LECTURE 7 (A. Agrachev)

Notes by D. Barilari 27/04/2021

Δ distribution

Lip with L^2 derivative.

$$\Omega = \{ \gamma : [0,1] \to M \mid \dot\gamma(t) \in \Delta_{\gamma(t)} \text{ a.e} \}$$

$$\Omega_{q_0} = \{ \gamma \in \Omega \mid \gamma(0) = q_0 \}$$

End point map $\quad E^1_{q_0} : \Omega_{q_0} \longrightarrow M$

$$E^1_{q_0}(\gamma) = \gamma(1)$$

restriction of the evaluation map at q_0

$$\boxed{\Omega^{q_1}_{q_0} = \{ \gamma \in \Omega_{q_0} \mid E^1_{q_0}(\gamma) = q_1 \}}$$

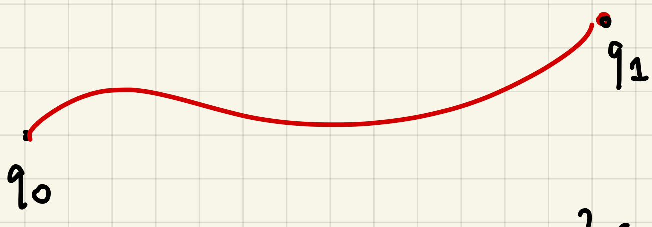
admissible curves connect q_0 & q_1

GENERALIZED LOOP SPACE

Recall that here no metric on the distr. Δ.
We want to understand the local structure of the space $\Omega^{q_1}_{q_0}$.

Let us start with some properties of $\Omega^{q_1}_{q_0}$

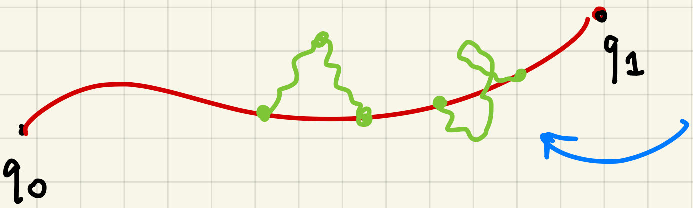
Consider an admissible curve in $\Omega^{q_1}_{q_0}$



$k = \dim \Delta$

$$\Omega_{q_0} \simeq L^2([0,1], \mathbb{R}^k)$$

Our curve is not isolated. thanks to Rashevski - Chow theorem we can perturb our curve and find arbitrary close admissible one



Recall $\Omega_{q_0}^{q_1} = (E_{q_0}^1)^{-1}(q_1)$ inside $\Omega_{q_0} \simeq L^2([0,1], \mathbb{R}^k)$

One can show that $\Omega_{q_0}^{q_1}$ is connected.

More precisely.

1. Every $\gamma \in \mathcal{O}_{\bar\gamma}$ neigh of $\bar\gamma$ in $\Omega_{q_0}^{q_1}$ (conn.) can be homotop. connected to $\bar\gamma$

2. Every pair $\gamma_0, \gamma_1 \in \mathcal{O}_{\bar\gamma}$ $\gamma_i \neq \bar\gamma$ $i=0,1$ can be homotop. connected avoiding $\bar\gamma$

Perturb using Rashevski Chow

Indeed one can connect avoiding any "finite dimensional obstacle"

Moreover the embedding

$$\Omega_{q_0}^{q_1} \longrightarrow C_{q_0}^{q_1}$$ is homotopic equivalence

↖ all curves from q_0 to q_1. not only admissible.

let us consider $\Omega_{q_0}^{q_1}$ in L^∞ topology for controls

If $\boxed{\bar\gamma \text{ is regular then } \Omega_{q_0}^{q_1} \text{ is a } n\text{-dim codimension submanifold of } \Omega_{q_0}}$

dim M.

this follows from implicit function theorem since $\bar\gamma$ is a regular point of $E_{q_0}^1$

If $\mathcal{O}_{\bar\gamma}$ is a neigh of $\bar\gamma$ in Ω_{q_0} endowed with L^p topology for controls $1 \le p \le \infty$

$$\Rightarrow \Omega_{q_0}^{q_1} \cap \mathcal{O}_{\bar\gamma} \approx \text{codim } n \text{ subspace of } L^\infty([0,1], \mathbb{R}^k)$$

