SUB-RIEMANNIAN GEOMETRY

(and the Rashevskii - Chow theorem).

A sub-Riemannian structure on $\mathbb{R}^n$ (or on a connected smooth manifold M) is given by a family $X_1, \dots, X_k$ of everywhere lin. indep. and globally def vector fields.

$2 \leq k \leq n$

We denote $D = \mathrm{Span}\{X_1, \dots X_k\}$ the distribution and we require that the bracket generating condition

$$\dim \mathrm{Lie}\, D|_x = \dim \mathrm{span}\left\{ [X_{i_1}, \dots, [X_{i_{m-1}}, X_{i_m}]]|_x : \begin{array}{l} m \in \mathbb{N} \\ 1 \leq i_j \leq k \end{array} \right\}$$

$$= n \qquad \forall x \in \mathbb{R}^n$$

On D one can define an inner product g by imposing that $X_1, \dots, X_k$ is an orthonormal basis

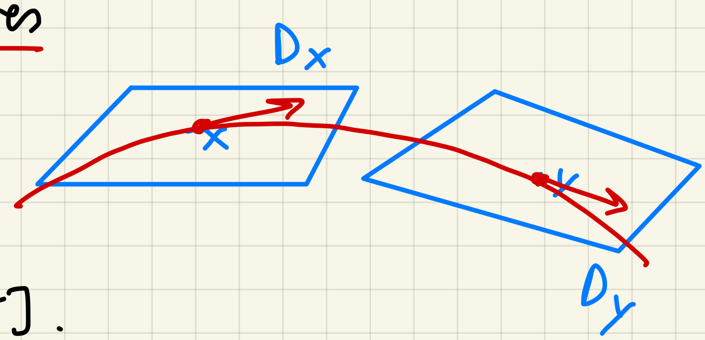
$$g(X_i, X_j) = \delta_{ij} \qquad i,j = 1 \dots k$$

We migth think to a sub-Riem structure as a triple (M, D, g) with D brack-gen distr.

g inner product on D

RK If $k = n \Rightarrow$ we have a Riemannian structure

Horizontal (or admissible) curves



Is a Lipschitz curve

$\gamma : [0,T] \to \mathbb{R}^n$ such that

$\dot\gamma(t) \in D_{\gamma(t)}$ a.e. $t \in [0,T]$.

In particular we can write

$$\boxed{\dot\gamma(t) = \sum_{i=1}^{k} u_i(t)\, X_i(\gamma(t))}$$

meas. & ess. bdd.

$u = (u_1, \dots, u_k) \in L^\infty([0,T], \mathbb{R}^k)$

We can define the length

$$\ell_{SR}(\gamma) = \int_0^T \|\dot\gamma(t)\|\, dt = \int_0^T g(\dot\gamma(t), \dot\gamma(t))^{1/2}\, dt =$$

(X_i o.n basis)

$$= \int_0^T \Big(\sum_{i=1}^k u_i(t)^2\Big)^{1/2} dt$$

$$= \int_0^T |u(t)|\, dt$$

$|u(t)|_{\mathbb{R}^k}$

The sub-Riemannian distance

$$d_{SR}(x,y) = \inf \{ \ell_{SR}(\gamma) \mid \dot\gamma(t) \in D_{\gamma(t)},\ \gamma(0) = x,\ \gamma(T) = y \}$$

Observe: $d_{SR} < +\infty$ on $\mathbb{R}^n \times \mathbb{R}^n$ iff $\forall\, x,y \in \mathbb{R}^n$

$\exists\, \gamma : [0,T] \to \mathbb{R}^n$ $\dot\gamma(t) \in D_{\gamma(t)}$ $\gamma(0) = x,\ \gamma(T) = y$

Lemma ① the length is invariant by monotone Lipschitz reparametrization

② Every admissible curve is the reparametrization of a curve with constant speed (unit)

Proof ① Change of variable formula in the AC class.

② One can define the length parameter

$$\varphi(t) = \int_0^t \|\dot\gamma(\tau)\| \, d\tau = \int_0^t |u(\tau)| \, d\tau.$$

and then define a new curve $\gamma_r(s) = \gamma(t)$ if $s = \varphi(t)$

Check: γ_r is well-def, Lipsch, and with unit speed.

