EXISTENCE OF LENGHT - MINIMIZERS

(and some examples)

Recall from the previous lectures:

A sub-Riemannian structure

- $M = \mathbb{R}^n$ or smooth manifold , $n \geq 3$
- $\{X_1, \dots, X_k\}$ glob. def vector fields
  lin ind
  $k \geq 2$
  satisfy the bracket gen. condition $\dim \text{Lie}\, D|_x = n$
  $k < n$

$D = \text{span}\{X_1, \dots, X_k\}$ distribution

$g(X_i, X_j) = \delta_{ij}$ metric on D

$\gamma : [0,T] \to \mathbb{R}^n$ Lipschitz such that $\dot\gamma(t) \in D_{\gamma(t)}$ a.e $t \in [0,T]$

horizontal curve.

length
$$l_{SR}(\gamma) = \int_0^T g(\dot\gamma(t), \dot\gamma(t))^{1/2} dt = \int_0^T \left(\sum_{i=1}^k u_i(t)^2\right)^{1/2} dt$$

$$\dot\gamma(t) = \sum_{i=1}^k u_i(t) X_i(\gamma(t))$$

$$d_{SR}(x,y) = \inf\left\{ \ell_{SR}(\gamma) \;\middle|\; \gamma \text{ horizontal join } x \;\&\; y\right\}$$

distance

<u>Rashevski - Chow thm</u> M connected $d_{SR}(x,y) < +\infty \quad \forall x, y$

topology (M, d_{SR}) = locally euclidean.

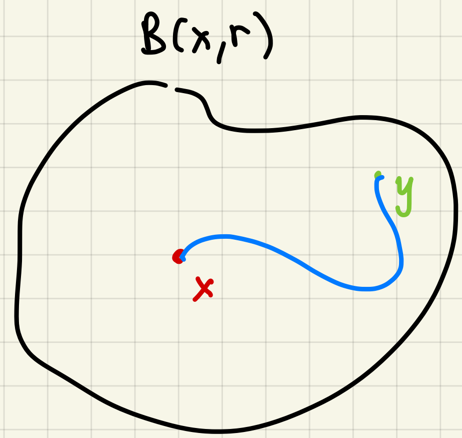
$B(x,r) = B_{SR}(x,r)$ = ball center x, rad $r > 0$

Corollary Given $x \in \mathbb{R}^n$ then $B_{SR}(x,r)$ has compact closure for $r>0$ small enough

Def A horizontal curve $\gamma : [0,T] \to M$ is a length minimizer if $\ell_{SR}(\gamma) = d_{SR}(\gamma(0), \gamma(T))$.

Theorem (existence of min)
Assume that $B_{SR}(x,r)$ has compact closure. then $\forall\, y \in B(x,r)$ $\exists$ length min joining x and y

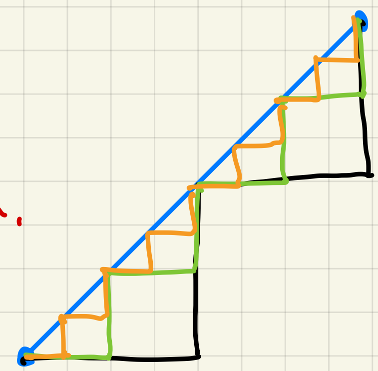
Remark One can prove that (M, d_{SR}) is complete if and only if $\forall\, x \in M$, $r>0$ $B_{SR}(x,r)$ has compact closure.



Crucial property: (cf. references)
Assume that $\gamma_n : [0,1] \to M$ horiz. const speed. such that $\gamma_n \rightrightarrows \gamma$ uniformly. then

$$\ell_{SR}(\gamma) \leq \liminf_{n \to +\infty} \ell_{SR}(\gamma_n)$$

semicontinuity of the length.



$$\ell(\gamma) = \sqrt{2} < 2 = \lim_n \ell(\gamma_n)$$

Proof of the theorem. Let $r>0$ st $\overline{B(x,r)}$ compact and $y\in B(x,r)$. Let $\{\gamma_n\}_n$ a minimizing seq.