If $\bar\gamma$ is singular then in general FALSE

Def $\bar\gamma$ is called RIGID if there exists $\mathcal{O}_{\bar\gamma}$ neigh. of $\bar\gamma$ st $\mathcal{O}_{\bar\gamma} \cap \Omega_{q_0}^{q_1} = \{\bar\gamma\}$

in the sense that $\mathcal{O}_{\bar\gamma} \cap \Omega_{q_0}^{q_1}$ contains only reparametrizations of the curve $\bar\gamma$

EXAMPLE (Isoperimetric problem)

A 1-form on $\mathbb{R}^2$

$c = \int_{x(\cdot)} A$

Admissible curves $\quad q_0 = (x_0, 0) \quad q_1 = (x_1, c)$

$$\Omega_{q_0}^{q_1} = \{ x(\cdot) : [0,1] \to \mathbb{R}^2, \; x(0) = x_0, \; x(1) = x_1 \}$$

with $\quad c = \int_{x(\cdot)} A$

?

Recall that $\displaystyle\int_{x(\cdot)} A = \int_0^1 \langle A(x(t)), \dot{x}(t) \rangle \, dt$.

look like ?

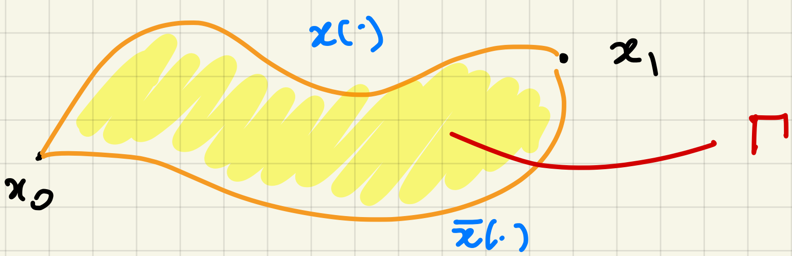
Recall $dA = b \, dx_1 \, dx_2$
where b is the function such that singular lives in $\{ b = 0 \}$

Let $\bar{x}(\cdot)$ a curve and $x(\cdot)$ with same endpoints.

$$\boxed{\int_{x(\cdot)} A - \int_{\bar{x}(\cdot)} A = \int_{\Gamma} b \, dx_1 \, dx_2}$$

STOKES FORMULA

$$\int_M d\omega = \int_{\partial M} \omega$$

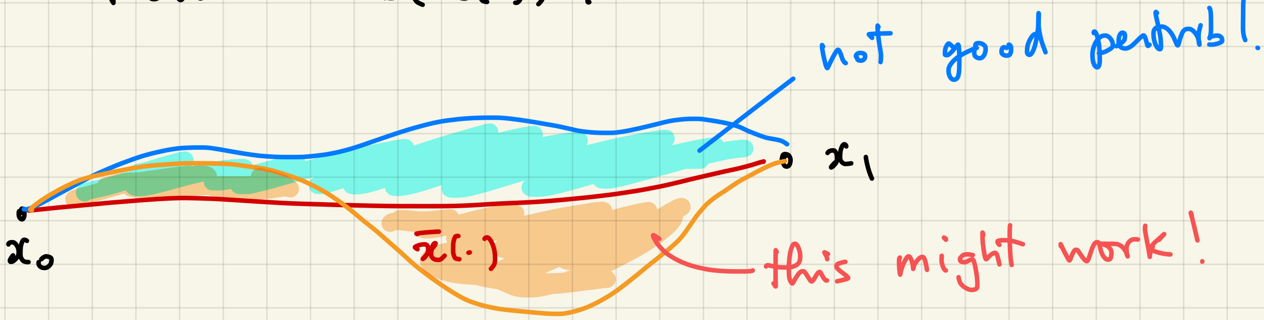


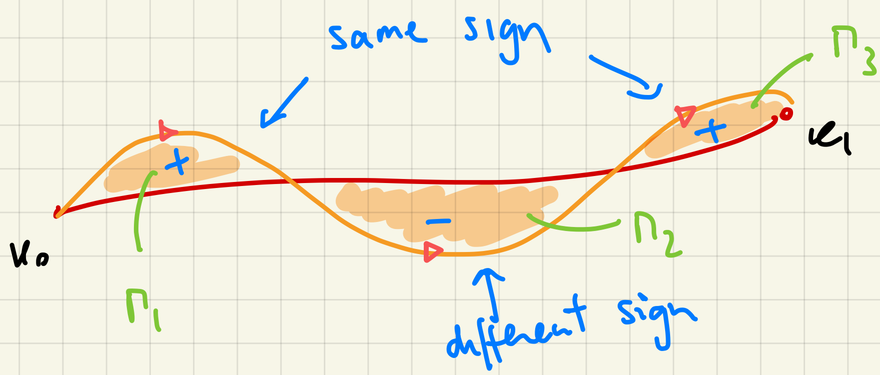
with correct orientation on ∂M induced !

