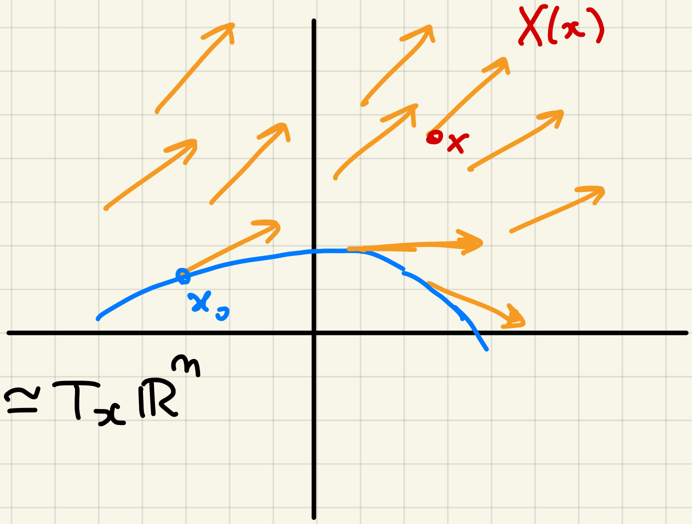
VECTOR FIELDS AND LIE BRACKETS

(also manifolds, Frobenius theorem ...)

Vector field in $\mathbb{R}^n$



$X : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

which is smooth (C^∞)

$x \in \mathbb{R}^n \quad \to \quad X(x) \in \mathbb{R}^n \simeq T_x\mathbb{R}^n$

Integral curve of X, is a solution to ODE

$$\begin{cases} \dot{x}(t) = X(x(t)) \\ x(0) = x_0 \end{cases} \Longrightarrow \quad \text{solution} \quad x(\cdot) :]-\varepsilon, \varepsilon[\longrightarrow \mathbb{R}^n$$

$$x(t) = e^{tX}(x_0)$$

exponential notation

$e^{tX} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ flow of X

$x_0 \longrightarrow e^{tX}(x_0)$

To be precise, e^{tX} is not defined in general for all t and all $x_0 \in \mathbb{R}^n$.

(Here a priori $|t| < \varepsilon$ and $\varepsilon = \varepsilon(x_0)$)

Fix $x_0 \in \mathbb{R}^n \to$ find $\varepsilon > 0$ and $\delta > 0$ such that $e^{tX} : B(x_0, \delta) \to \mathbb{R}^n \quad \forall\, |t| < \varepsilon$.

<u>def</u> We say that X is complete if we can choose for every x_0 " $\varepsilon = +\infty$ ".

" $B(x_0,\delta) = \mathbb{R}^n$ "

namely all solutions are defined for all times.

