elliptic or hyperbolic.

$$\dot{\lambda} = u_1 \vec{h}_1 + u_2 \vec{h}_2 \qquad \langle p, X_1(q)\rangle = 0$$
$$h_1 = h_2 = 0 \longrightarrow \langle p, X_2(q)\rangle = 0$$

$\rightarrow$ unique $p = p(q)$ find an ODE on M.

$n=3$ $\quad \Delta^\perp \simeq M$ (for every q unique such p up to multiplication)

so $\quad \dot{\lambda} = u_1 \vec{h}_1 + u_2 \vec{h}_2$

is an equation on $\Delta^\perp$

can be described as equation in M

We can recover u_1, u_2 by higher order differentiation $\frac{d}{dt} h_{12}(\lambda_t) = 0$

$$\Rightarrow \quad u_1 h_{112} + u_2 h_{212} = 0 \qquad \Rightarrow (u_1, u_2) = (-h_{221}, h_{112})$$

So we have out of tangency points

$$\dot{\lambda} = h_{221}\vec{h}_1 + h_{112}\vec{h}_2$$

which indeed projects on M.

equation of abnormal out of tangency

$$\boxed{\dot{q} = \langle p, X_{221}(q)\rangle X_1(q) + \langle p, X_{112}(q)\rangle X_2(q)}$$

and p is unique (up to multiplications)

by $\langle p, X_1(q)\rangle = 0$

$\langle p, X_2(q)\rangle = 0$

tangency are exactly critical points of the ODE. (where the coeff of the v.f vanish).

So abnormals on N are solution of an ODE of the form

$$\dot{q} = V(q)$$

tangency pts

at equilibrium $V(q_0) = 0$ the linearization

$$\dot{\xi} = \frac{dV}{dq}(q_0)\,\xi$$ is intrinsic

as well as $\operatorname{div} V(q_0)$ are well-def.

and indeed $\operatorname{div} V(q_0) = 0$. $\left(= \operatorname{trace} \frac{dV}{dq}(q_0)\right)$

Generically $\frac{dV}{dq}(q_0) \neq 0$. as a matrix.

So that $\operatorname{spec} \dfrac{dV}{dq}(q_0) = \{\pm\alpha\}$ or $\{\pm i\alpha\}$

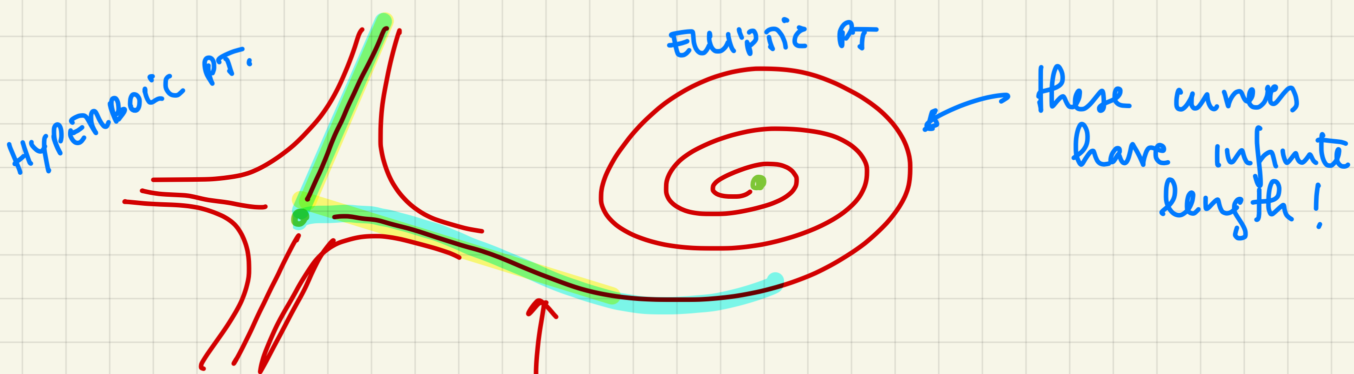
spectrum.

with $\alpha \neq 0$.

We get the above picture (*)

(cf also discussion sec 12.6.1. on the book)

Remark Here we also see appearance of non smooth abnormals when we have saddles. When we pass through sing. point.



Indeed in this setting this might not happen in the focus since infinite lenght.

no abnormal arriving to elliptic points.

QUESTION : SARD CONJECTURE

Recall that γ singular if γ critical point of $E^1_{q_0}$ where $E^t_{q_0} : \Omega_{q_0} \longrightarrow M$

end-point map.

means $D_\gamma E^1_{q_0}$ not surjective

Sard conjecture : the image of singular curves under $E^1_{q_0}$ has measure zero

OPEN PB.

this is related to the classical Sard thm. which states the following

can require less.

If $f : \mathbb{R}^n \to \mathbb{R}^m$ of class C^∞
then $f(C)$ has measure zero in $\mathbb{R}^m$
where $C = \{ x \in \mathbb{R}^n \mid Df(x) \text{ not surjective} \}$ critical points

We can say "measure of critical values is zero"

This is false in infinite dimension and the Sard conjecture asks whether this is true for the specific class of end-point map.

RK this is related to a basic consideration

Given $f: \mathbb{R}^n \to \mathbb{R}^m$ we want to solve the equation $f(x) = y$

the set $\{x \mid f(x) = y\}$ for y in a full measure set in $\mathbb{R}^m$ not critical then the set of solution is C^∞ manifold.

A recent result in this direction for distributions in $\mathbb{R}^3$ has been obtained for analytic distributions

$$\boxed{\text{Abnormals are concatenations of } C^1 \text{ curves.}}$$

[Belotto, Figalli, Parusinski, Rifford]