A finite dimensional anteprima

NORMAL VS ABNORMAL

Prop Assume $L : \mathbb{R}^n \to \mathbb{R}$ smooth
$F : \mathbb{R}^n \to \mathbb{R}^m$ smooth.

If $x_0 \in \mathbb{R}^n$ solves the problem

$$\inf \{ L(x) \mid F(x) = y \}$$

row vectors

then $\exists \ (\lambda_0, \nu) \neq (0,0) \in (\mathbb{R}^m \times \mathbb{R})^*$ st.

$$\boxed{\lambda_0 \cdot DF(x_0) + \nu \cdot DL(x_0) = 0}$$

row vector.

this means $\lambda_0 \, DF(x_0)[w] + \nu \, DL(x_0)[w] = 0 \quad \forall w.$

Proof If x_0 is a min then
$L(x_0) \leq L(x) \quad \forall x$ such that $F(x) = y$

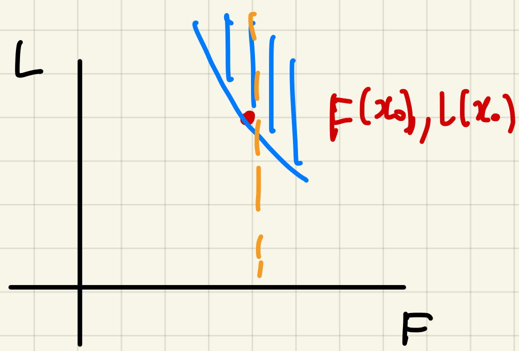
In particular the extended map

$$\bar{F} : \mathbb{R}^n \to \mathbb{R}^{m+1} \qquad \bar{F}(x) = (F(x), L(x))$$

cannot be open at x_0

So the $D\bar{F}(x_0)$ not surjective

$$D\bar{F}(x_0) = (DF(x_0), DL(x_0))$$



① Observe that the pair $(\lambda_0, \nu) \neq (0,0)$ is defined up to mult. by a scalar

If $\{F(x) = y\}$ is a regular level set (if $DF(x)$ surj $\forall x : F(x) = y$) then necessarily $\nu \neq 0$.

We can fix $\nu = -1$ and write

$$DL(x_0) = \lambda_0 \cdot DF(x_0) = \sum_{i=1}^{m} \lambda_i \, DF_i(x)$$

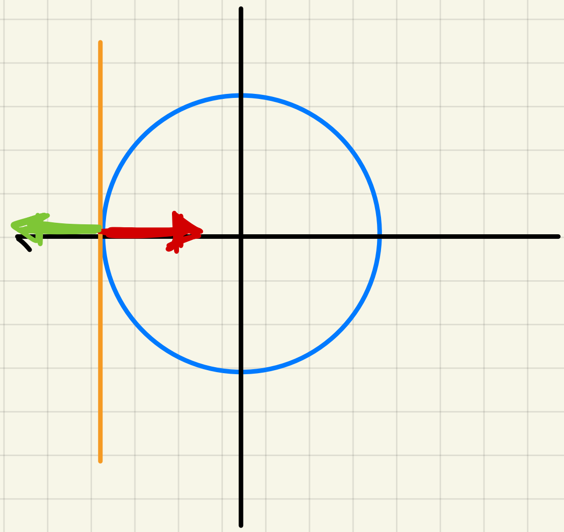
where $\lambda_0 = (\lambda_1, \dots, \lambda_m)$

Lagrange mult. rule.

