THE HAMILTONIAN VIEWPOINT

The goal of today lecture is to reinterpret the first order conditions for length-min param. with constant speed $\gamma : [0,T] \to M$ as a single equation for the pair

$$\lambda(t) = (\gamma(t), p(t)) \quad \text{in} \quad T^*M \simeq \mathbb{R}^n \times \mathbb{R}^n$$

$$\boxed{\begin{cases} \dot\gamma(t) = \sum_{i=1}^{k} u_i(t)\, X_i(\gamma(t)) \\ p(t) = (\Phi_{0,t}^{-1})^* p_0 \end{cases}}$$

$$p_0 \in T^*_{x_0} M, \quad x_0 = \gamma(0)$$

with either (N) $\langle p(t), X_i(\gamma(t)) \rangle = u_i(t)$ a.e $t \in [0,T]$, $i = 1, \dots, k$

(A) $\langle p(t), X_i(\gamma(t)) \rangle = 0$

In case (A) $p_0 \neq 0$ and so $p(t) \neq 0$ $\forall t \in [0,T]$.

Variational and adjoint equation

X vector field

$$(E) \quad \dot x(t) = X(x(t)), \qquad x(0) = x_0$$

If such a solution $x(t)$ defined on $[0,T]$ then for initial cond. close to x_0 also we have a sol def on $[0,T]$.

We can write for $x \in U$ neigh of x_0

$$(*) \quad \frac{\partial}{\partial t} \Phi_{0,t}(x) = X(\Phi_{0,t}(x)) \qquad \begin{array}{l} x \in U \\ t \in [0,T] \end{array}$$

Using smoothness wrt initial condition we can differentiate $(*)$ at x_0 in the dir of v_0

$$\frac{\partial}{\partial t} \underbrace{D\Phi_{0,t}(x_0)[v_0]}_{v(t)} = DX(\underbrace{\Phi_{0,t}(x_0)}_{x(t)}) \underbrace{D\Phi_{0,t}(x_0)[v_0]}_{v(t)}$$

$$\begin{cases} \dot{v}(t) = DX(x(t))\, v(t) \\ v(0) = v_0 \end{cases}$$

linearized equation of (E) along $x(t)$.

$\rightarrow \quad v(t) = D\Phi_{0,t}(x_0)[v_0]$

the flow of the linearized equation is the linearization of the flow

<u>Remarks</u>| Consider two nonautonomous eq in $\mathbb{R}^n$

$$(**) \qquad \dot{v} = A(t)\, v \qquad\qquad \dot{p} = -A(t)^T p$$

are called <u>adjoint equations</u>

the scalar product between two sol. is constant

$$\frac{d}{dt} \langle p, v \rangle = \langle \dot{p}, v \rangle + \langle p, \dot{v} \rangle$$

depends on t
solves (**)

$$= - \langle A(t)^T p, v \rangle + \langle p, A(t) v \rangle$$

$$= 0 .$$

Moreover one can check if

$$v(t) = M(t) v_0 \quad \text{then} \quad p(t) = N(t) p_0$$

$$\text{where} \quad N(t) = (M(t)^{-1})^T .$$

In particular if

$$\dot{v}(t) = DX(x(t))\, v(t)$$

$$\dot{p}(t) = - DX(x(t))^T p(t)$$

$$\text{with} \quad p(t) = {}^t\!\left(D\Phi_{0,t}(x_0)\right)^{-1} p_0$$

$$= \left(\Phi_{0,t}^*\right)^{-1} p_0$$

notation for $p(t)$ as a column.

If we consider $p(t)$ as a row vector

$$\dot{p}(t) = - p(t) \cdot DX(x(t))$$

the second crucial observation is that

$$\begin{cases} \dot{x} = X(x) \\ \dot{p} = -p \cdot DX(x) \end{cases} \iff \begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}(x,p) \\ \dot{p} = -\dfrac{\partial H}{\partial x}(x,p) \end{cases}$$

where $H(x,p) = h_X(x,p) = p \cdot X(x)$

$$= \sum_{i=1}^{n} p_i X_i(x)$$

linear Hamiltonian in p.

