# LECTURE 8 (A. Agrachev)

Notes by
D. Barilari
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this is the last lecture but the material on this part on singular curves is really vast. One can have a look to Chapter 12 of the Book.

little bit techical. but today we discuss some ideas.

We introduced $2^{nd}$ derivative (the good part of it) and we discussed rigid curves.
Now we apply to end-point map.

$$\boxed{E^1_{q_0} : \Omega_{q_0} \longrightarrow M \qquad E^1_{q_0}(\gamma) = \gamma(1).}$$

end-point = evaluation applied to curves starting at $q_0$

If we want to charact. rigidity we cannot apply directly the ideas.
We deal with parametrized curves.

If $\varphi : [0,1] \longrightarrow [0,1]$ reparametrization (change of var).
$\dot\varphi > 0$, $\varphi(0) = 0$, $\varphi(1) = 1$.

Consider $\underline{\gamma \circ \varphi}$ reparam. curve.

We have $E^1_{q_0}(\gamma \circ \varphi) = E^1_{q_0}(\gamma)$

for all such $\varphi$ so that the end point map do not see reparam.

Consider a family $\{\varphi_\varepsilon\}_\varepsilon$ of reparam that is $C^\infty$ wrt $\varepsilon$. $\varphi_0(t) = t$

the space of repar is $\infty$-dim

$$\left[\begin{array}{l} \dfrac{\partial^k}{\partial \varepsilon^k} E^1_{q_0}(\gamma \circ \varphi_\varepsilon) = 0 \qquad \forall\, k \geq 1 \\ \left.\dfrac{\partial \varphi}{\partial \varepsilon}\right|_{\varepsilon=0} \in \operatorname{Ker} D_\gamma E^1_{q_0} \cap \operatorname{Ker} D^2_\gamma E^1_{q_0} \end{array}\right]$$

check it!

$\Rightarrow$ the Hessian we built is very degenerate

the fact that reparam are allowed is obstacle to rigidity (because they stay in the kernel)

If we consider unparametrized curves then we have bad space in the $L^2$ topology.

the idea is to restrict study to curves with particular (selected) reparametriz.

We need to fix, given a param. curve, a fixed param. for every curve in the neighborhood. $\longrightarrow$ basic idea

Find in an $L^2$-top. in controls is difficult.

We can do it in $L^\infty$-topology in controls.

Now we look more carefully to our map.

Use controls as coordinates as follows:

$\gamma \in \Omega_{q_0}$ $\quad \gamma = x_u$ where

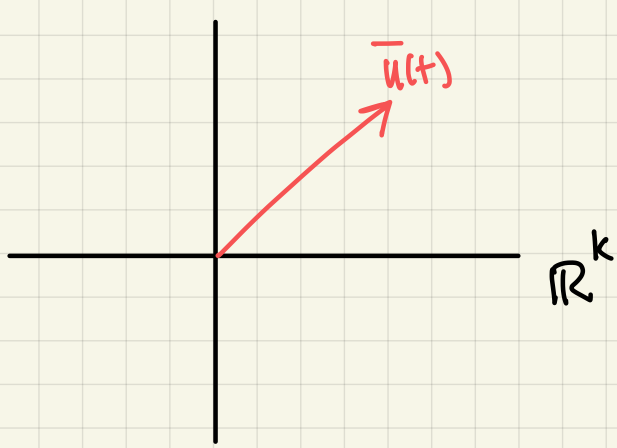
$$\dot{x}_u = X_{u(t)}(x_u(t))$$

where $X_u = \sum_{i=1}^{k} u_i X_i$

so that $\boxed{x_u \text{ is sol of } \quad \dot{x}(t) = \sum_{i=1}^{k} u_i(t) X_i(x(t))}$

if $\gamma = x_u \implies \gamma \circ \varphi = x_{\dot{\varphi}(u \circ \varphi)}$ exercise

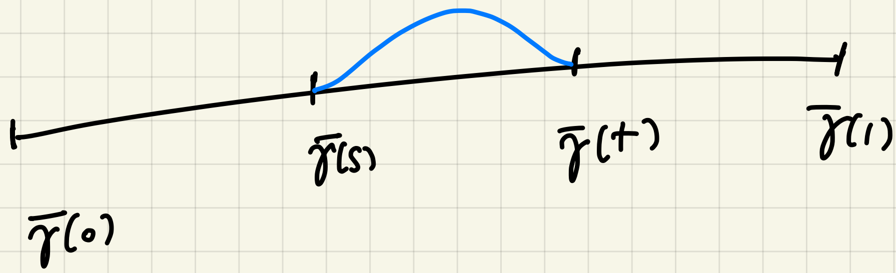
(ie. the control associated to $\gamma \circ \varphi$ is $\dot{\varphi}(u \circ \varphi)$)




space of controls

$\boxed{\text{Assume fixed } x_{\bar{u}} \text{ abnormal}}$

let us consider our curve




curves that are different from $\bar\gamma$ only on the segment.