Theorem (Rashevskii – Chow, 1938-39)

Let (M, D, g) be a sub-Riemannian manifold

Then
- $\forall\, x, y \in M$ $\exists$ horizontal curve joining x and y
- (M, d_{SR}) is a metric space. The d_{SR}-topology is equivalent to the manifold topology (locally euclidean)

⚠ Indeed what we find in the proof is that $d_{SR} : M \times M \to \mathbb{R}$ is continuous

but it is never C^∞ on $M \times M$.

As a conseq. d_{SR} balls are never C^∞. if $\underline{k < n}$.

The proof (is long) and start with a lemma about flows of vector fields.

Proposition Under previous assumptions, for every $x_0 \in M$ and every V neigh of the origin in $\mathbb{R}^n$

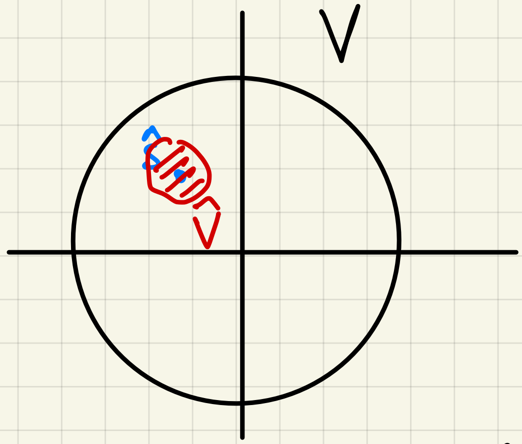
there exists
- $\hat{s} = (\hat{s}_1, \dots, \hat{s}_n) \in V$
- a choiche $\underbrace{X_{i_1}, \dots, X_{i_n}}_{n \text{ vector fields}}$ in D

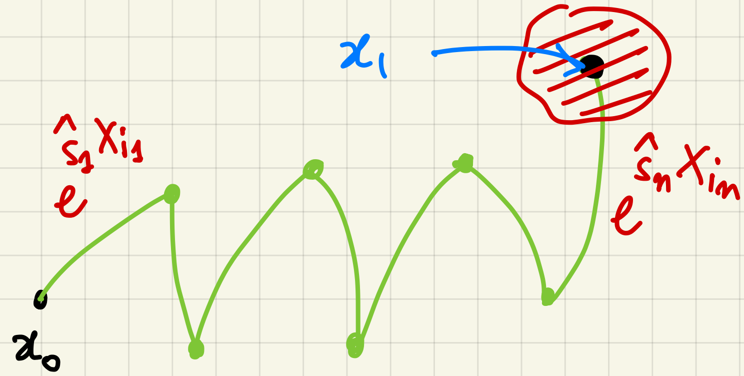
$n > k$

such that the map

$$\boxed{\psi : \mathbb{R}^n \to M \qquad \psi(s_1 \dots s_n) = e^{s_n X_{i_n}} \circ \dots \circ e^{s_1 X_{i_1}}(x_0)}$$

is a local diffeo at $\hat{s}$.





<u>Rk</u> In general $\hat{s} \neq 0$ because the image of the differential of ψ at $s=0$

$$\left.\frac{\partial \psi}{\partial s_j}\right|_{s=0} = X_{i_j}(x_0) \qquad \operatorname{im} D_0\psi \subseteq D_{x_0}$$

$$\operatorname{im} D_0 \psi \neq T_{x_0} M$$

when $k < n$.

The prop says that there exists $\hat{V} \subset V$ such that $\psi : \hat{V} \longrightarrow \psi(\hat{V})$ is a diffeo. (ψ covers a neighborhood of x_1 on M)

We use then a trick which is a kind of "return method" (in control theory)

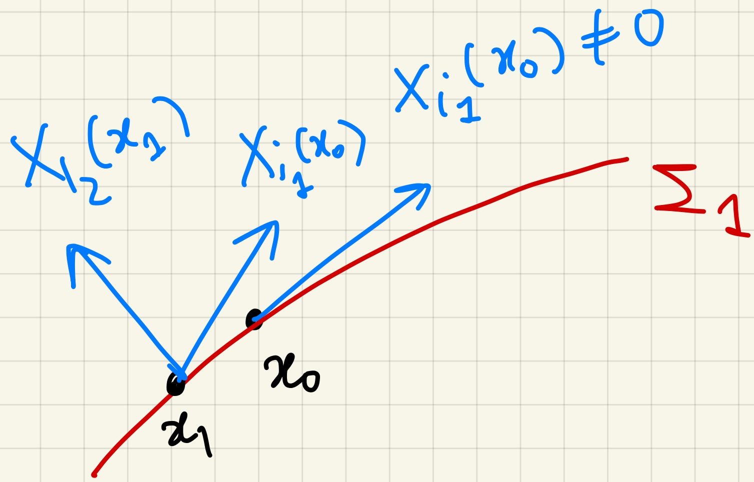
Define the map $\hat{\psi} : \mathbb{R}^n \longrightarrow M$ — the order is crucial!