We will also denote by $\vec{H}$ the Hamiltonian vector field

$$\vec{H} = \frac{\partial H}{\partial p}(x,p)\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}(x,p)\frac{\partial}{\partial p}$$

in such a way that $\lambda = (x,p)$

$$\dot{\lambda} = \vec{H}(\lambda) \iff \begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}(x,p) \\ \dot{p} = -\dfrac{\partial H}{\partial x}(x,p) . \end{cases}$$

a single equation in T^*M, of dim $2n$.

Notation Given $X_1, \dots, X_k$ vector fields of the sub-Riemannian structure

$$h_{X_i}(p,x) = p \cdot X_i(x).$$

Theorem Let $\gamma : [0,T] \to M$ length minimizer param with constant speed. Then it admits a lift $\lambda(t) = (\gamma(t), p(t))$ satisfying.

$$(*) \qquad \boxed{\dot{\lambda}(t) = \sum_{i=1}^{k} \bar{u}_i(t)\, \vec{h}_{X_i}(\lambda(t))}$$

Moreover at least one of the following is satisf

$$(N) \qquad h_{X_i}(\lambda(t)) = \bar{u}_i(t) \qquad i = 1, \dots, k$$

$$(A) \qquad h_{X_i}(\lambda(t)) = 0 \qquad i = 1, \dots, k. \qquad \boxed{\lambda_0 \neq 0}$$

continuous. a.e def. $\lambda_0 \in T^*_{\gamma(0)}M$

$$(*) \iff \begin{cases} \dot{x}(t) = \sum_{i=1}^{k} \bar{u}_i(t)\, X_i(x(t)) \\ \dot{p}(t) = -p(t) \cdot \sum_{i=1}^{k} \bar{u}_i(t)\, DX_i(x(t)) \end{cases}$$

Corollary 1 A horizontal curve (param by const speed.) $\gamma : [0,T] \to M$ admits a lift $\lambda(t) = (\gamma(t), p(t))$ satisfying (*) and (N) if and only if

$$\dot\lambda(t) = \vec{H}(\lambda(t)) \qquad H = \frac{1}{2}\sum_{i=1}^{k} h_{X_i}^2$$

$$H(p,x) = \frac{1}{2}\sum_{i=1}^{k} (p \cdot X_i(x))^2$$

quadratic in p.

Moreover H is constant along $\lambda(t)$ and

$$H(\lambda(t)) = \frac{1}{2}\|\dot\gamma(t)\|^2$$

Corollary 2 Every length minimizer (param by const speed) which admits a lift satisfying (N) is of class C^∞.

Proof of the Corollary 1

the condition (N)

$$\dot\lambda(t) = \sum_{i=1}^{k} \bar u_i(t)\, \vec{h}_{X_i}(\lambda(t)) = \sum_{i=1}^{k} h_{X_i}(\lambda(t))\, \vec{h}_{X_i}(\lambda(t))$$

$$= \vec{H}(\lambda(t))$$

use that $\overrightarrow{\frac{1}{2}f^2} = f\vec{f}$

(exercise!)

Proposition Assume $\dim M = 3$ and that D has rank 2 and satisfies

$$D + [D, D] = TM$$

|| kind of strong brack. gen assumpt.

which means if $D = \text{span}\{X_1, X_2\}$

$$\dim \text{span}\{X_1, X_2, [X_1, X_2]\}\big|_x = 3 \qquad \forall x$$

then there exists no abnormal length minimizer joining distinct points.

Proof Assume that there exists one then the lift satisfies

$$\dot{\lambda}(t) = \sum_{i=1}^{2} u_i(t)\, \vec{h}_{X_i}(\lambda(t))$$

$u(t) \neq 0$ because distinct points

$$h_{X_1}(\lambda(t)) = h_{X_2}(\lambda(t)) = 0 \qquad \forall t \in [0, T].$$

these are differentiable a.e on $[0,T)$.

a.e. $$0 = \frac{d}{dt} h_{X_1}(\lambda(t)) = u_2(t)\big(p(t) \cdot [X_1, X_2](\gamma(t))\big)$$

a.e $$0 = \frac{d}{dt} h_{X_2}(\lambda(t)) = -u_1(t)\big(p(t) \cdot [X_1, X_2](\gamma(t))\big)$$

then since $(u_1(t), u_2(t)) \neq (0,0)$ we have that

$$\left\{\begin{aligned} &p(t)\cdot[X_1,X_2](\gamma(t))=0\\ &p(t)\cdot X_1(\gamma(t))=0\\ &p(t)\cdot X_2(\gamma(t))=0\end{aligned}\right. \qquad \left.\begin{aligned}&\\&\end{aligned}\right] h_{X_1}=h_{X_2}=0.$$