So to check that two curves (with same endp. in $\mathbb{R}^2$) belong to the same loop space (in $\mathbb{R}^3$) we need

$$\int_\Gamma b\, dx_1\, dx_2 = 0 \qquad \text{where } \Gamma = \text{region spaned by two curves.}$$

Assume that $b(\bar{x}(\cdot)) \neq 0$.



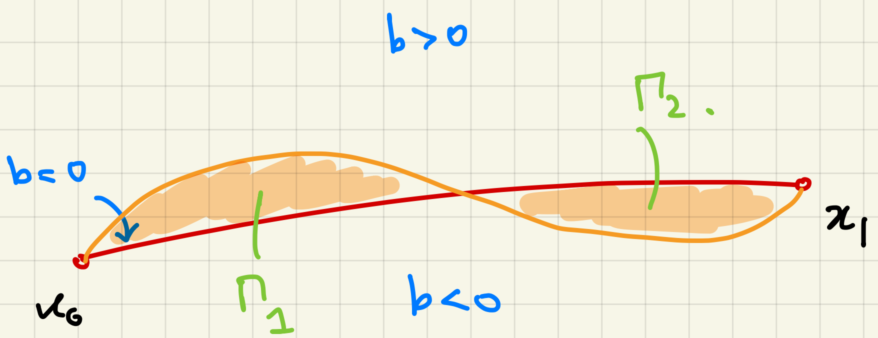


$$\int_x A - \int_{\bar{x}} A = \int_{\Gamma_1} b\, dx - \int_{\Gamma_2} b\, dx + \int_{\Gamma_3} b\, dx = 0$$

We need to balance the b-Area with sign! to have good perturbations.

Assume now $b(\bar{x}(\cdot)) = 0$ along the curve
(we have singular)

Moreover $db \neq 0$ on the curve
this implies that it is NICE abnormal

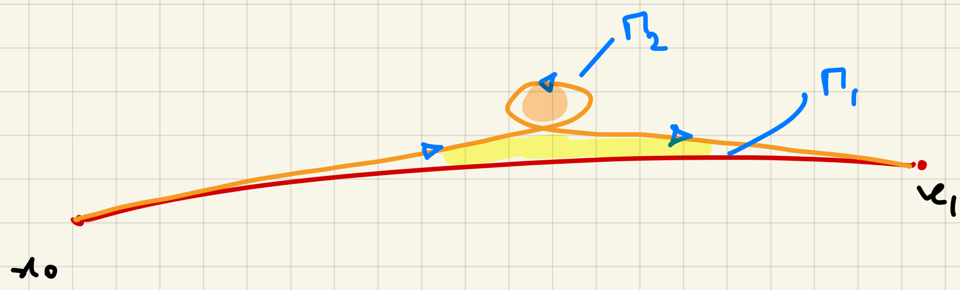


We would like to keep the difference fixed.

$$\int_{x(\cdot)} A - \int_{\bar{x}(\cdot)} A = \int_{\Gamma_1} b\, dx - \int_{\Gamma_2} b\, dx$$

orientation change but also sign of b changes. No chance if your curve has no self intersection.

Something that has self intersection you are far in the L^2 topology
The curve is rigid !



close in L^1 top.
but not L^∞ topology.
↑
think!

$$\int_x A - \int_{\bar{x}} A = \int_{\Gamma_1} b\,dx - \int_{\Gamma_2} b\,dx$$

RK If you have a metric on Δ then these become local minimizers for any metric.

So rigidity $\overset{\text{essentially}}{\Longleftrightarrow}$ local min. for any metric.

Go back to the general study of $\Omega_{q_0}^{q_1} = (E_{q_0}^1)^{-1}(q_1)$

the study is local $\Omega_{q_0}^{q_1} \cap \mathcal{O}_{\bar{\gamma}}$ ← neigh of $\bar{\gamma}$.

At critical points we can study 2^{nd} derivative. The property we want to catch here is purely ∞-dim. So difficult to build intuition on finite dim analogous. situation.

Still let us try some general reasoning.

$\varphi : U \to \mathbb{R}^m$ $\qquad u \in U$, $\varphi(u) = 0$.

$\operatorname{im} D_u\varphi = \mathbb{R}^m \subset \mathbb{R}^n$

fin dim space mani

$m < n$ $\qquad$ critical !