THREE EXAMPLES We are in $\mathbb{R}^2$

$$\inf \{ L(x,y) \mid F(x,y) = 0 \}$$

① $L(x,y) = x$ $\qquad F(x,y) = x^2 + y^2 - 1$



inf reached at $(-1,0)$

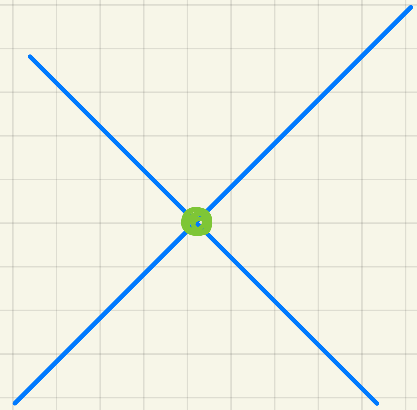
$\nabla L(-1,0) = (+1,0)$

$\nabla F(-1,0) = (-1,0)$

$\nu = -1 \qquad \lambda_0 = -1$

NORMAL

(2) $L(x,y) = x^2 + y^2$ $\quad$ $F(x,y) = x^2 - y^2$



inf reached at $(0,0)$

$DL(0,0) = (0,0)$

$DF(0,0) = (0,0)$

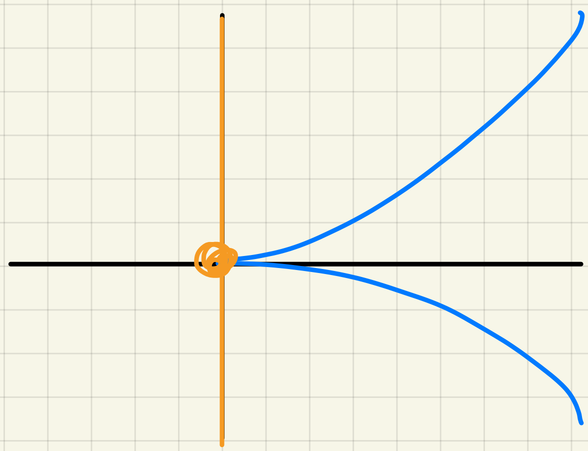
$\nu = -1$ $\quad$ λ_0 arbitrary $\leftarrow$ a family

$\nu = 0$ $\quad$ $\lambda_0 = 1$ $\leftarrow$ one solution.

NORMAL and ABNORMAL

(3) $L(x,y) = x$ $\quad$ $F(x,y) = x^3 - y^2$

inf reached at $(0,0)$



$DL(0,0) = (1,0)$

$DF(0,0) = (0,0)$

necessarily $\boxed{\nu = 0}$

$\lambda_0 = 1$

STRICTLY ABNORMAL

Remark In the geometric setting, if

$L : M \to \mathbb{R}$
$F : M \to N$ where M, N smooth manif.

the pair $(\lambda_0, \nu) \neq (0,0)$ satisfying

$$\lambda_0 \cdot D_{x_0} F + \nu \cdot D_{x_0} L = 0$$

should be an element $(\lambda_0, \nu) \in T^*_{F(x_0)} M \times \mathbb{R}^*$

since $D_{x_0} F : T_{x_0} M \longrightarrow T_{F(x_0)} M$

$\lambda_0 : T_{F(x_0)} M \to \mathbb{R}$

Theorem let $\gamma : [0,T] \to M$ be a length-minim param with constant speed satisfying

$$\dot{\gamma}(t) = \sum_{i=1}^{k} \bar{u}_i(t) \, X_i(\gamma(t)) \qquad (*)$$

Denote $\Phi_{0,t}$ flow of the ODE $(*)$

$\Phi_{0,t} : x_0 \longmapsto x_{\bar{u}}(t)$ where $x_{\bar{u}}(t)$ solves $(*)$ with $x_{\bar{u}}(0) = x_0$.

There exists $p_0 \in T^*_{x_0} M$ such that setting

$$p(t) = {}^t(D_{x_0} \Phi_{0,t})^{-1} \, p_0 \in T^*_{\gamma(t)} M$$

satisfies at least one of the following conditions

(N) $\quad \bar{u}_i(t) = p(t)\cdot X_i(\gamma(t)) = \langle p(t), X_i(\gamma(t)) \rangle$

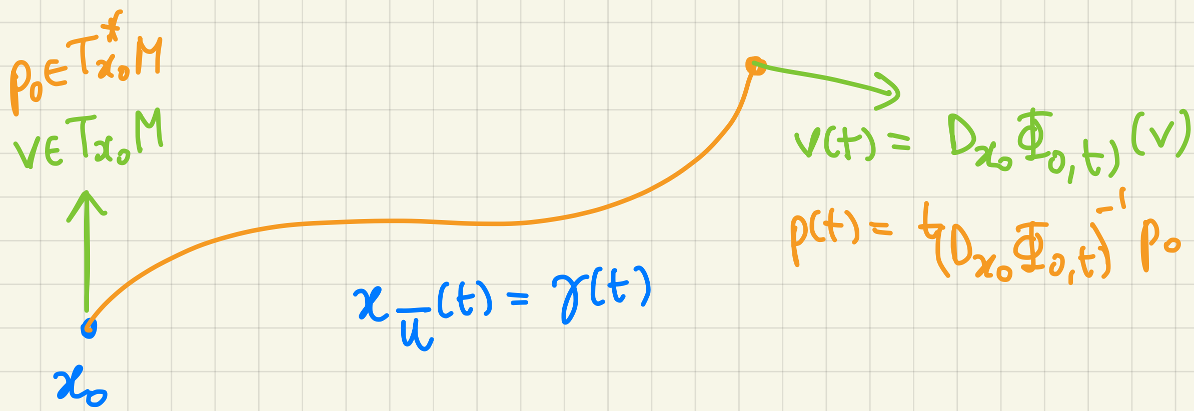
(A) $\quad 0 = p(t)\cdot X_i(\gamma(t))$ with $p_0 \neq 0$.