$D+[D,D]=TM \Rightarrow p(t)\equiv 0$. <u>Contradiction!</u>

<u>Proof of the identity:</u>

$$\frac{d}{dt}h_{X_1}(\lambda(t)) = \frac{d}{dt}\,p(t)\cdot X_1(\gamma(t))$$

$$= \dot{p}(t)\cdot X_1(x(t)) + p(t)\cdot DX_1(\gamma(t))\,\dot{\gamma}(t)$$

$$= \underbrace{-\,p(t)\cdot\sum_{i=1}^{2}u_i(t)\,DX_i(\gamma(t))}_{\dot{p}\text{ from Ham eq.}}\,X_1(\gamma(t))$$

$$+\,p(t)\cdot DX_1(\gamma(t))\underbrace{\sum_{i=1}^{2}u_i(t)\,X_i(\gamma(t))}_{\dot{\gamma}\text{ in Ham eq.}}.$$

$$\overset{\text{the } i=1 \text{ term simplify}}{=} u_2(t)\,p(t)\cdot\big(-DX_2(\gamma(t))\,X_1(\gamma(t)) + DX_1(\gamma(t))\,X_2(\gamma(t))\big)$$

$$= -\,u_2(t)\,p(t)\cdot[X_1,X_2](\gamma(t)).$$

Corollary If $\dim M = 3$ and $\operatorname{rank} D = 2$

Abnormal length minizers joining distinct points (if they exist) must be horizontal curves contained in the Martinet set

$$\mathcal{M} = \{ x \in M \mid D_x + [D,D]|_x \neq T_x M \}$$

THREE EXAMPLES (all of them $\dim M = 3$, $\operatorname{rank} D = 2$)

(1) **Heisenberg group** In $\mathbb{R}^3$ we consider

$$D = \operatorname{span}\{X, Y\} \qquad X = \partial_x - \frac{y}{2}\partial_z \qquad Y = \partial_y + \frac{x}{2}\partial_z$$

$[X,Y] = \partial_z$ so

$$\dim \operatorname{span}\{X, Y, [X,Y]\} = 3 \quad \forall (x,y,z).$$

So there are no abnormal length min joining distinct points.

Hence we can recover all length min through the solution of a single Ham. eq.

② <u>Martinet structure</u> In $\mathbb{R}^3$ we consider

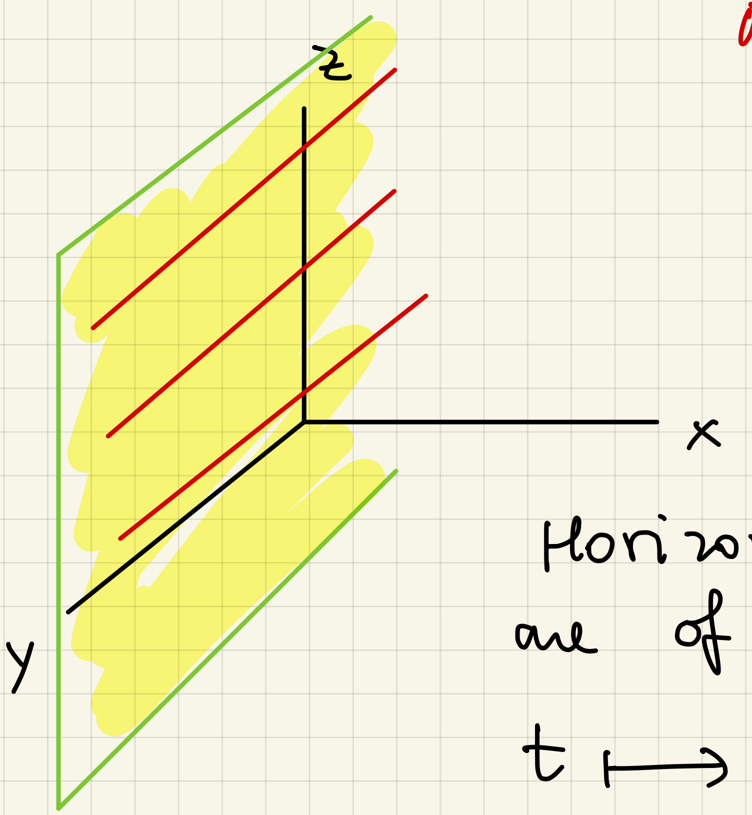
$D = \text{span}\{X, Y\}$ $\quad X = \partial_x \quad Y = \partial_y + \frac{x^2}{2}\partial_z$

here $[X,Y] = x\,\partial_z$.