$$\hat{\psi}(s_1, \dots, s_n) = e^{-\hat{s}_1 X_{i_1}} \circ \dots \circ e^{-\hat{s}_n X_{i_n}} \psi(s_1, \dots, s_n)$$

in particular $\hat{\psi}(\hat{s}_1, \dots, \hat{s}_n) = x_0$.

$\nwarrow$ 2n pieces!

Very sketch of the Proof



① $\exists\ X_{i_1}(x_0) \neq 0$ otherwise $X_i(x_0) = 0$ $\forall i = 1 \dots k$

$\Rightarrow [X_i, X_j](x_0) = 0$.

② $\exists\ X_{i_2}$ not everywhere tangent to Σ_1 otherwise no brack. gen. ass...

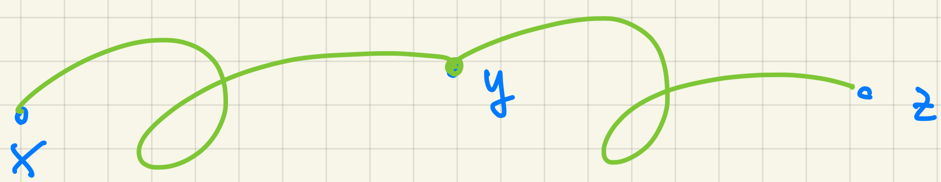
Consequence

Given x_0 $\exists\ U_{x_0}$ (neigh of x_0) : $d_{SR}(x_0, y) < +\infty$ $\forall y \in U_{x_0}$.

We want to prove:

1. d_{SR} is symmetric and sat. triang inequality
2. $d_{SR}(x,y) < +\infty \quad \forall x,y \in M$
3. $d_{SR}(x,y) = 0$ implies $x = y$.
4. $\forall x_0 \ \forall \varepsilon > 0 \ \exists U_{x_0} \ : \ U_{x_0} \subset B_{SR}(x_0, \varepsilon)$
5. $\forall x_0 \ \forall U_{x_0} \ \exists \delta > 0 \ : \ B_{SR}(x_0, \delta) \subset U_{x_0}$

(1) $d_{SR}(x,y) = d_{SR}(y,x)$ because the reversed param. of a hor. curve is hor.



Similarly the triangular inequality holds.

(2) Use the local result of the Prop + the fact that M is connected and d_{SR} symmetric.

(3) $\Leftarrow$ (5) indeed $x \neq y \Rightarrow \exists \ U_x \not\ni y$

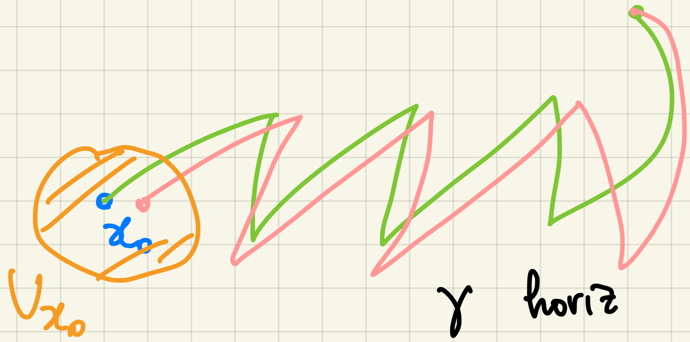
$\Rightarrow \ d_{SR}(x,y) \geq \delta > 0$.

(4) We use our proposition.