<u>Remark</u> Notice that if $\Phi^{\bar{u}}_{0,t} : x_0 \longrightarrow \overset{\gamma}{x(t)}$

then $D_{x_0}\Phi_{0,t} : T_{x_0}M \longrightarrow T_{x(t)}M$

$$^t(D_{x_0}\Phi_{0,t}) : T^*_{x(t)}M \longrightarrow T^*_{x_0}M$$

$$^t(D_{x_0}\Phi_{0,t})^{-1} : T^*_{x_0}M \longrightarrow T^*_{x(t)}M$$



In what follows we will use the notation from diff. geometry

$$D_{x_0}\Phi_{0,t} \rightsquigarrow (\Phi_{0,t})_* \qquad \text{pushforward.}$$

$$^t(D_{x_0}\Phi_{0,t})^{-1} \rightsquigarrow (\Phi^*_{0,t})^{-1} \qquad \text{(inverse of the) pull back}$$

Proof Recall $T>0$ is fixed, $\bar{u}$ fixed.

$E_{x_0,T}$ end point map.

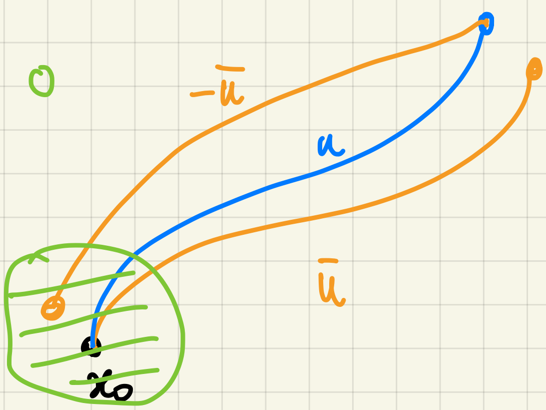
$$E_{x_0,T} : u(\cdot) \longmapsto x_u(T)$$

$$\begin{cases} x(0) = x_0 \\ \dot{x}(t) = \sum_{i=1}^{k} u_i(t) X_i(x(t)) \end{cases}$$

$\gamma(t) = x_{\bar{u}}(t)$

u close to $\bar{u}$
or
v close to 0

$$u(\cdot) = \bar{u}(\cdot) + v(\cdot)$$



Define $y_u(t) := \Phi_{0,t}^{-1}(x_u(t))$

(depends on u or on v also)

Claim 1 The curve $y(t) = (y_u(t) = y_v(t))$ satisfies

$$\boxed{\dot{y}(t) = \sum_{i=1}^{k} v_i(t)\, Y_i^t(y(t))}$$

where $Y_i^t := (\Phi_{0,t})_*^{-1} X_i$

<u>RK</u> Given $\Phi : M \to N$ diffeo

$X \in Vec(M)$ $\qquad\qquad$ $Vec(N)$ $\ni$

$$X = \sum_{i=1}^{n} X_i(x) \frac{\partial}{\partial x_i} \qquad\qquad \Phi_* X = \sum_{j=1}^{n} X_i(\Phi^{-1}(y)) \frac{\partial \Phi_j}{\partial x_i}(\Phi^{-1}(y)) \frac{\partial}{\partial y_j}$$

We can build <u>return end point map</u>

$$F_{x_0,T}(u) := \left(\Phi^{\bar{u}}_{0,T}\right)^{-1} E_{x_0,T}(u).$$

crucial $F_{x_0,T}(\bar{u}) = x_0$ ← back to the initial point!