consider $\lambda_1 \cdot D^2_{\bar u} E^1_{q_0} \Big|_{\{\bar u + v \,/\, \operatorname{supp} v \subseteq [s,t]\}}$ (*)

we would like to check rigidity of the piece on $[s,t]$.

variation is only on $[s,t]$

Assume that the form (*) is sign-indefinite for <u>any</u> $[s,t]$. ($\operatorname{ind}_+ > 0$, $\operatorname{ind}_- > 0$)

then $\lambda_1 D^2_{\bar u} E^1_{q_0}$ has infinite index (both positive and negative)

(the same is true for restrictions on segment)

<u>Morale</u> : if sign-indef on any segment then infinite positive and neg indices

To justify this let us introduce for

$\dot x = X_{\bar u(t)}(x) \rightsquigarrow \Phi_{s,t} : x(s) \longmapsto x(t)$

flow DEP ON $\bar u$. $\longrightarrow$ flow of the ODE. $\bar u$

Given $x_u(t)$ we set $y_v(t) = \Phi^{-1}_{0,t}(x_{\bar u + v}(t))$

We have $\dot{y}(t) = Y^t_v(y(t))$

where $Y^t_v = (\Phi^{-1}_{0,t})_* X_v = \sum v_i(t) (\Phi^{-1}_{0,t})_* X_i$

We have the first variation formula (cf $1^{st}$ part of the course)

$$\boxed{D_{\bar{u}} E^1_{q_0}(v) = (\Phi_{0,1})_* \int_0^1 Y^t_{v(t)}(q_0)\, dt}$$

CF FORMULAS IN $1^{ST}$ PART OF COURSE

we have also the following formula (cf Chapt 12) defined on the Ker $D_{\bar{u}} E^1_{q_0}$

$$\boxed{\lambda_1 D^2_{\bar{u}} E^1_{q_0}(v,v) = \lambda_1 (\Phi_{0,1})_* \iint_{0 \le \tau \le t \le 1} [Y^\tau_{v(\tau)}, Y^t_{v(t)}](q_0)\, dt\, d\tau.}$$

FORMULA FOR THE $2^{nd}$ derivative
→ on the Kernel of $1^{st}$ one ←

Recall $\lambda_t = \lambda_1 \circ (\Phi_{t,1})_* \overset{def}{=} (\Phi^*_{t,1}) \lambda_1$

$$\boxed{\lambda_1 D^2_{\bar{u}} E^1_{q_0}(v,v) = \iint_{0 \le \tau \le t \le 1} \langle \lambda_0 , [Y^\tau_{v(\tau)}, Y^t_{v(t)}](q_0) \rangle \, dt\, d\tau.}$$

REWRITING THE PREVIOUS FORM.

Now we would like to check on this form.
(already restricted on ker of $1^{st}$ deriv)

Now we study small pieces of trajectories.
ie for $v$ concentrated on small segments.

$\underline{\underline{Q}}$: is it true that is sign-def or not?
If not by previous discussion index is $+\infty$.

Take some $s \in [0,1]$. Take $w : [0,1] \to \mathbb{R}^k$

$$v_\varepsilon(t) = w\left(\frac{t-s}{\varepsilon}\right)$$

Sample variation rescaled and "fast" around $s$.

Applying the formula to $v_\varepsilon$ and expanding wrt $\varepsilon$ we have the following

$$= \boxed{\varepsilon^2 \iint_{0 \le \tau \le t \le 1} \langle \lambda_0, [Y^s_{w(\tau)}, Y^s_{w(\tau)}] \rangle \, dt \, d\tau. + O(\varepsilon^3)}$$

the main order term ↑ after change of variable we freeze the "upper times" at $\tau = t = 1$.

We can rewrite the main order term.

$$= \varepsilon^2 \iint_{0 \le \tau \le t \le 1} \langle \lambda_s, [X_{w(t)}, X_{w(\tau)}](\bar\gamma(s)) \rangle \, d\tau \, dt + O(\varepsilon^3)$$

$$= \varepsilon^2 \, Q^0(w) + O(\varepsilon^3)$$

$$\boxed{\int_0^1 w(t)\,dt = 0}$$

stay in the kernel at the limit.