$\gamma_n : x \rightsquigarrow y$

$$\gamma_n : [0,1] \to M \quad \text{hor} \qquad \ell(\gamma_n) \longrightarrow d_{SR}(x,y)$$

const. speed.

Since $d_{SR}(x,y) < r \implies$ we can assume $\ell(\gamma_n) < r$ $\forall n\in\mathbb{N}$

$$\implies \gamma_n([0,1]) \subset \overline{B(x,r)} =: K$$

We can repeat our estimates of last time to prove

$$|\gamma_n(t) - \gamma_n(s)| \le \int_s^t |\dot\gamma_n(\tau)|\, d\tau$$

$\dot\gamma_n = \sum u_i X_i(\gamma_n)$

Cauchy - Schwartz

$$\le \int_s^t \Big|\sum_{i=1}^{k} u_{i,n}(\tau)\, X_i(\gamma_n(\tau))\Big|\, d\tau$$

$$\le C_K\, \ell(\gamma_n)\, |t-s| \le C_K\, r\, |t-s|$$

so the γ_n are equilipschitz.

By Ascoli Arzela thm $\exists$ subsequence $\{\gamma_n\}$ same symbol

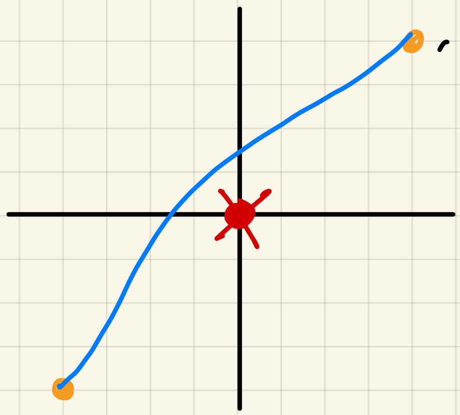
$\gamma_n \rightrightarrows \gamma$. By semicont.

$$\ell_{SR}(\gamma) \le \liminf_n \ell_{SR}(\gamma_n) = d_{SR}(x,y)$$

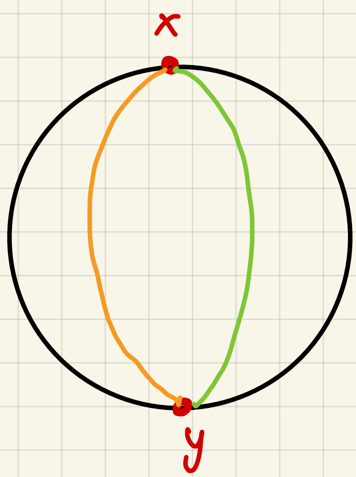
$\implies \gamma$ is a length - minimizer.

Remark ① In general we do not have existence of minimizers for every pair of points.

example. euclidean case $M = \mathbb{R}^2 \setminus \{0\}$
↑ not complete.



② Minimizers, if they exist, are not unique in general.



EXAMPLES ① The Heisenberg group

This is the sub-Riemannian structure on $M = \mathbb{R}^3$ defined by

$$X = \partial_x - \frac{y}{2}\partial_z \qquad Y = \partial_y + \frac{x}{2}\partial_z$$

so that $D = \operatorname{span}\{X, Y\}$ and X, Y are g-o.n.

D is bracket generating since

$$[X,Y] = \left(\partial_x - \frac{y}{2}\partial_z\right)\left(\partial_y + \frac{x}{2}\partial_z\right)$$

$$- \left(\partial_y + \frac{x}{2}\partial_z\right)\left(\partial_x - \frac{y}{2}\partial_z\right) =$$

$$= \partial_x\left(\frac{x}{2}\right)\partial_z + \partial_y\left(\frac{y}{2}\right)\partial_z = \partial_z$$

$\dim \operatorname{span}\{X, Y, [X,Y]\} = 3$ at every point since

$$\begin{array}{c} \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{y}{2} & \frac{x}{2} & 1 \end{pmatrix} = 1 \neq 0 . \\ \qquad\quad X \quad\; Y \quad [X,Y] \end{array}$$

By Rashevski- Chow theorem: $\forall\, q_0, q_1 \in \mathbb{R}^3$

$q_0 = (x_0, y_0, z_0)$
$q_1 = (x_1, y_1, z_1)$ $\exists$ hor. curve joining them.