The extended map $\bar{F} = (F_{x_0,T}, J)$ cannot be open in a neighborhood of $\bar{u}$

Thanks to the "lagrange multipliers rule" there exists $(p_0, \nu) \neq (0,0)$ such that

$$\boxed{p_0\, D_{\bar{u}} F_{x_0,T}[v] + \nu\, D_{\bar{u}} J[v] = 0} \quad \forall v \in L^\infty$$

Now $J(u) = \int_0^T \sum_{i=1}^k u_i(t)^2\, dt$ ← quadratic.

$$D_{\bar{u}} J(v) = \int_0^T \sum_{i=1}^k \bar{u}_i(t)\, v_i(t)\, dt$$

$$J(\bar{u}+v) = J(\bar{u}) + D_{\bar{u}} J[v] + o(\|v\|)$$

On the other hand.

Claim 2 $$D_{\bar{u}} F_{x_0,T}[v] = \int_0^T \sum_{i=1}^k v_i(t)\, Y_i^t(x_0)\, dt$$

In the case $\nu = -1$ we get

$$D_{\bar{u}} J[v] = \langle p_0, D_{\bar{u}} F_{x_0,T}[v] \rangle$$

$$\int_0^T \sum \bar{u}_i(t) v_i(t) dt = \langle p_0, \int_0^T \sum_{i=1}^k v_i(t) Y_i^t(x_0) \, dt \rangle$$

$$= \int_0^T \sum_{i=1}^k v_i(t) \langle p_0, Y_i^t(x_0) \rangle \, dt$$

true for all variations $v \in L^\infty$

We Deduce

$$\bar{u}_i(t) = \langle p_0, Y_i^t(x_0) \rangle .$$

$$= \langle p_0, \left[(\Phi_{0,t})_*^{-1} X_i \right](x_0) \rangle$$

moving the transpose to the left

$$= \langle (\Phi_{0,t}^*)^{-1} p_0, X_i(\Phi_{0,t}(x_0)) \rangle$$

$$= \langle p(t), X_i(\gamma(t)) \rangle .$$

The case $\nu = 0$ is similar. Since $(p_0, \nu) \neq (0,0)$, in this case $p_0 \neq 0$!

Proof of Claim 1

Set $\quad y(t) = \Phi_{0,t}^{-1}(x(t)) \quad$ where

$$\dot{x}(t) = \sum_{i=1}^{k} u_i(t)\, X_i(x(t)) \qquad \boxed{u = \bar{u} + v}$$

better to write $\quad x(t) = \Phi_{0,t}(y(t))$

$$\dot{x}(t) = \frac{d}{dt}\Big(\Phi_{0,t}(y(t))\Big)$$

flow of $\bar{u}$

$$= \Big(\frac{\partial}{\partial t}\Phi_{0,t}\Big)(y(t)) + (\Phi_{0,t})_* \, \dot{y}(t)$$

$$\sum_{i=1}^{k} u_i(t)\, X_i(x(t)) = \sum_{i=1}^{k} \bar{u}_i(t)\, X_i\big(\underbrace{\Phi_{0,t}(y(t))}_{=\, x(t)}\big) + (\Phi_{0,t})_* \, \dot{y}(t)$$

$$(\Phi_{0,t})_* \, \dot{y}(t) = \sum_{i=1}^{k} v_i(t)\, X_i(x(t)) \qquad \Phi_{0,t}^{-1}(x(t))$$

$$\dot{y}(t) = \sum_{i=1}^{k} v_i(t) \Big[(\Phi_{0,t})_*^{-1} X_i\Big](y(t))$$

Proof of Claim 2

To compute the differential of $F_{x_0,T}$ at $\bar{u}$ we do

$$\underbrace{F_{x_0,T}(\bar{u}+v)}_{=\, y_v(T)} - \underbrace{F_{x_0,T}(\bar{u})}_{=\, x_0} = D_{\bar{u}} F_{x_0,T}(v) + o(\|v\|)$$

in L^∞

$$\underset{\substack{\shortparallel \\ y_v(0)}}{y_v(T) - x_0} = \int_0^T \dot{y}_v(t)\, dt$$

$$= \int_0^t \sum_{i=1}^{k} v_i(t)\, Y_i^t(y_v(t))\, dt$$

here $Y_i^t(y_v(t))$ depends on v, the 0^{th} order term (wrt v) is $Y_i^t(y_0(t)) = Y_i^t(x_0)$

$$= \underbrace{\int_0^t \sum_{i=1}^{k} v_i(t)\, Y_i^t(x_0)\, dt}_{D_{\bar{u}} F_{x_0,T}[v]} + o(\|v\|) .$$