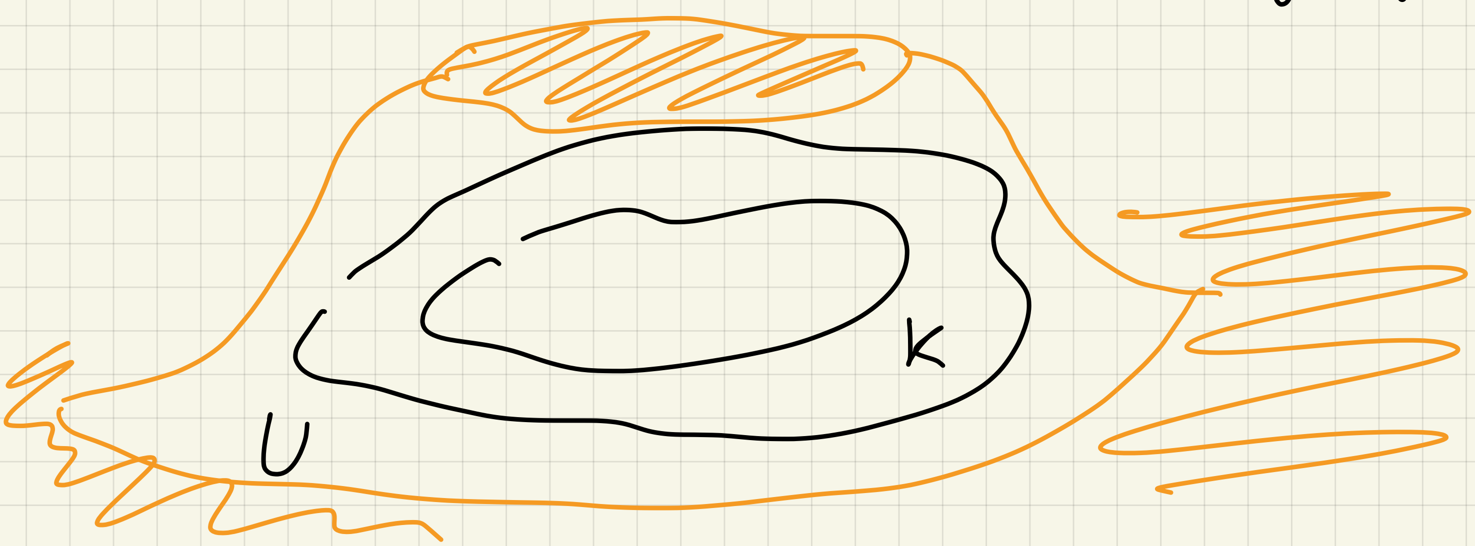
In what follows we assume all vector fields are complete.

For local purposes, namely if we are interested in what happens in a compact K, it is not restrictive to assume X complete.

One can multiply X with a bump function

$Y = \psi X$ $\qquad\qquad$ $\psi \equiv 1$ on K

$\psi \equiv 0$ on U neigh of K



the exponential notation highlights

$$e^{0X} = id \qquad e^{tX} \circ e^{sX} = e^{(t+s)X}$$

$$(e^{tX})^{-1} = e^{-tX}$$

② let us write a vector field like this

$$X : \mathbb{R}^n \xrightarrow{C^\infty} \mathbb{R}^n \qquad X(x) = (X_1(x), \dots, X_n(x))$$

as a vector $\quad X(x) = \sum_{i=1}^{n} X_i(x)\, e_i \leftarrow$ canonical basis