Take $V = \left\{ s \in \mathbb{R}^n \ \middle| \ \sum_{i=1}^{m} |s_i| < \frac{\varepsilon}{2} \right\}$

Fix $U_{x_0} = \hat{\varphi}(\hat{V})$

For $y \in U_{x_0}$



γ horiz made by $2n$ pieces. each of which is an integral curve of a vector field X_{i_j}

$$d_{SR}(x_0, y) \leq \ell(\gamma)$$

$$\leq |s_1| + \dots + |s_n| + |\hat{s}_1| + \dots + |\hat{s}_n|$$

but $s, \hat{s} \in \hat{V} \subseteq V$

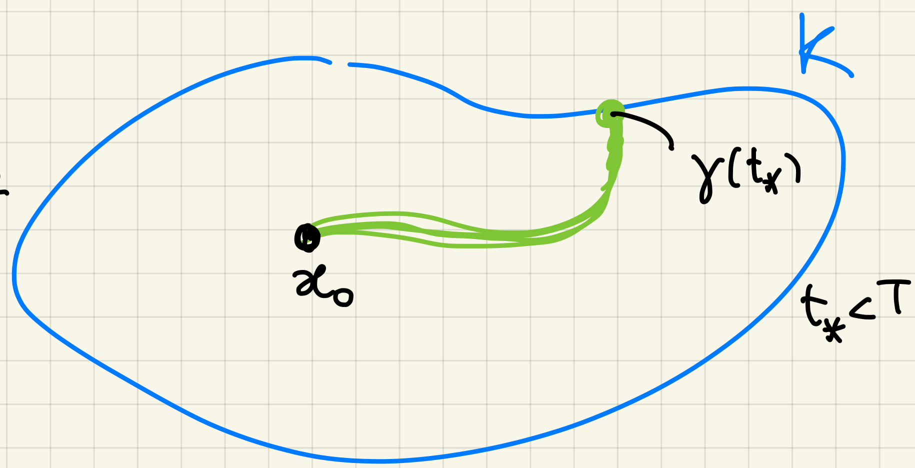
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \Longrightarrow \qquad \boxed{U_{x_0} \subseteq B(x_0, \varepsilon)}$$

<u>Lemma</u> let $x_0 \in M$ and $K \subset M$ a compact, $x_0 \in \mathring{K}$ then $\exists\, \delta_K > 0$: every horiz curve starting from x_0 and $\ell(\gamma) \leq \delta_K$ is contained in K.

<u>Proof</u> Set

$$C_K = \max_K \left(\sum_{i=1}^{k} |X_i(x)|^2 \right)^{1/2}$$

norm in $\mathbb{R}^n$ (or in coord)



And fix $\delta_K > 0$ such that $\text{dist}(x_0, \partial K) > C_K \delta_K$

By contradiction assume there exists

$$\gamma : [0,T] \to M \qquad \ell(\gamma) \leq \delta_K \qquad \gamma(0) = x_0$$

and such that $t_* = \sup \{ t \in [0,T] \mid \gamma|_{[0,t]} \subseteq K \}$

$$t_* < T$$

we have

$$|\gamma(t_*) - \gamma(0)| \leq \int_0^{t_*} |\dot{\gamma}(t)| \, dt$$

$$|u| = \left(\sum u_i^2 \right)^{1/2}$$

$$\leq \int_0^{t_*} \sum_{i=1}^{K} |u_i(t) X_i(\gamma(t))| \, dt$$

$$\text{(Cauchy-Schw)} \leq C_K \int_0^{t_*} |u(t)| \, dt \leq C_K \delta_K < \operatorname{dist}(x_0, \partial K)$$

<u>Proof of (5)</u> Fix $x_0 \in M$ and K compact neigh.

Fix $\varepsilon > 0$ and set $\delta := \min \left\{ \delta_K, \frac{\varepsilon}{C_K} \right\}$.

we want to prove that $B(x_0, \delta) \subseteq U_{x_0}$

where $U_{x_0} = \{ |x - x_0| < \varepsilon \}$

We show $|x - x_0| < \varepsilon$ whenever $d_{SR}(x_0, x) < \delta$

$d_{SR}(x_0, x) = \inf \{ \ell(\gamma) \mid \dots \}$

let γ_n be a sequence of admissible curves

$$\ell(\gamma_n) \longrightarrow d_{SR}(x_0, x) \qquad \ell(\gamma_n) \le \delta \quad \forall n$$

By our lemma $\quad \ell(\gamma_n) \le \delta \le \delta_K$

$$\Longrightarrow \quad \gamma_n([0,T]) \subseteq K$$

Repeating the estimate

$$|x - x_0| = |\gamma_n(T) - \gamma_n(0)| \le C_K \, \ell(\gamma_n)$$

For $n \to +\infty$

$$|x - x_0| \le C_K \, d_{SR}(x, x_0) \le C_K \delta < \varepsilon$$

<u>Corollary</u> $\quad d_{SR}$ continuous on $M \times M$.