If we consider involution $w(t) \mapsto w(1-t) = \tilde w(t)$ then one can see that

$$Q^0(\tilde w) = - Q^0(w) \Rightarrow$$ both indices are equal

## Theorem (Goh condition)

from '60 Goh without complete proof

If the singular curve $\bar\gamma$ is rigid then there exist $\lambda_t \in T^*_{\bar\gamma(t)} M$ abnormal extremal such that

$$\boxed{\langle \lambda_t, [X_i, X_j](\bar\gamma(t)) \rangle = 0 \quad \forall\, i,j = 1 \dots k}$$

GOH CONDITION

<u>Pf</u>: if not zero then localizing we can find space where the quad form is positive and negative.

RK this means that $H(\lambda_t) \equiv 0$

where $H = \left( \{ h_i, h_j \}_{i,j=1 \dots n} \right)$

Remember : $\lambda(t)$ abnormal $\Rightarrow \lambda(t) \in \mathrm{Char}_\Delta$

where $\mathrm{Char}_\Delta = \{ \lambda \in \Delta^\perp \mid \underline{\mathrm{Ker}\, H_\lambda} \neq 0 \}$

nice $\iff$ ker to be as small as possible.

To be rigid we need the kernel to be all the space.

$\begin{pmatrix} 0 & h_{12} \\ -h_{12} & 0 \end{pmatrix}$

If rank $\Delta = 2$ then $\lambda \in \mathrm{Char}_\Delta \iff H(\lambda) = 0$.

So if rank $\Delta = 2$ we have that all extremals satisfy Goh conditions. Are candidate to be rigid.

Indeed they are: if rank $=2$ all nice are rigid and so local minimizer

On the other hand, if $k > 2$, then Goh condition is exceptional. Here nice are NOT RIGID. and NOT LOC. MIN. (in general)

If Goh condition is not satisfied and curve is strictly abnormal then no local min!

crucial, no normal

If we go back at the formulas of the expansion with respect to $\varepsilon$, if the $\varepsilon^2$ term is zero we get a condition by observing the main $\varepsilon^3$ term.

theorem (Generalized Legendre condition)

If Goh condition is satisfied and moreover

$$\boxed{\langle \lambda_t , [[X_{\bar{u}(t)}, X_w], X_w] \rangle \geq \alpha \|w\|^2 \quad \forall w \perp \bar{u}}$$

then sufficiently small pieces of trajectories are rigid; and length min for every choice of a metric on $\Delta$. ] geodesics

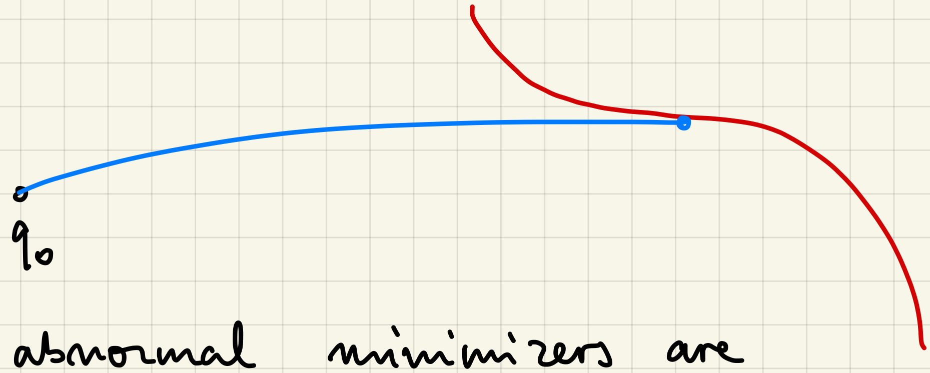
RK It appears from the quantity

$$< \lambda_0, [\tfrac{d}{dt} Y_w^t, Y_w^t](q_0)>$$

In the case $k=2$, there is just 1 vector orthog to $\bar{u}$ and the thm says that a form of 1-variable

Exercise : Nice abnormal extremals satisfy the generalized Legendre cond.

## Some final comments about $d_{SR}$




- strict abnormal minimizers are tangent to sub-Riemannian spheres

in general for normal the covector at smooth points is "orthogonal" to the sphere (it annihilates the front)

- what analytic property it determines?

$$S(x_0, r) = d_{SR}(x_0, \cdot)^{-1}(r)$$ level set

$d_{SR}$ is not lipschitz at a point reached only by abnormal min — check on Chapt 11/12

$d_{SR}(x_0, \ )$ is smooth on an open dense set of $M \setminus \{x_0\}$.

$\Rightarrow$ Open problem: what is the structure of the sphere close to points where abnormal arrive?