rename $\quad e_i := \frac{\partial}{\partial x_i}$

$$\boxed{X(x) = \sum_{i=1}^{n} X_i(x) \frac{\partial}{\partial x_i}}$$

Formally X acts as 1^{st} order diff. operator on smooth functions

$$Xf(x) = \sum_{i=1}^{n} X_i(x) \frac{\partial f}{\partial x_i}$$

So $\quad X : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$

linear operator on the vector space $C^\infty(\mathbb{R}^n)$

but it is also a derivation $\quad C^\infty(\mathbb{R}^n) \leftarrow$ algebra

$$\boxed{X(fg) = X(f)\, g + f \cdot X(g)}$$

LEIBNITZ RULE

About smooth manifolds

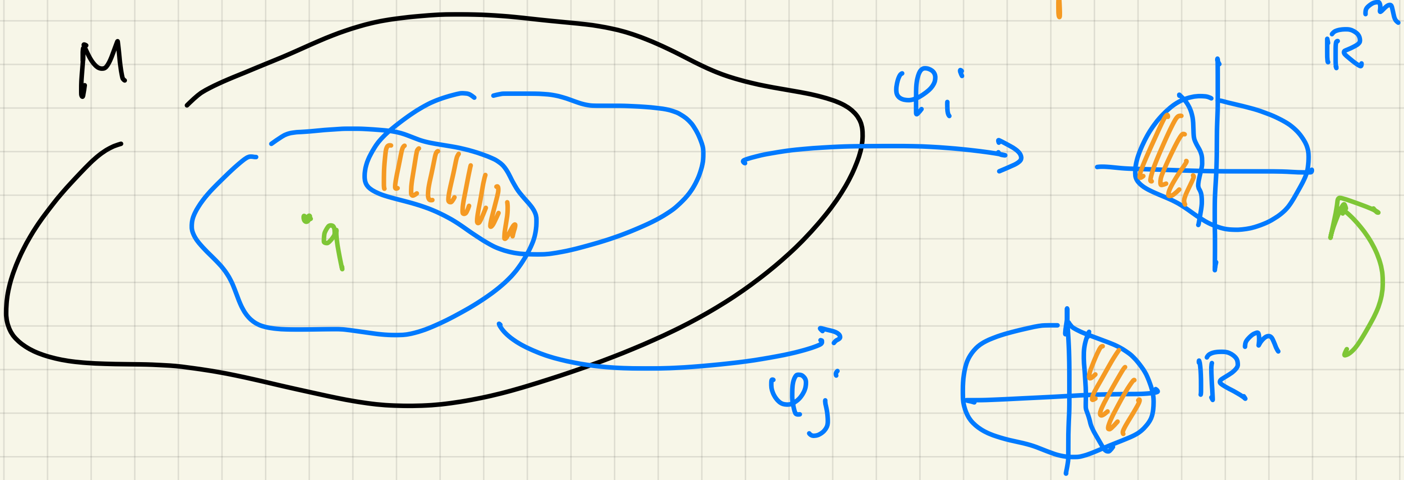
M smooth (C^∞) manifold. of dimension n

<u>def</u> M is a topological space
(is Hausdorff and 2^{nd} countable)

with open cover $\{(U_i,\varphi_i)\}_{i\in\mathbb{N}}$ $\quad M \subseteq \bigcup_{i\in\mathbb{N}} U_i$

and $\varphi_i : U_i \longrightarrow \varphi_i(U_i) =: V_i \subseteq \mathbb{R}^n$ homeomorph.

open.



$$\varphi_i \circ \varphi_j^{-1} : \underset{\mathbb{R}^n}{\underset{\cap}{\varphi_j(U_i \cap U_j)}} \longrightarrow \underset{\mathbb{R}^n}{\underset{\cap}{\varphi_i(U_i \cap U_j)}} \qquad C^\infty.$$

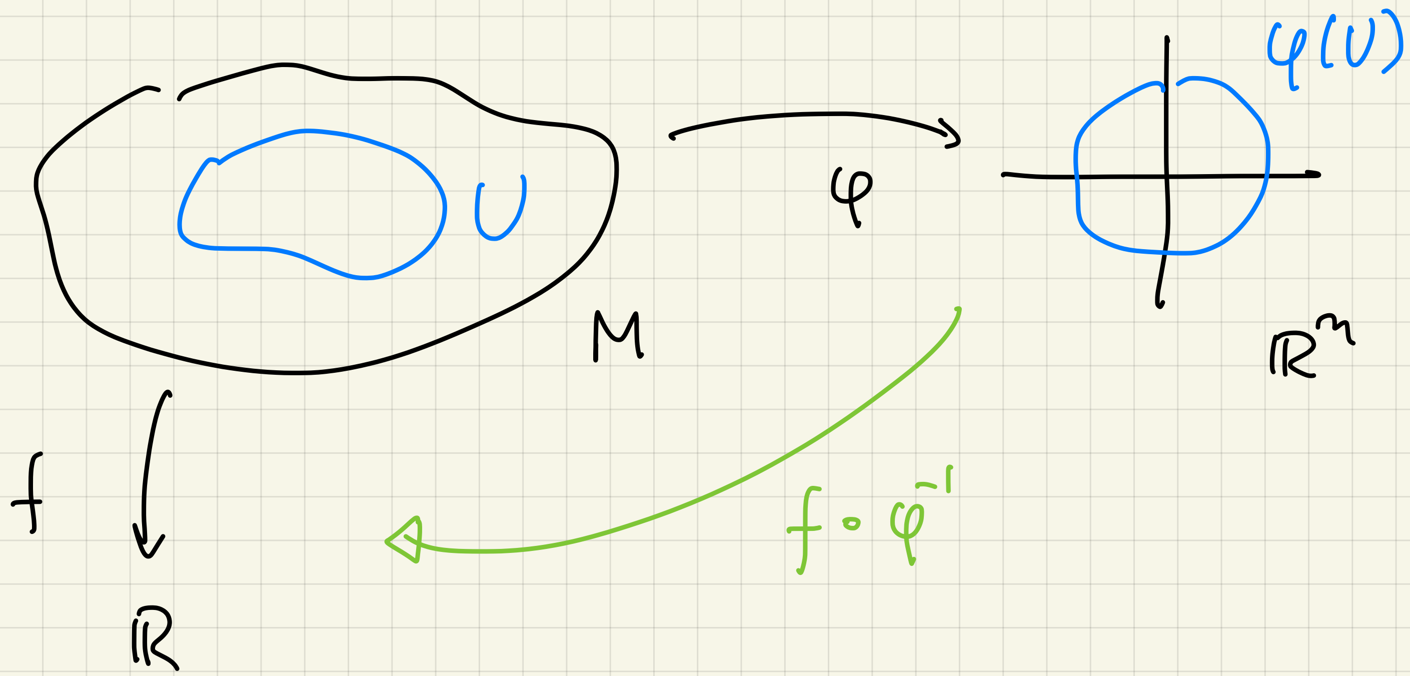
If $\varphi : U \to \mathbb{R}^n$ is a chart then