What are horizontal curves?

$$\dot\gamma(t) = u_1(t)\, X_1(\gamma(t)) + u_2(t)\, X_2(\gamma(t)) \longleftrightarrow \begin{cases} \dot x = u_1 \\ \dot y = u_2 \\ \dot z = -\frac{y}{2} u_1 + \frac{x}{2} u_2 \end{cases}$$

$$\dot\gamma(t) \in D_{\gamma(t)}$$

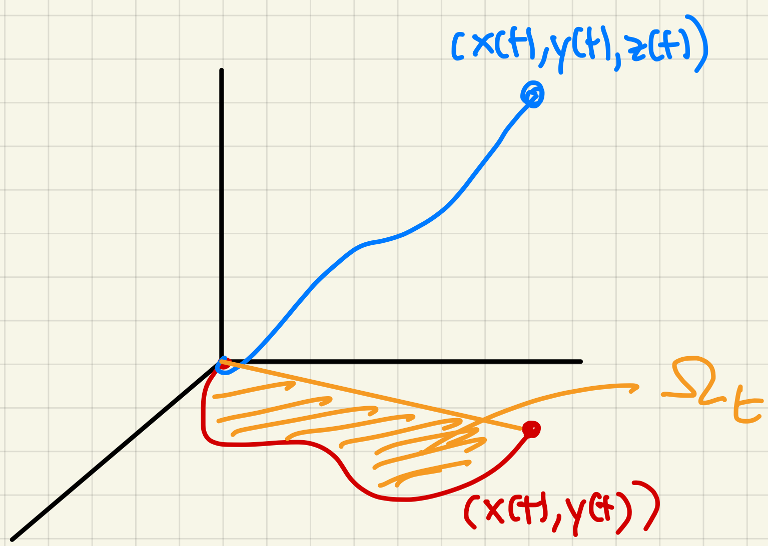
In particular, given u_1, u_2 $\rightsquigarrow$ we can find x, y

and then $x(t) = \cancel{x_0} + \int_0^t u_1(s)\,ds \qquad y(t) = \cancel{y_0} + \int_0^t u_2(s)\,ds$

$$z(t) = \cancel{z_0} + \int_0^t \left(-\frac{y(s)}{2} u_1(s) + \frac{x(s)}{2} u_2(s) \right) ds.$$