We split φ into two parts, remember we are interested in $\varphi^{-1}(0) \cap B_u$, B small ball neigh of u. We want to know if 0 is isolated or not in $\varphi^{-1}(0)$.

Rk If point is not critical $\Rightarrow$ never isolated due to impl. f. thm.

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^{n-m}$$

$\varphi_1 : U \to \mathbb{R}^m$

$\varphi_2 : U \to \mathbb{R}^{n-m}$

such that u regular pt of φ_1

moreover we write $u = (v, w)$ such that

$\operatorname{Ker} D_u\varphi = \{(v, 0)\}$

v is $\dim U - m$.

$w \in \mathbb{R}^m$

Hence we can write

$$\varphi(u) = \begin{pmatrix} \varphi_1(v,w) \\ \varphi_2(v,w) \end{pmatrix}$$

$\boxed{\dfrac{\partial \varphi_1}{\partial w} \text{ invertible}}$ $\quad$ by construction

We apply a smooth non linear change of variable !

using implicit f. thm on φ_1 and get

$$\varphi(u) \approx \begin{pmatrix} w \\ \varphi_2(v,w) \end{pmatrix}$$

after a change of variables

to have a meaningful 2nd derivative

$$\mathcal{O}_{\bar u} \cap \varphi^{-1}(0) \approx \{ (v,0) : \varphi_2(v,0) = 0 \}$$

Instead of φ we have to consider

$$\varphi_2 \Big|_{\operatorname{Ker} D_u\varphi}$$ its 2nd derivative is the Hessian.

can be written in coord. indep way.

also in ∞-dim. T. works.

<u>def</u> $\operatorname{Hess}_u \varphi = D^2_{\bar u} \varphi_1 \Big|_{\operatorname{Ker} D_u\varphi}$

quadratic form

<u>Prop</u> Assume there exists covector $\lambda \in T^*_{\varphi(u)} M$ such that $\lambda \cdot D_u\varphi = 0$

and $\lambda D^2_{\bar u} \varphi \Big|_{\operatorname{Ker} D_u\varphi} > 0$ (sign definite)
(as quadratic form)

in coord $\langle p, \frac{\partial \varphi_2}{\partial v^2}(\cdot,\cdot) \rangle$

Then $\bar u$ is isolated in the level set $\varphi^{-1}(0)$.

Notice $$\boxed{\lambda\, D^2_{\bar u}\varphi\Big|_{\ker D_{\bar u}\varphi} : v \longmapsto \left\langle p, \frac{\partial^2 \varphi_2}{\partial v^2}(v,v)\right\rangle}$$

second derivative only in the direction of the kernel.

$$\frac{\partial^2}{\partial \varepsilon^2}\Big|_{\varepsilon=0} \langle p, \varphi_2(\bar v + \varepsilon v, \bar w)\rangle$$

$\bar u = (\bar v, \bar w)$

The assumption: $\lambda\, D^2_{\bar u}\varphi(v,v) \geq \alpha \|v\|^2$

→ miss something here due to connection unstable!

What about necessary conditions?

Let $\boxed{Q_\lambda = \lambda\, D^2_{\bar u}\varphi\big|_{\ker D_{\bar u}\varphi}}$ this Hessian

Given $Q: V \to \mathbb{R}$ quadratic form we set

$$\boxed{\begin{aligned} \operatorname{ind}_+ Q &= \max\{\dim W \mid W \subset V : Q|_W > 0\} \quad \text{POSITIVE INDEX} \\ \operatorname{ind}_- Q &= \max\{\dim W \mid W \subset V : Q|_W < 0\} \quad \text{NEGATIVE INDEX} \end{aligned}}$$

number of positive and negative entries in the diagonal form of Q

Theorem If both $\mathrm{ind}_{\pm} Q_\lambda \geq \mathrm{corank}\, D_{\bar{u}}\varphi$ $\forall \lambda \in \mathrm{Im}\, D_u \varphi^{\perp}$, $\lambda \neq 0$.

Then u is not rigid.

Rk Sign definite $\Rightarrow$ one between ind_+ or ind_- is zero

Here we look to sign indefinite cases.
If these indices are big enough then we are not rigid

CF. Also Chapter 12 in Book.

Corollary If $\mathrm{corank}\, D_{\bar{u}}\varphi = 1$ (*) and Q_λ is sign-indefinite then $\bar{u}$ is not rigid.

Recall that here (*) λ is unique up to multiplier.

A positive result: under conditions of the theorem
$\forall\, \mathcal{O}_{\bar{u}}$, $\mathcal{O}_{\bar{u}} \cap \varphi^{-1}(0)$ contains a regular point of φ