$$\varphi(q) = (x_1(q), \dots, x_n(q))$$

$$x_i : U \to \mathbb{R}$$

coordinates are smooth functions on M.

$\frac{\partial}{\partial x_i}$ is a tangent vector in the sense that it is an element of $T_x M$ = derivations at x.



$$\left(\frac{\partial}{\partial x_i}\right) f = \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}$$

a true function of n variables.

$$\hat{f} := f \circ \varphi^{-1} \qquad \hat{f}(x_1, \ldots, x_n)$$

Given a chart $\varphi : U \longrightarrow \mathbb{R}^n$ associated with coordinates $\{x_1, \ldots, x_n\}$

$$T_x M = \operatorname{span}\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}\bigg|_x$$

$$X(x) = \sum_{i=1}^{n} X_i(x) \frac{\partial}{\partial x_i} \qquad X(x) \in T_x M.$$

If we consider a smooth function $f : U \to \mathbb{R}$ and we write its different.

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, dx_i$$

dual basis

we can see that setting $\langle dx_i , \frac{\partial}{\partial x_j} \rangle = \delta_{ij}$ (*)

$$\langle df , X \rangle = \langle \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i , \sum_{j=1}^{n} X_j(x) \frac{\partial}{\partial x_j} \rangle$$

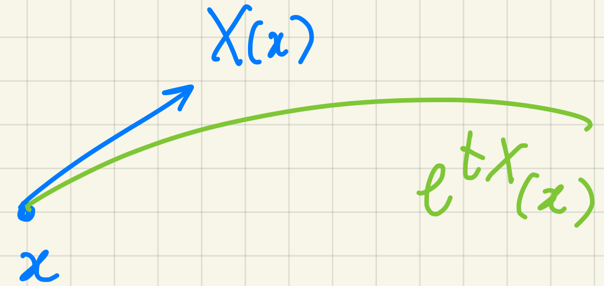
$$\overset{(*)}{=} \sum_{i=1}^{n} X_i(x) \frac{\partial f}{\partial x_i} = Xf$$

this gives a last geometric interpretation of what is the function Xf

We can compute the derivative of f along integral curves

$$\frac{d}{dt}\Big|_{t=0} f(e^{tX}(x)) = \langle df|_x , \overbrace{\frac{d}{dt}\Big|_{t=0} e^{tX}(x)}^{X(x)} \rangle = \langle df|_x , X(x) \rangle$$

$$= Xf(x)$$



RK If M smooth manifold.