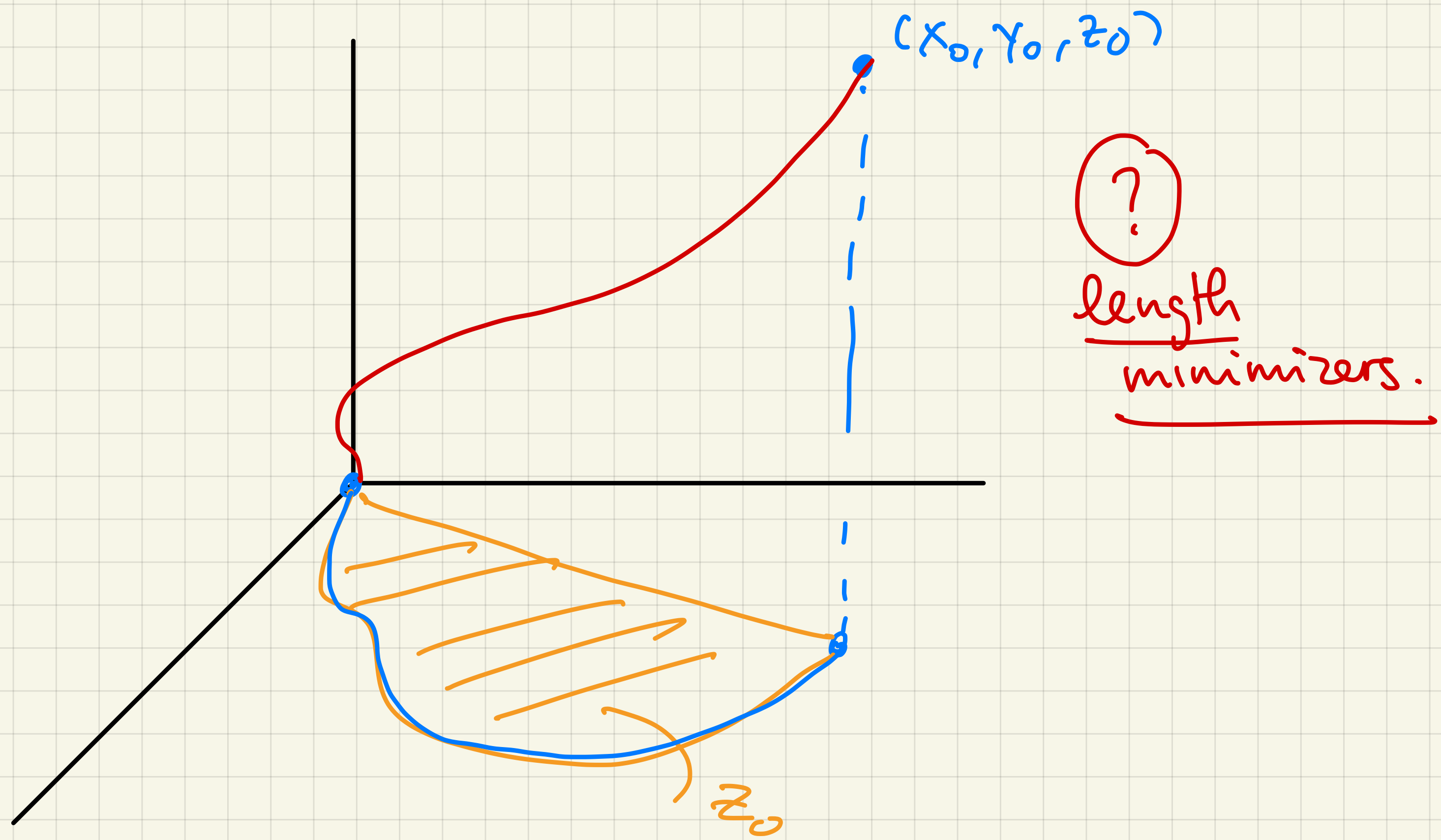
fix $q_0 = (x_0, y_0, z_0) = (0,0,0)$

$$\underline{z(t)} = \int_0^t \frac{1}{2}\left(-y\dot x + x \dot y\right) ds$$



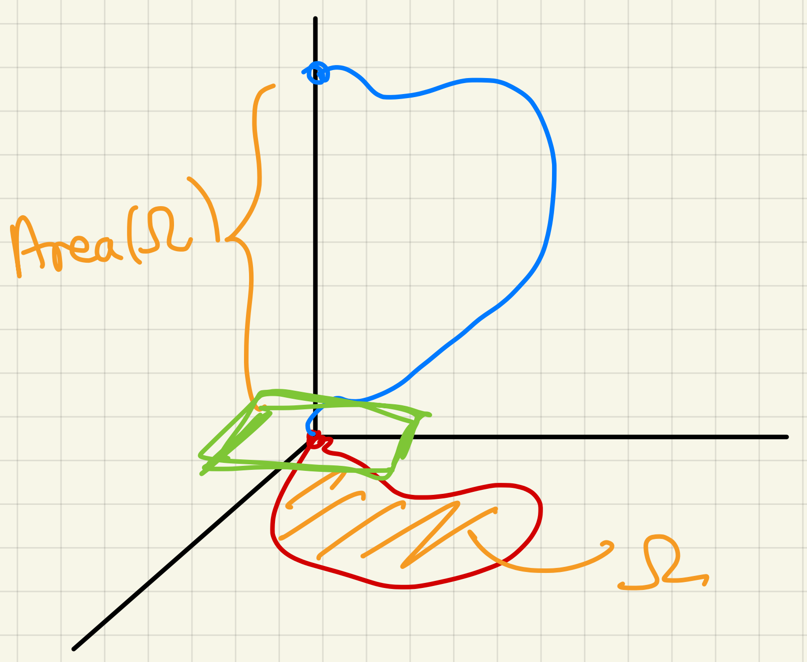
<u>Green formula</u>

$= \text{Area}(\Omega_t)$



Recall
$$\ell_{SR}(\gamma) = \int_0^T \left(u_1(t)^2 + u_2(t)^2 \right)^{1/2} dt$$
$$= \int_0^T \left(\dot{x}(t)^2 + \dot{y}(t)^2 \right)^{1/2} dt = \ell_{Eucl.}(\pi(\gamma))$$

$$\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longrightarrow (x,y)$$



$$D = \text{span}\{X, Y\}$$
$$X = \partial_x - \frac{y}{2}\partial_z$$
$$Y = \partial_y + \frac{x}{2}\partial_z$$
$$D|_{(0,0,0)} = \text{span}\{\partial_x, \partial_y\}$$

② (Bracket generating vs non integrability)