so $\dim \text{span}\{X, Y, [X,Y]\} = 3 \iff x \neq 0$

In other words $\mathcal{M} = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\}$

Martinet set



When $x = 0$

$X = \partial_x \quad Y = \partial_y$

Horizontal curves in $\mathcal{M}$ are of the form

$t \longmapsto (0, t, z_0)$.

Those curves are length-minimizers and are abnormal.

Taking $\tilde{X}, \tilde{Y}$ in such a way that

$\tilde{D} = \text{span}\{\tilde{X}, \tilde{Y}\} = D$ but $\tilde{g} \neq g$

one can show that these curves are strictly abnormal length minim.
(Richard Montgomery, 1994)

③ Corners! Consider in $\mathbb{R}^3$

$$X = \partial_x - \frac{xy^2}{4}\partial_z \qquad Y = \partial_y + \frac{x^2y}{4}\partial_z$$

$$[X, Y] = \partial_x\left(\frac{x^2y}{4}\right)\partial_z + \partial_y\left(\frac{xy^2}{4}\right)\partial_z$$

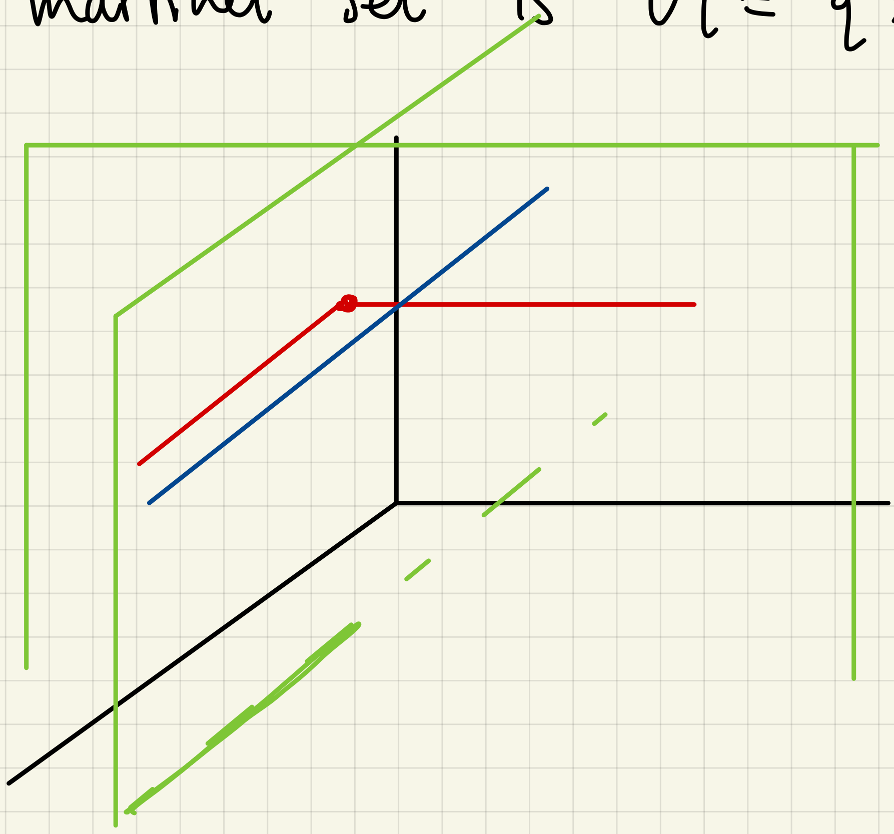
$$= xy\,\partial_z$$

the martinet set is $\mathcal{M} = \{xy = 0\}$

on $\mathcal{M}$

$X = \partial_x$

$Y = \partial_y$.



the corner curve satisfies the necessary condition for being an abnormal length min but it is not a length minimizer.

Open: Are all abnormal length minimizers smooth?

Thm Corners are not length minimizers (Hakavouri - Le Donne 2015?)