$T_x M =$ tangent space at $x \in M$
(space of derivations at $x \in M$)

$$T_x^* M := (T_x M)^*$$ cotangent space

if we have a chart $\{x_1, \dots, x_n\}$

$$T_x^* M = \mathrm{span}\{dx_1, \dots, dx_n\}\big|_x$$

Lie Brackets & Frobenius theorem

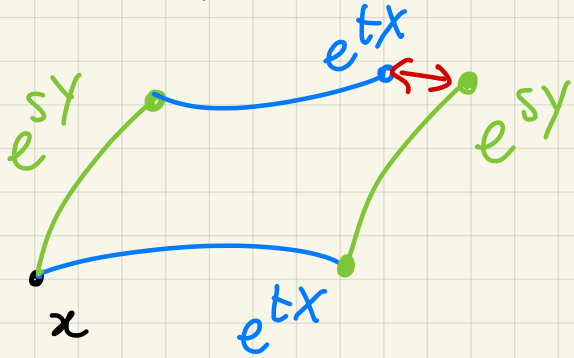
Given two vector fields X, Y

$$X = \sum_{i=1}^{n} X_i(x) \frac{\partial}{\partial x_i} \qquad Y = \sum_{i=1}^{n} Y_i(x) \frac{\partial}{\partial x_i}$$

do they commute?

→ as derivations $XYf = YXf$?

→ as flows $e^{tX} e^{sY}(x) = e^{sY} \circ e^{tX}(x)$



Indeed one can compute $XYf - YXf \overset{?}{=} 0$

$$\sum_{i=1}^{n} X_i(x) \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} Y_j(x) \frac{\partial f}{\partial x_j}\right) - \sum_{i=1}^{n} Y_i(x) \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} X_j(x) \frac{\partial f}{\partial x_j}\right)$$

$$= \sum_{i,j=1}^{n} \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i}\right) \frac{\partial f}{\partial x_j} \overset{?}{=} 0$$

$$[X,Y] = \sum_{i,j=1}^{n} \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i}\right) \frac{\partial}{\partial x_j}$$

the Bracket beetween X and Y
(it is again a 1st order diff operator)
so it is a vector field.