Frobenius $D = \{X_1 \dots X_k\}$ st.

$$[X_i, X_j] = \sum_{l=1}^{k} c_{ij}^l X_l \iff D \text{ involutive} \text{ a integrable.}$$

D integrable $\overset{def}{\iff}$ $\exists\, S$ k-dim sub. $T_x S = D_x \quad \forall x \in S$

locally def

Consider the rank 2 distribution

$D = \operatorname{span}\{X, Y\}$ on $\mathbb{R}^3$

$X = \partial_x$

$Y = \partial_y + \phi(x)\,\partial_z$

$$\phi(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is C^∞ but not analytic.

One can see that

$[X, Y] = \phi'(x)\,\partial_z$ (vanish at $x=0$)

$[Y, [X, Y]] = 0$

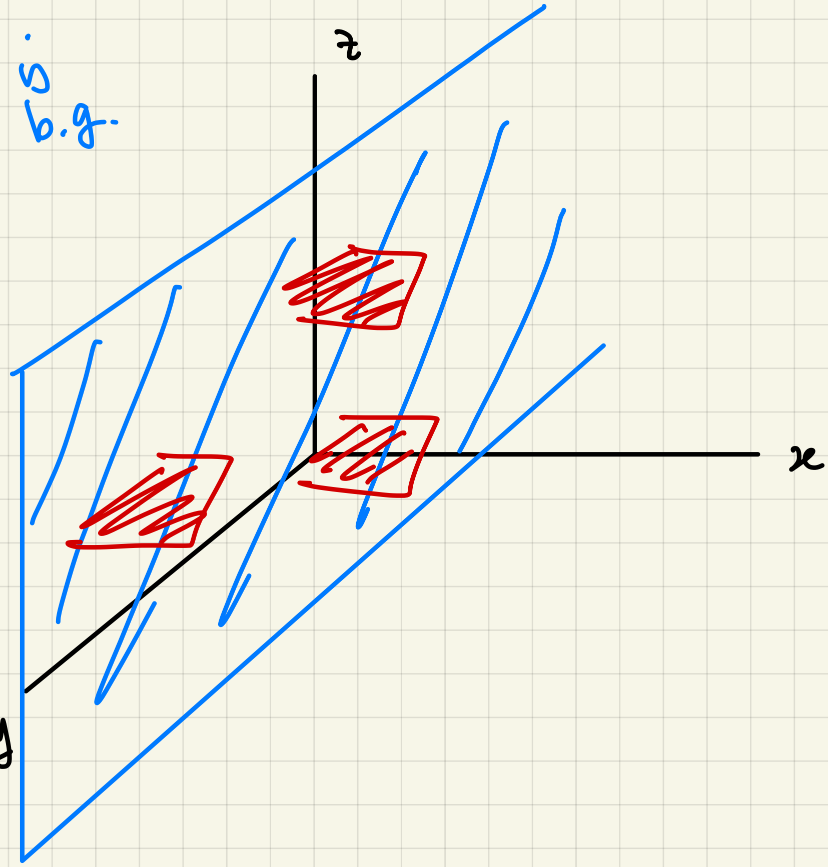
$[\underbrace{X, \dots,}_{j} [X, Y]] = \phi^{(j)}(x)\,\partial_z$ (vanish at $x=0$)

D is not bracket generating on $\{x=0\}$ S

where D is not b.g.