Prop $e^{-tY} e^{-tX} e^{tY} e^{tX}(x) = x + t^2 [X,Y](x) + o(t^2)$

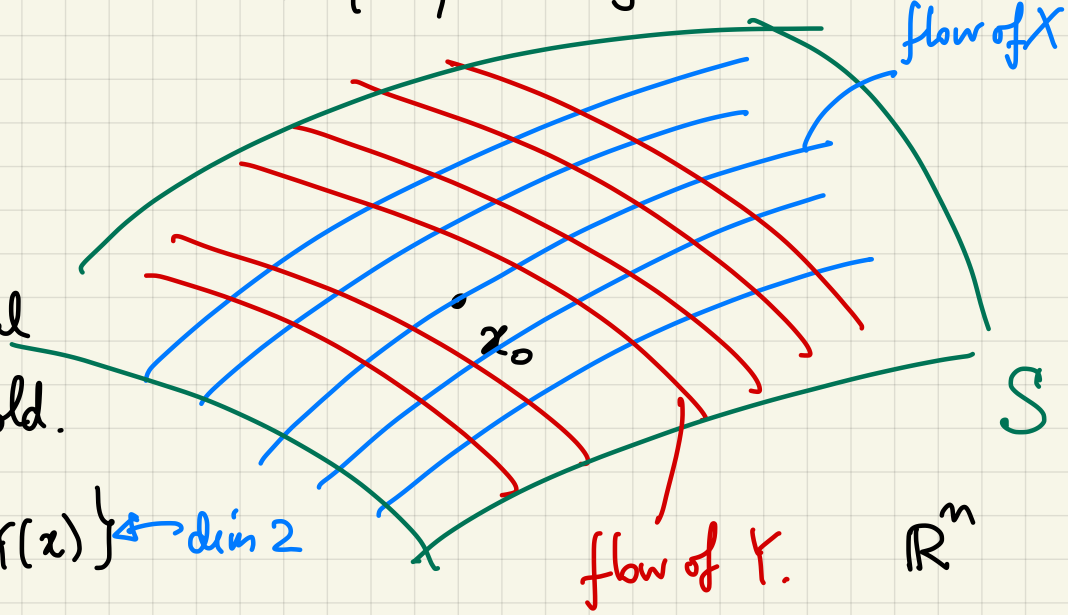
"parking manoeuvre"
with X and Y

Prop $[X,Y] = 0 \iff e^{tX} \circ e^{sY} = e^{sY} \circ e^{tX} \quad \forall s,t.$

the idea is that if $[X,Y]=0$ then combining flows of X and Y starting from a point x_0 we are confined on the set

$$S=\{ e^{tX}\circ e^{sY}(x_0) \mid t,s\in\mathbb{R} \}$$

Indeed one can prove that S is a 2 dimensional immersed submanifold.



$T_xS = \operatorname{span}\{X(x),Y(x)\} \leftarrow$ dim 2

(implicitly assuming that X,Y are linearly independent)

Theorem (Frobenius) let us consider smooth $X_1,\dots,X_k$ lin ind. (at every point) vector fields in $\mathbb{R}^n$ (or manifold)

We have two equivalent properties

① the family $X_1,\dots,X_k$ is involutive, i.e.

$$(*)\quad [X_i,X_j]=\sum_{\ell=1}^{k} c_{ij}^{\ell}\, X_\ell \qquad \forall i,j=1\dots k$$

for some $c_{ij}^{\ell}\in C^\infty(\mathbb{R}^n)$

② $\forall x_0 \in \mathbb{R}^n$ there exists a k-dimensional submanifold S such that

locally defined.

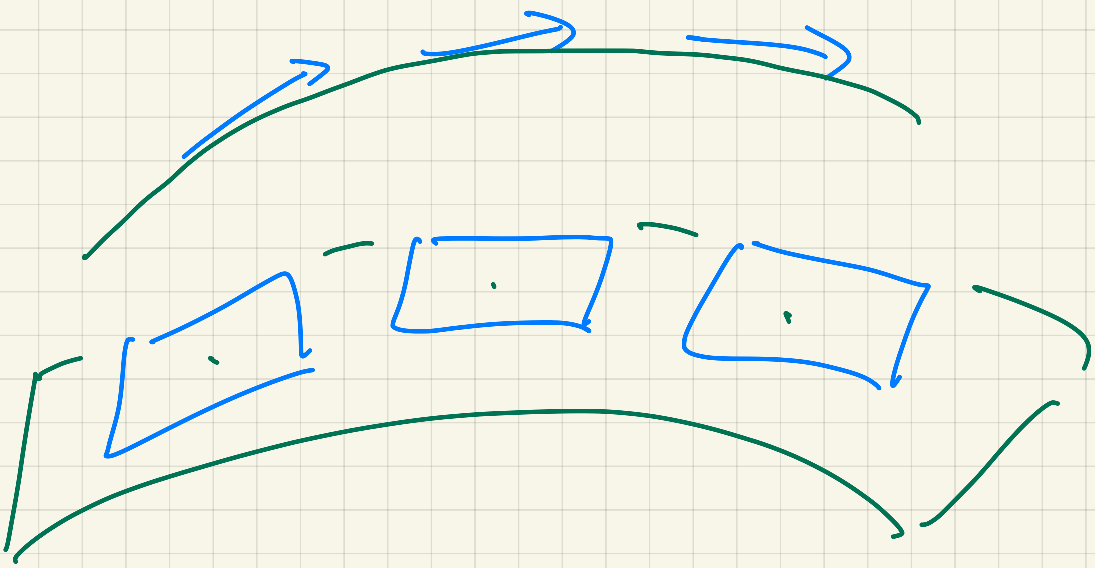
$$T_x S = \text{span}\{X_1, \dots, X_k\}|_x \quad \forall x \in S$$

$D_x = \text{span}\{X_1, \dots, X_k\}|_x$ family of k-dim subspaces of $\mathbb{R}^n$

DISTRIBUTION

① $\iff$ $[D, D] \subseteq D$ INVOLUTIVITY.

② $\iff$ $\exists S : T_x S = D_x$ INTEGRABILITY.



the lemma says

② $\Rightarrow$ ①.

<u>Lemma</u> If two vector fields are tangent to a submanifold S, then $[X,Y]$ is also tangent to S.

Examples (1) In $\mathbb{R}^3 = \{(x,y,z)\}$ we consider

$$X = \frac{\partial}{\partial x} \qquad Y = \frac{\partial}{\partial y}$$

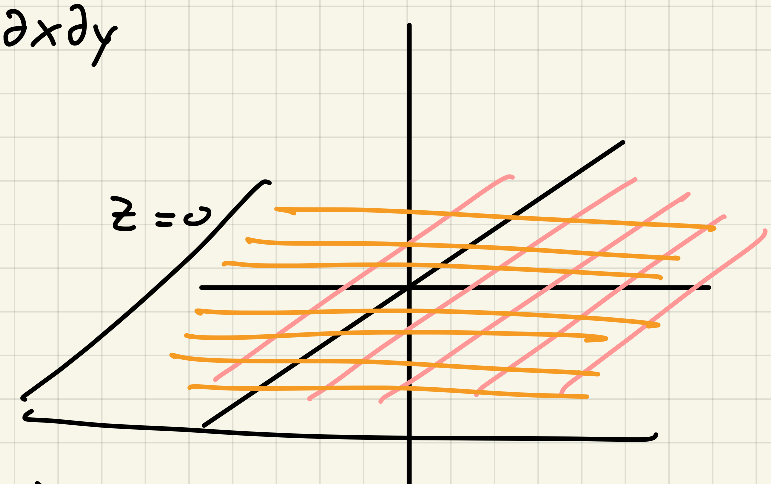
In this case $[X,Y] = 0$

$$XYf = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial^2 f}{\partial x \partial y} \rightarrow [X,Y]f = 0.$$

$$YXf = \quad \cdots \quad = \frac{\partial^2 f}{\partial x \partial y}$$

$$e^{tX}(x_0,y_0,z_0) = (x_0+t, y_0, z_0)$$

$$e^{tY}(x_0,y_0,z_0) = (x_0, y_0+t, z_0)$$



$D = \text{span}\{X,Y\} \quad [D,D] \subseteq D.$

(2) In $\mathbb{R}^2 \times S^1$ (or $\mathbb{R}^3$) $\quad (x, y, \theta)$

$$X = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \qquad Y = \frac{\partial}{\partial \theta}$$

here
$$[X,Y] = \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)\frac{\partial}{\partial \theta} - \frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)$$

so $\quad [X,Y] = \sin\theta \dfrac{\partial}{\partial x} - \cos\theta \dfrac{\partial}{\partial y}$.

it is not true that $[D,D] \subseteq D$.

Indeed $\quad \dim \operatorname{span}\{X, Y, [X,Y]\}\big|_q = 3$

$q = (x, y, \theta)$.

3 Example of the ball rolling on the plane

$$M = \mathbb{R}^2 \times SO(3) \leftarrow \text{dim } 5$$

$D = \operatorname{span}\{X, Y\}$ the two allowed movements.

In this case (check formulas from 1st lecture)

$$\dim \operatorname{span}\{X, Y, [X,Y]\} = 3$$

at every point

$$\dim \operatorname{span}\{X, Y, [X,Y], [X,[X,Y]], [Y,[X,Y]]\} = 5$$

at every point.