$\downarrow$

$\{x=0\}$



at $x=0$

$X = \partial_x$

$Y = \partial_y$

At $S = \{x=0\}$ D does not satisfy $T_q S = D_q$

<u>Comment</u>: Here we have C^∞ not analytic structure which is not bracket gen. at every point but it is non integrable.

If M and D are analytic then we have equivalence bracket gen. at every point

$\Updownarrow$

there exist no integral subman for D.

LENGTH - MINIMISERS : a first characterisation

A length-min is $\gamma : [0,T] \to M$ joining x and y

$$\ell_{SR}(\gamma) = d_{SR}(\gamma(0), \gamma(T))$$

① every hor. curve is the reparam. of a hor curve with const speed (and ℓ_{SR} invariant)

② curves with constant speed realize the equality in this Cauchy-Schwartz qine.

$$\left(\int_0^T \|\dot\gamma(t)\| \, dt \right)^2 \leq \left(\int_0^T 1 \, dt \right) \left(\int_0^T \|\dot\gamma(t)\|^2 \, dt \right)$$

$$\underset{\ell(\gamma)^2}{\parallel} \qquad \leq 2T\, J(\gamma)$$

$$J(\gamma) = \frac{1}{2} \int_0^T \|\dot\gamma(t)\|^2 \, dt$$ ← sub-Riem. energy.

To find length minimisers from x to y given $T>0$ we have to solve

$$\inf \{ J(\gamma) \mid \gamma \in H^T_{x,y} \}$$

where $H^T_{x,y}$ = hor curves betw x & y def $[0,T]$.

Observe that this is an optimal control problem

$$(*)\begin{cases} \inf \underbrace{\boxed{\int_0^T \sum_{i=1}^k u_i(t)^2 \, dt}}_{J(u)} \\ x(0)=x \ , \ x(T)=y \\ \boxed{\dot{x}(t)=\sum_{i=1}^k u_i(t)\, X_i(x(t))} \end{cases}$$

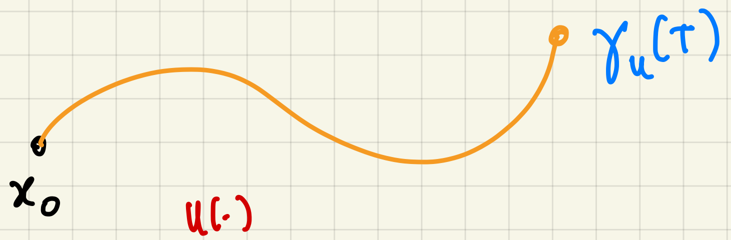
cost is quadratic in u

dynamic linear in u

If we imagine J as a function of u and we introduce the end-point map

$$E_{x_0,T} : u(\cdot) \longmapsto \gamma_u(T)$$

solution of the ODE $(*)$



changed x & y with. x_0 & x_1

So that our problem becomes

$$\inf \{ J(u) \mid E_{x_0,T}(u) = x_1 \}$$

inf function on a level set

A finite dimensional anteprima

<u>Prop</u> Assume $L : \mathbb{R}^n \to \mathbb{R}$ smooth
$F : \mathbb{R}^n \to \mathbb{R}^m$ smooth.

If $x_0 \in \mathbb{R}^n$ solves the problem

$$\inf \{ L(x) \mid F(x) = y \}$$

row vectors

then $\exists \ (\lambda_0, \nu) \neq (0,0) \in (\mathbb{R}^m \times \mathbb{R})^*$ st.

$$\lambda_0 \cdot DF(x_0)[w] + \nu \cdot DL(x_0)[w] = 0 \quad \forall w.$$

row vector.